CFNS Summer School 2021
Accelerator Physics for EIC

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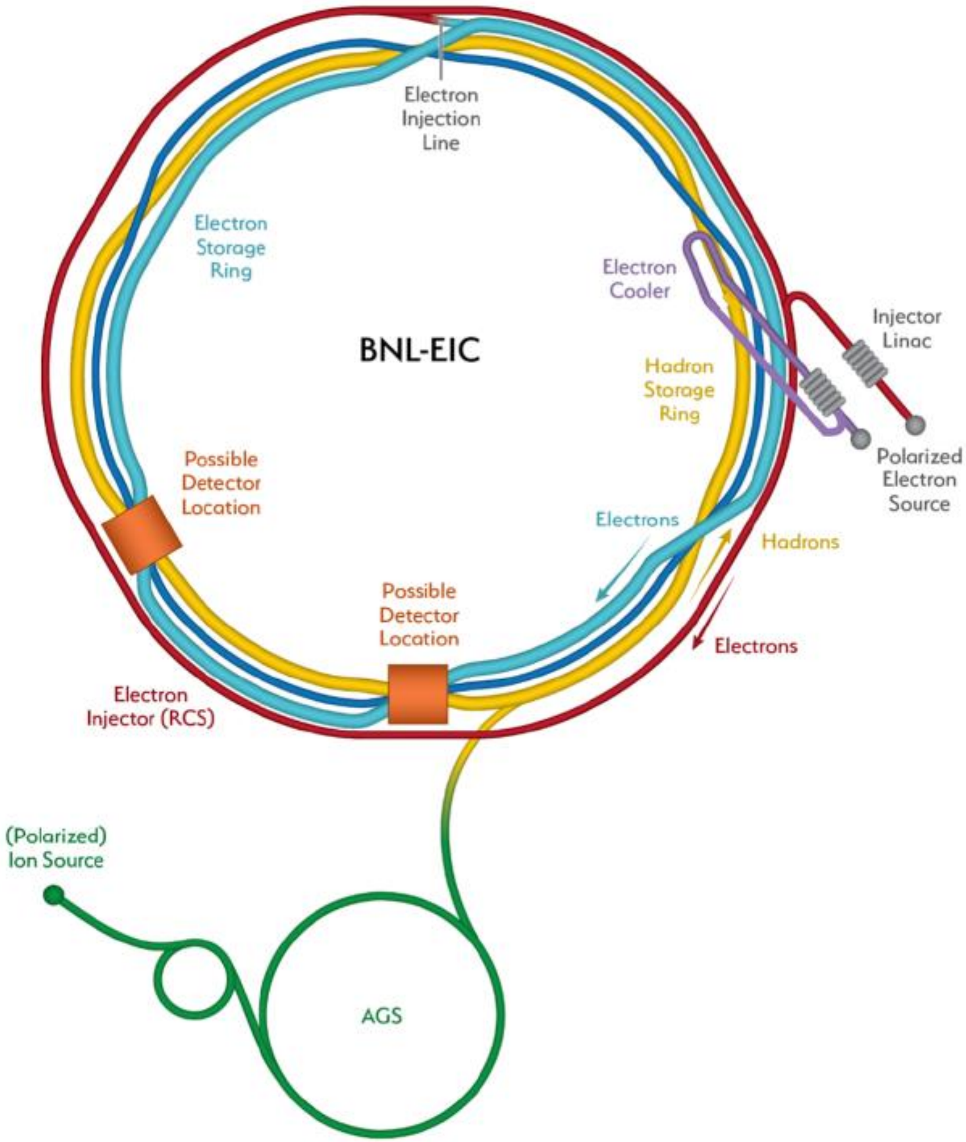
Content

- **Introduction and accelerator fundamental**
  - Overview of US EIC current design
  - Accelerator physics fundamentals
- **Collider accelerator physics**
  - Luminosity, beam-beam effect
- **Spin dynamics**
  - Spin dynamics in circular accelerators
- **Synchrotron radiation and its applications**

US Electron Ion Collider

- 2nd high energy electron ion collider in the world for further exploring the physics inside the nucleon
- Build on top of existing RHIC complex, add an additional electron storage ring for collision, rapid cycling electron ring and its pre-injectors
- Both beams are designed to be polarized



Comparison between HERA and US-EIC

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	US-EIC[1]		HERA[2]	
	hadron	lepton	hadron	lepton
species	p, He, …	e	p	e
energy [GeV]	275 (p)	10	920	27.6
# of bunches	1160	1160	174	174
Bunch intensity$[10^{10}]$	6.9	17.2	7.2	2.9
Emittance[nm]	11.3/1.0	20.0/1.3	5.1/5.1	20/3.4
Beta* [cm]	80/7.2	45/5.6	245/18	63/26
Bunch length [cm]	6	0.7	19	1
Luminosity $[10^{34} cm^{-2} s^{-1}]$	1.0		0.004	

Unique challenges with EIC

- High luminosity and High polarization of both beams

[1] EIC Concept Design Report, 2021
[2] V. Ptitsyn, From HERA to Future Electron-Ion Colliders, Proceeding of PAC07, 2007. https://www.bnl.gov/isd/documents/36707.pdf

Why particle accelerator?: Engine of Discovery

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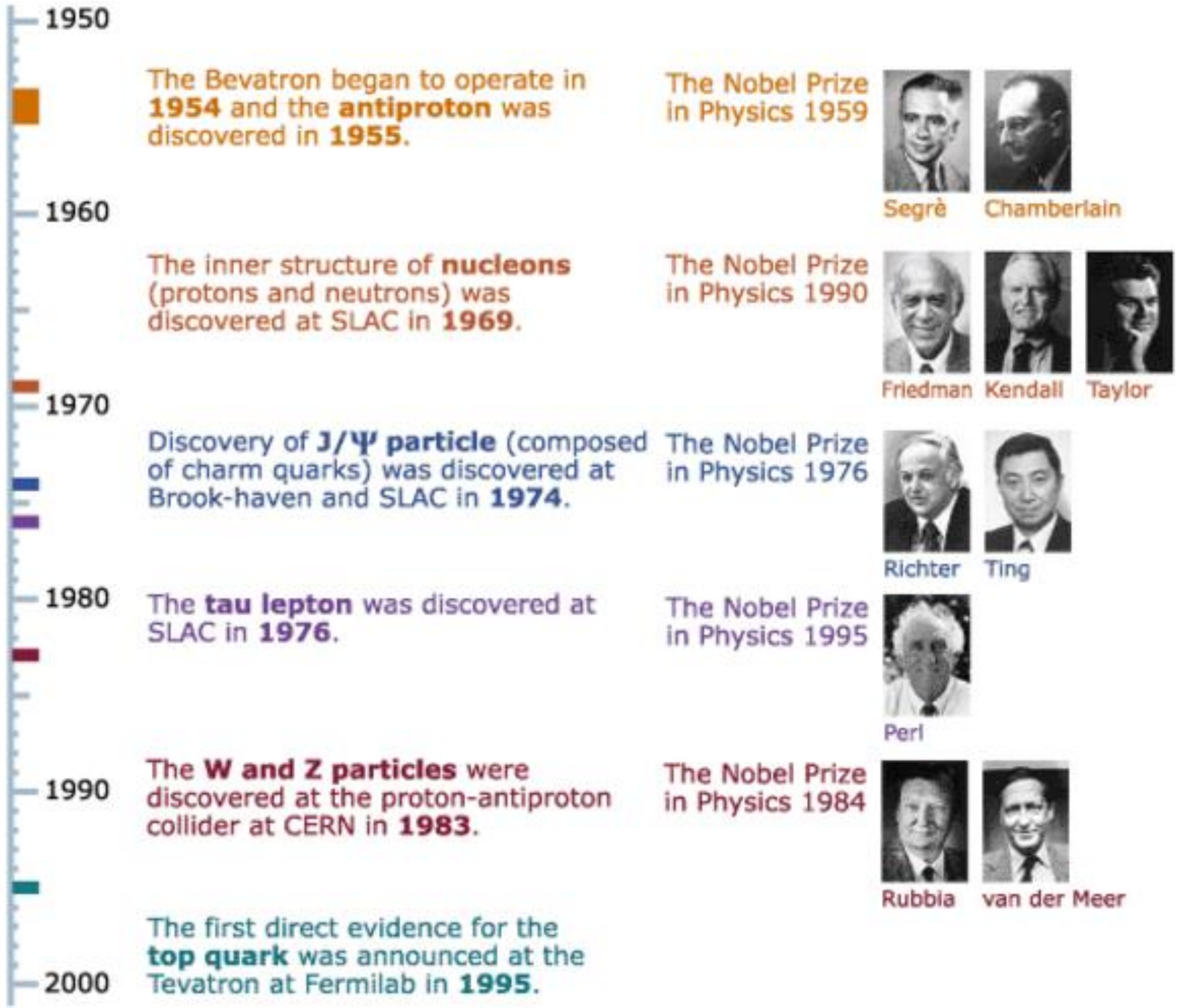
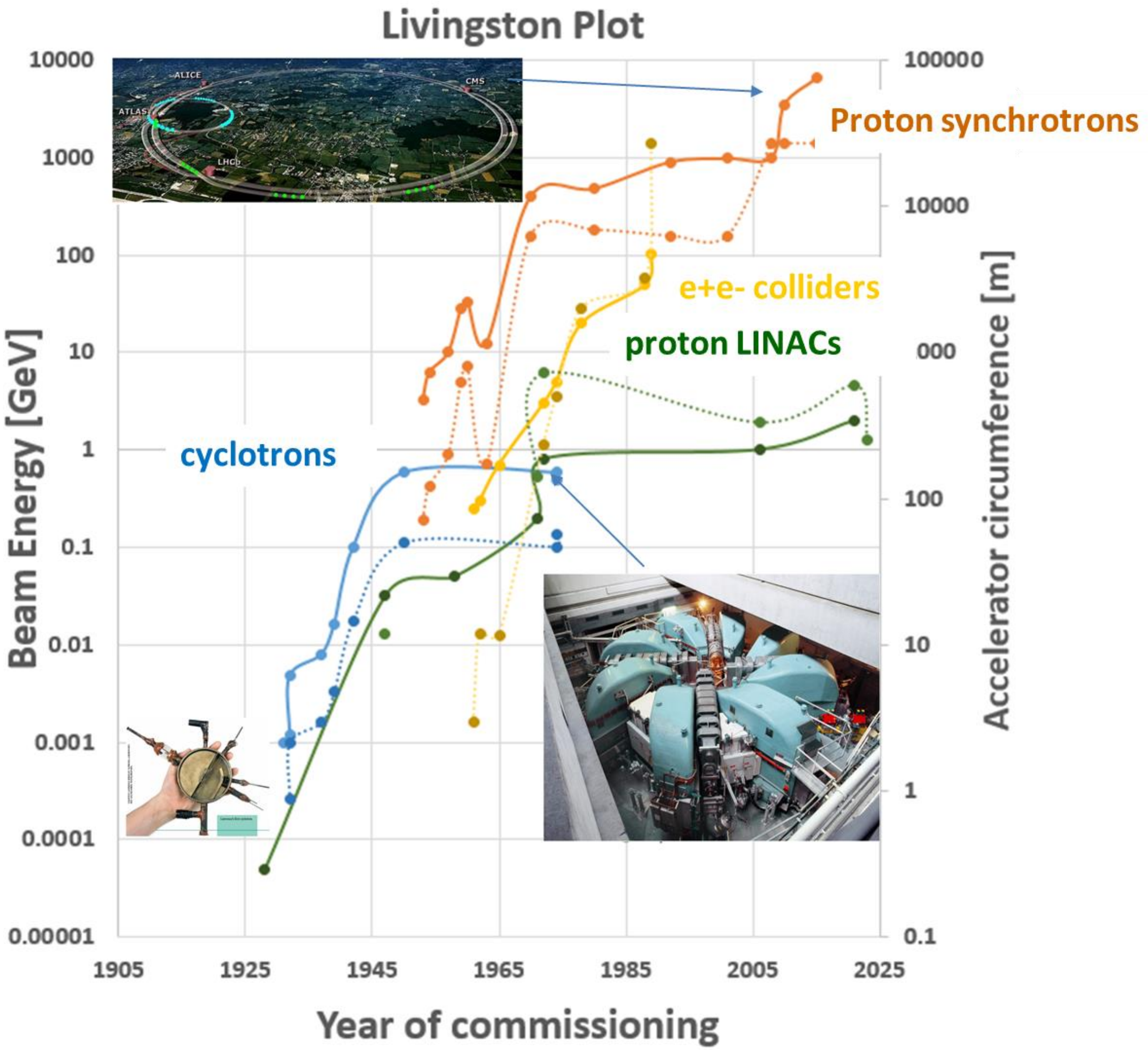


http://www.nobelprize.org/educational/physics/accelerators/discoveries-1.html

Timeline of Accelerators



Legends of Accelerator Development

- John D. Cockcroft and Ernest Walton
  - Invented Cockcroft-Walton Accelerator

- Rolf Wideroe: a nuclear physicist and engineer
  - Invented concept of LINAC when he was a Ph.D student at RWTH-Aachen
  - Invented the principle of Betatron

- Ernest Orlando Lawrence: a nuclear physicist
  - Invented/implemented cyclotron
  - Nobel Laureate in 1939

Legends of Accelerator Development

- E. Courant, H. S. Snyder, M. Stanley Livingston: Accelerator physicists
  - Invented strong focusing principle
- Bruno Touschek: particle physicist
  - father of the 1st e+e- collider (AdA)
- Simon van der Meer: particle physicist and engineer
  - Invented/implemented Stochastic Cooling, and Nobel Laureate in 1984

Modern High Energy Accelerators

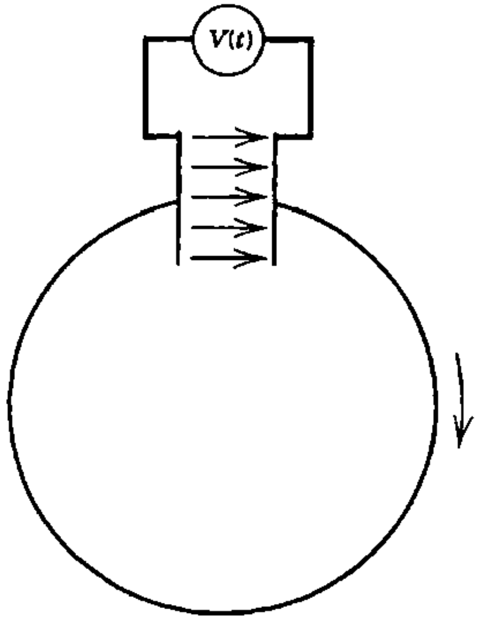
- Linear Accelerator
- Cyclotron
- Betatron
- Synchrotron
- Plasma wake field acceleration

RF field based Acceleration

Can we accelerate particles with DC electric field?

- Of cause, we can!
  - But, can't reach high energy

- what about in a circular machine?



For a circular machine

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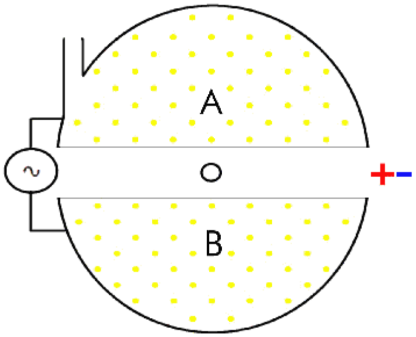
- Use time varying electric field to accelerate particles
  - RF cavity or a group of cavities
  - This is also required for longitudinal focusing for storage
- Cyclotron: fixed B field, spiral trajectory

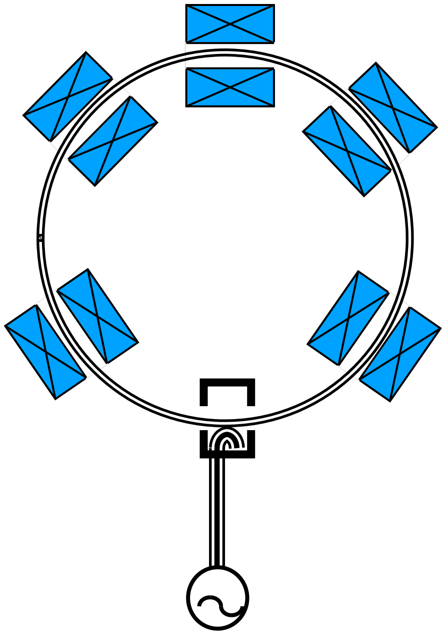


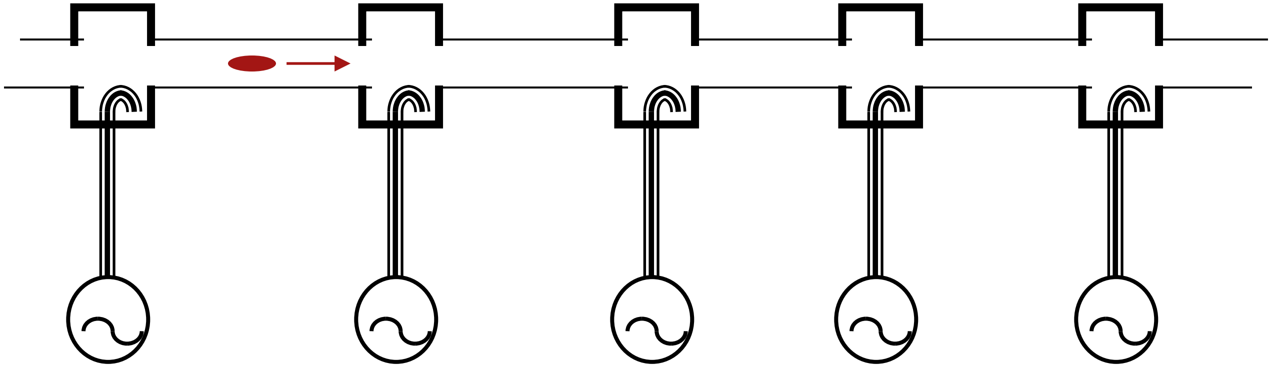
- Synchrotron:
  - Fixed orbit with matched B field $\frac{dB_0}{dt} = \frac{V sin\phi_s}{\rho L}$
  - Rate of energy gain for a synchronous particle $\frac{dE_0}{dt} = \frac{v}{L} eV sin\phi_s$
  - The rate of momentum change: $\frac{dp_0}{dt} = \frac{eV}{L} sin\phi_s$

RF synchronism

Synchronism condition

$$\Delta\tau = N/f_{rf}$$

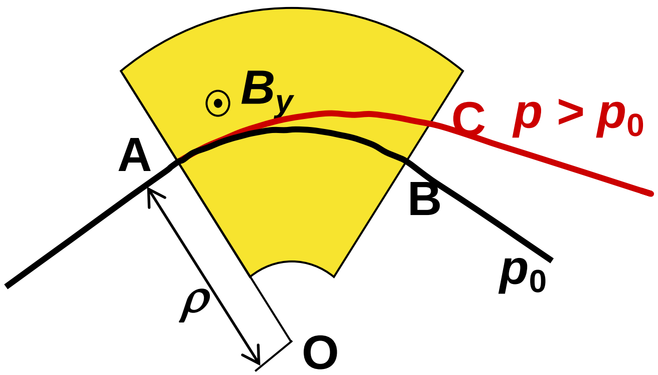




RF synchronism

For a circular machine

$$\rho = \frac{p}{qB_y} = \frac{\beta\gamma m_0 c}{qB_y}$$

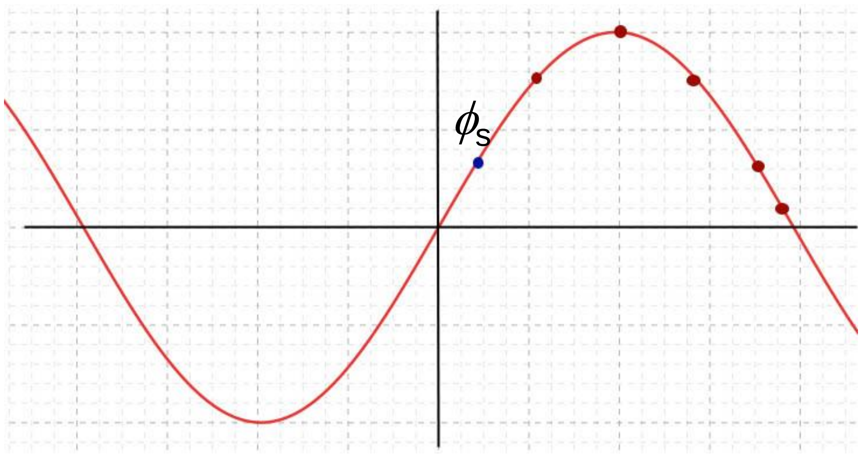


Time between each cavity passage is $\tau = L/v$, where v is the velocity and L is the circumference for a circular accelerator. Hence, we have

$$\frac{\Delta\tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v}$$

RF phase for each pass

$$\phi_{n+1} = \phi_n + \omega_{rf}(\tau + \Delta\tau)_{n+1}$$



The field of one RF cavity is $\epsilon = \epsilon_0 \sin\left(\phi_{rf}(t) + \phi_s\right)$, where

- $\boldsymbol{\phi_{rf}(t) = h\omega_0 t}$ is the phase of the RF field
- $\boldsymbol{\omega_0 = {}^{\beta_0 c}/_{R_0}}$ is angular revolution frequency of the synchronous particle
- $\boldsymbol{R_0}$ is the average radius

For the synchronous particle, the energy gain after passing the cavity is

$$\Delta E_0 = q\epsilon_0 \int_{-\frac{g}{2\beta_0 c}}^{\frac{g}{2\beta_0 c}} sin(h\omega_0 t + \phi_s)\beta_0 c\, dt,$$

where g is the width of the cavity gap

This then yields $\Delta E_0 = e\epsilon_0 gTsin\phi_s = \boldsymbol{eVsin\phi_s}$

The acceleration rate of the synchronous particle is

$$\boldsymbol{\dot{E}_0 = \frac{\omega_0}{2\pi} eVsin\phi_s}$$

Longitudinal Motion

The acceleration rate for a non-synchronous particle is

$$\dot{E} = \frac{\omega}{2\pi} qVsin\phi$$

where $E = E_0 + \Delta E$, $\omega = \omega_0 + \Delta\omega$ and $\phi = \phi_s + \Delta\phi$ are the energy, angular revolution frequency and the RF phase of the non–synchronous particle, respectively.

With this, we have $\phi - \phi_s = h(\omega - \omega_0)t$. Hence, $\dot{\phi} = h\Delta\omega$

As we know, $\frac{\Delta\omega}{\omega_0} = \frac{\beta R_0}{\beta_0 R} - 1$. With $\frac{\beta}{\beta_0} = 1 + \frac{1}{\gamma^2}\delta + o(\delta^2)$ and $R = R_0(1 + \gamma_t^2\delta + o(\delta^2)$ where $\delta = \frac{\Delta p}{p}$ is the momentum spread, one can show $\dot{\phi} = h\omega_0\frac{\Delta\omega}{\omega_0} = \mathrm{h}\omega_0\left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)\delta$, or

$$\dot{\phi} = h\omega_0\eta\delta\text{, where } \eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

Longitudinal equation of motion

With $\boldsymbol{\delta} = \frac{1}{\beta_0^2}\frac{\Delta E}{E_0}$, we have

$$\frac{\dot{\Delta E}}{\omega_0} = \frac{eV}{2\pi}(\sin\phi - \sin\phi_s) \text{ and } \dot{\phi} = h\eta\frac{\omega_0^2}{\beta_0^2 E_0}\frac{\Delta E}{\omega_0}$$

This yields the 2nd order differential equation

$$\ddot{\phi} = h\eta\frac{\omega_0^2}{2\pi\beta_0^2 E_0} qV(\sin\phi - \sin\phi_s)$$

$$(\frac{\ddot{\Delta E}}{\omega_0}) = h\eta\frac{qV\omega_0^2}{2\pi\beta_0^2 E_0}\cos\phi\frac{\Delta E}{\omega_0}$$

For small $\Delta\phi = \phi - \phi_s$,

$$\ddot{\Delta\phi} = h\eta\frac{\omega_0^2}{2\pi\beta_0^2 E_0} qV\cos\phi_s\Delta\phi$$

Longitudinal equation of motion

Clearly, a stable motion of the phase requires $\boldsymbol{\eta cos\phi_s < 0}$

- below transition, $\eta < 0$, $\phi_s < \pi/2$
- above transition, $\eta > 0$, $\pi > \phi_s > \pi/2$

and $\Delta\phi(t) = \Delta\hat{\phi}\cos(Q_s\omega_0 t + \chi)$, where

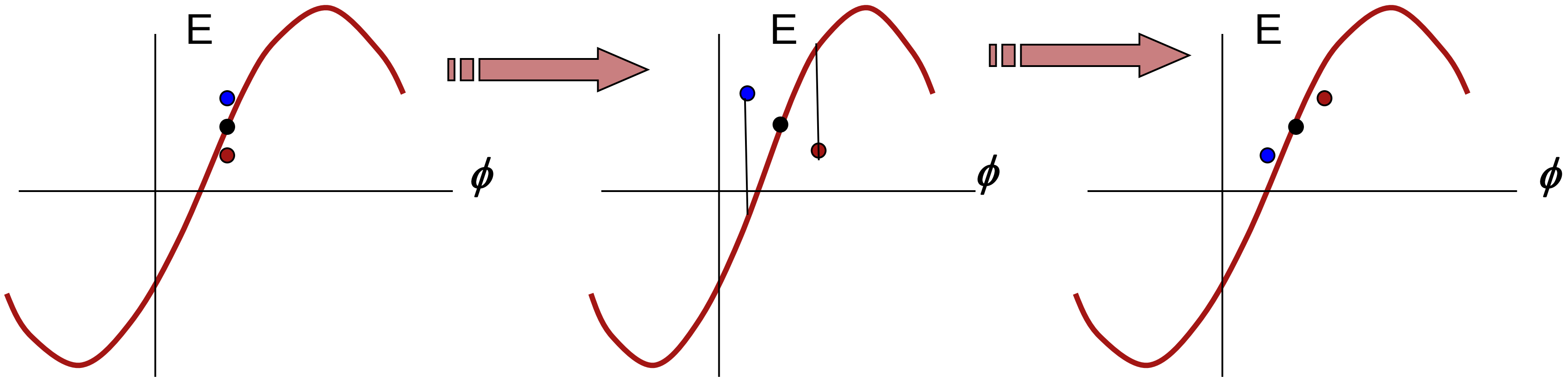
$$\boldsymbol{Q_s = \frac{\omega_s}{\omega_0} = \sqrt{\frac{hqV|\eta cos\phi_s|}{2\pi\beta_0^2 E_0}}}$$

is the synchrotron tune, i.e. # of synchrotron oscillations in one orbital revolution

Example:

for accelerating protons in RHIC, harmonic is 360, RF voltage is 12MV, and $\gamma_t = 22.89$, the synchrotron tune at injection energy 28.3 GeV is 0.00455.

Synchrotron Oscillation



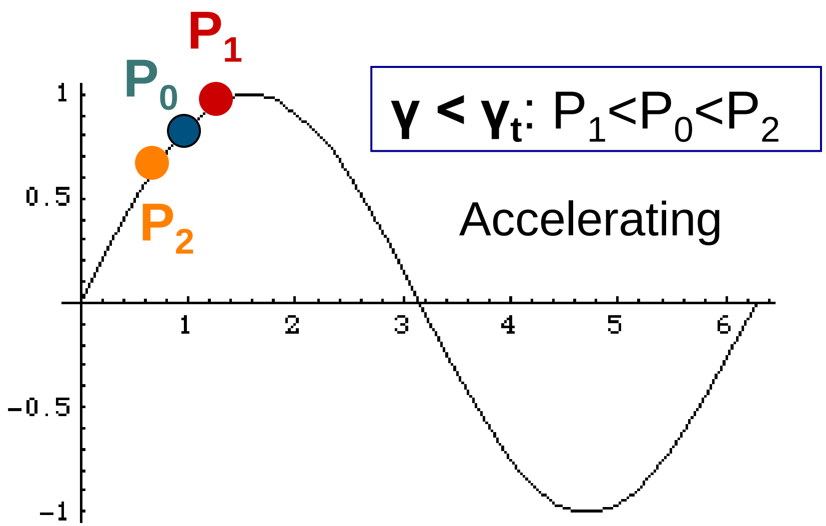
$$\ddot{\phi} = h\eta \frac{\omega_0^2}{2\pi\beta_0^2 E_0} qV(sin\phi - sin\phi_s)$$

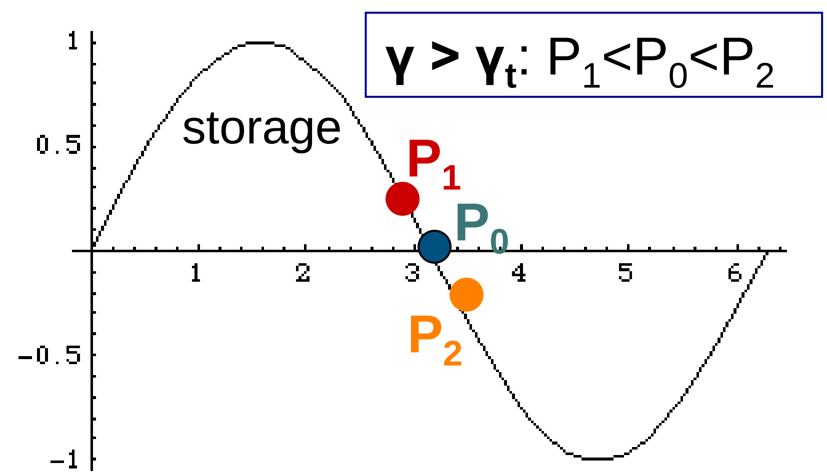
Transition energy

In previous slide (#15), we define the average radius $R = R_0(1 + \alpha_0\delta + \alpha_1\delta^2 + \cdots)$, where R_0 is the average radius of the reference particle, $\delta = \frac{\Delta p}{p}$ is the momentum spread, and $\dot{\boldsymbol{\phi}} = \boldsymbol{h\omega_0\eta\delta}$, where $\boldsymbol{\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}}$.

We then have $\frac{\Delta L}{L} \approx \frac{1}{\gamma_t^2}\delta$, and where γ_t is known as **transition energy.**

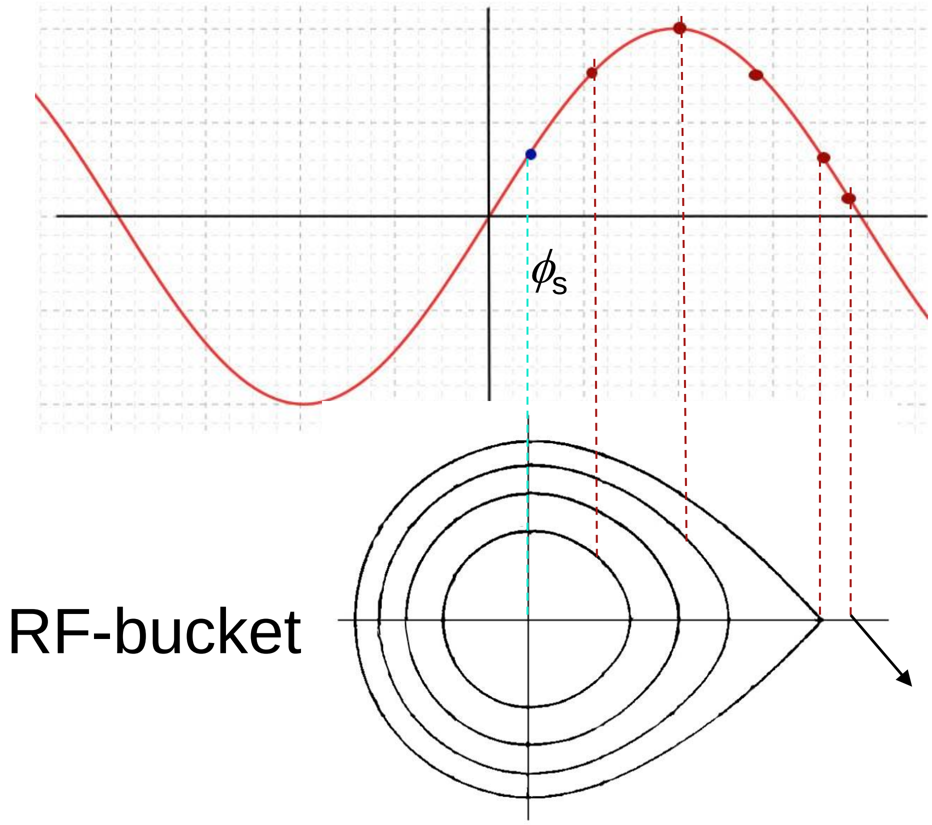
As phase stability requires $\boldsymbol{\eta cos\phi_s < 0}$, when $\gamma = \gamma_t$ the time between each cavity passage becomes independent of energy. This is also known as isochronous condition



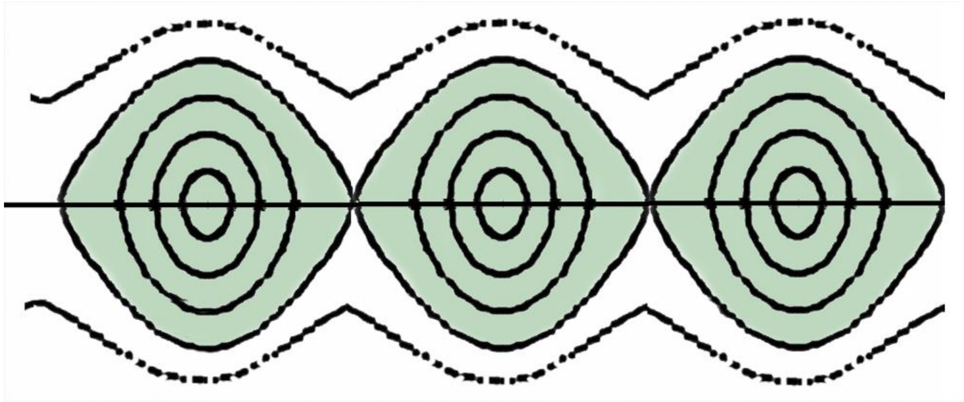


Longitudinal equation of motion

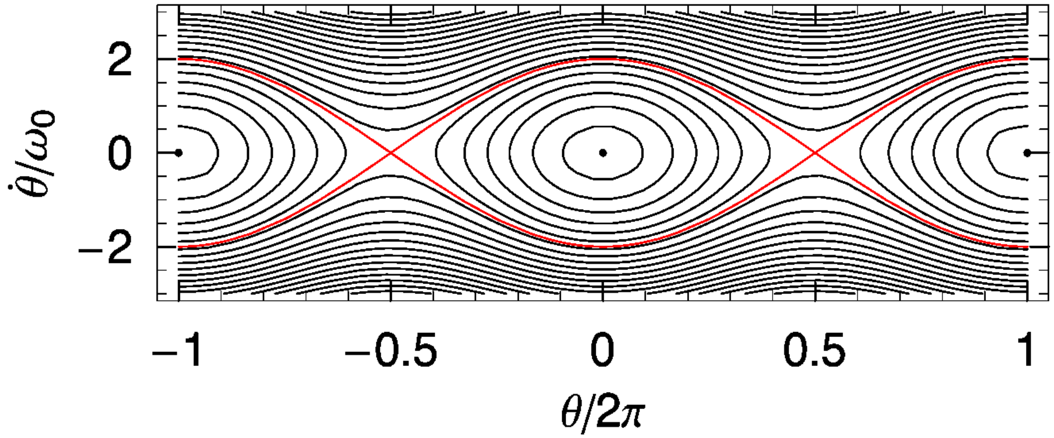
With $\frac{1}{2}(h\omega_0\eta\delta)^2 + h\eta\frac{\omega_0^2}{2\pi\beta_0^2 E_0}eV(cos\phi + \phi\sin\phi_s) = const.$, the phase space of (δ, ϕ) forms a closed curve, RF bucket



Stationary bucket

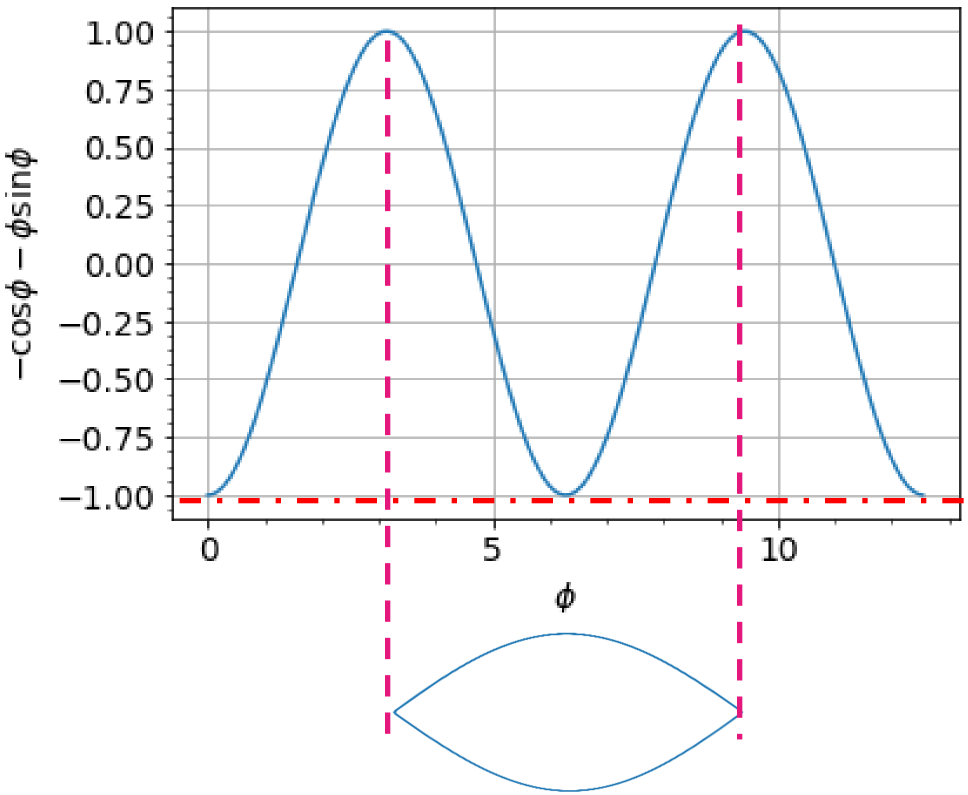


We've seen this behavior for the pendulum

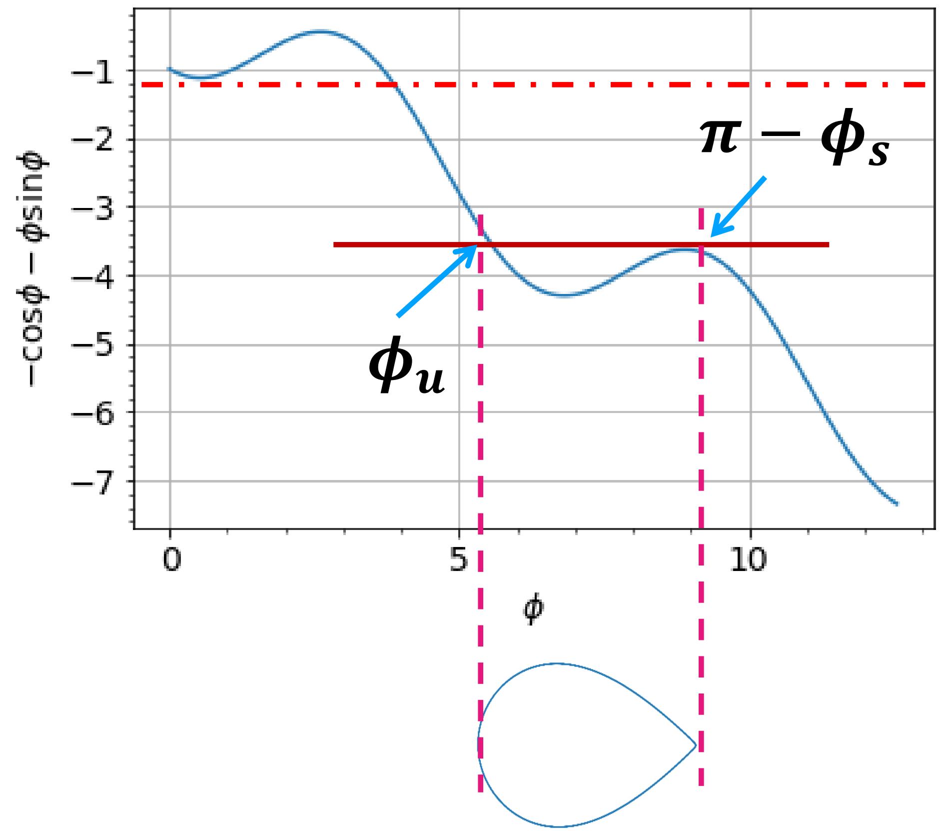


Potential Energy of an RF bucket

$\phi_s = 0.0$

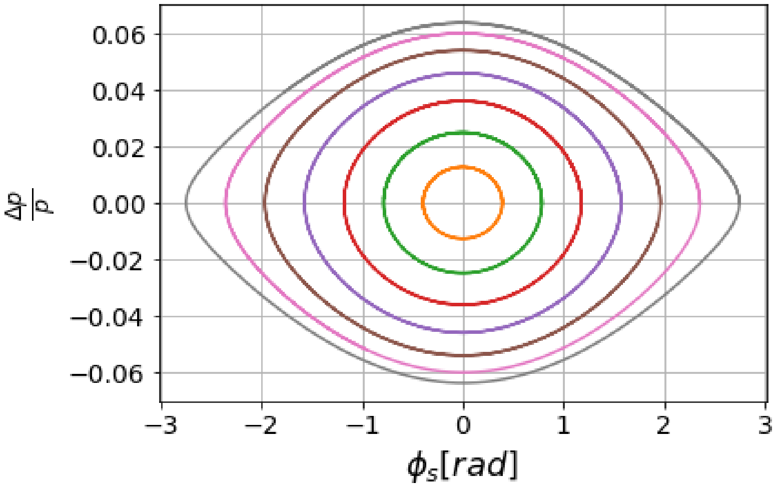


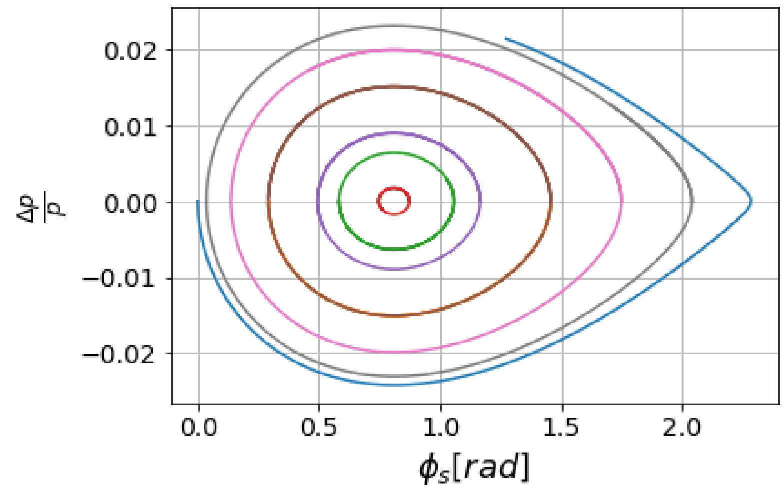
$\phi_s = 0.53$

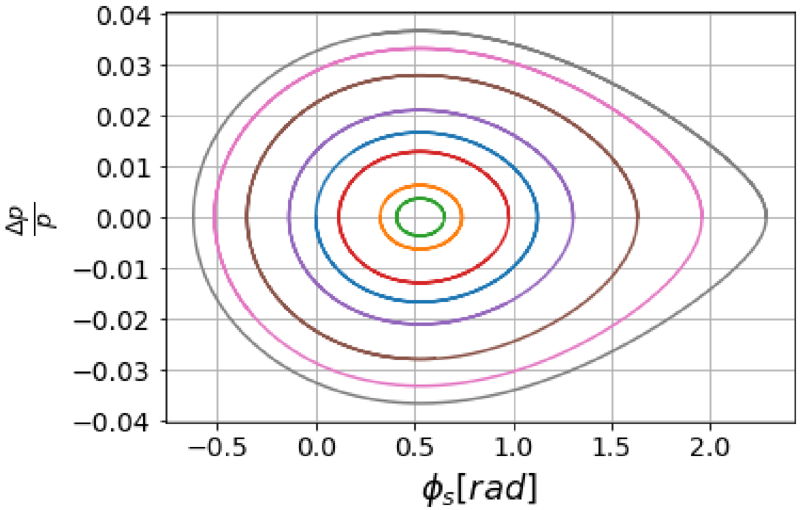


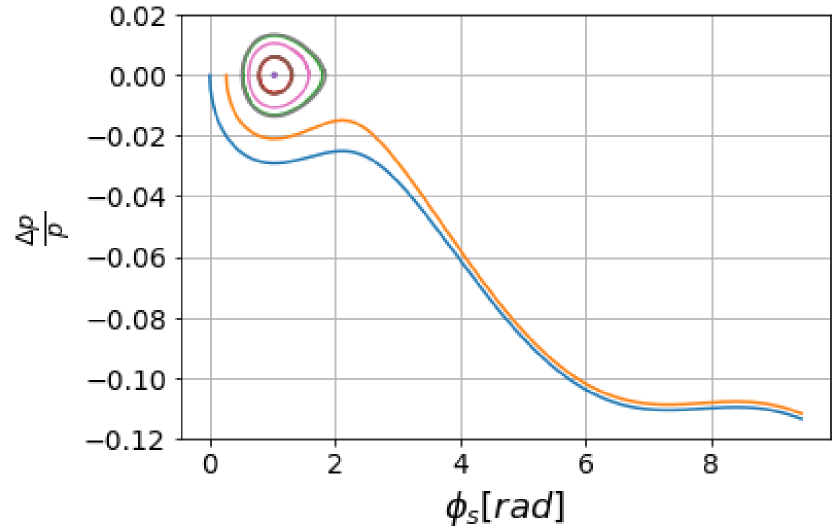
- ϕ_u is the turning point
- while ϕ_s is the stable fixed point, $\pi - \phi_s$ is the unstable fixed point
- bucket length is $\pi - \phi_s - \phi_u$

RF Bucket w. different synchrotron phase









Synchrotron motion at large amplitude

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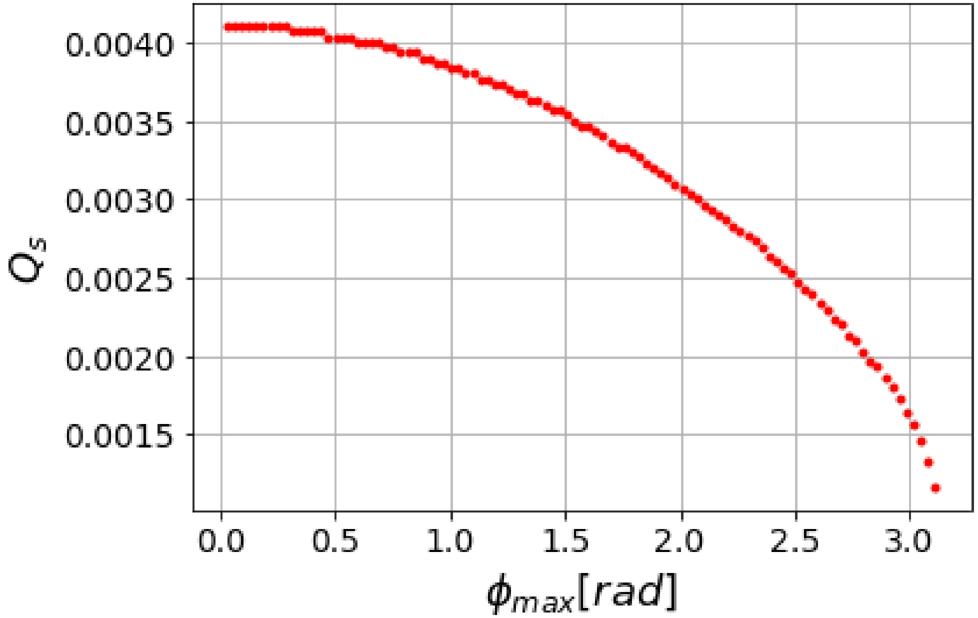
- The linear approximation is only applicable to the particles that are moving very closely around the phase of the synchronous particle
- In reality, it often happens that the beam fills up the full bucket. Hence, one has to deal with the intrinsic non-linear aspect of the longitudinal motion
- In general, the synchrotron tune depends on the amplitude of the synchrotron oscillation
  - 0-amp tune: $Q_{s0} = \sqrt{\frac{hqV|\eta cos\phi_s|}{2\pi\beta_0^2 E_0}}$
  - in general, $Q_s = Q_{s0}\frac{\pi}{2K\left(sin^2\frac{\phi m}{2}\right)}$

    $K(b) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-nx^2}}$

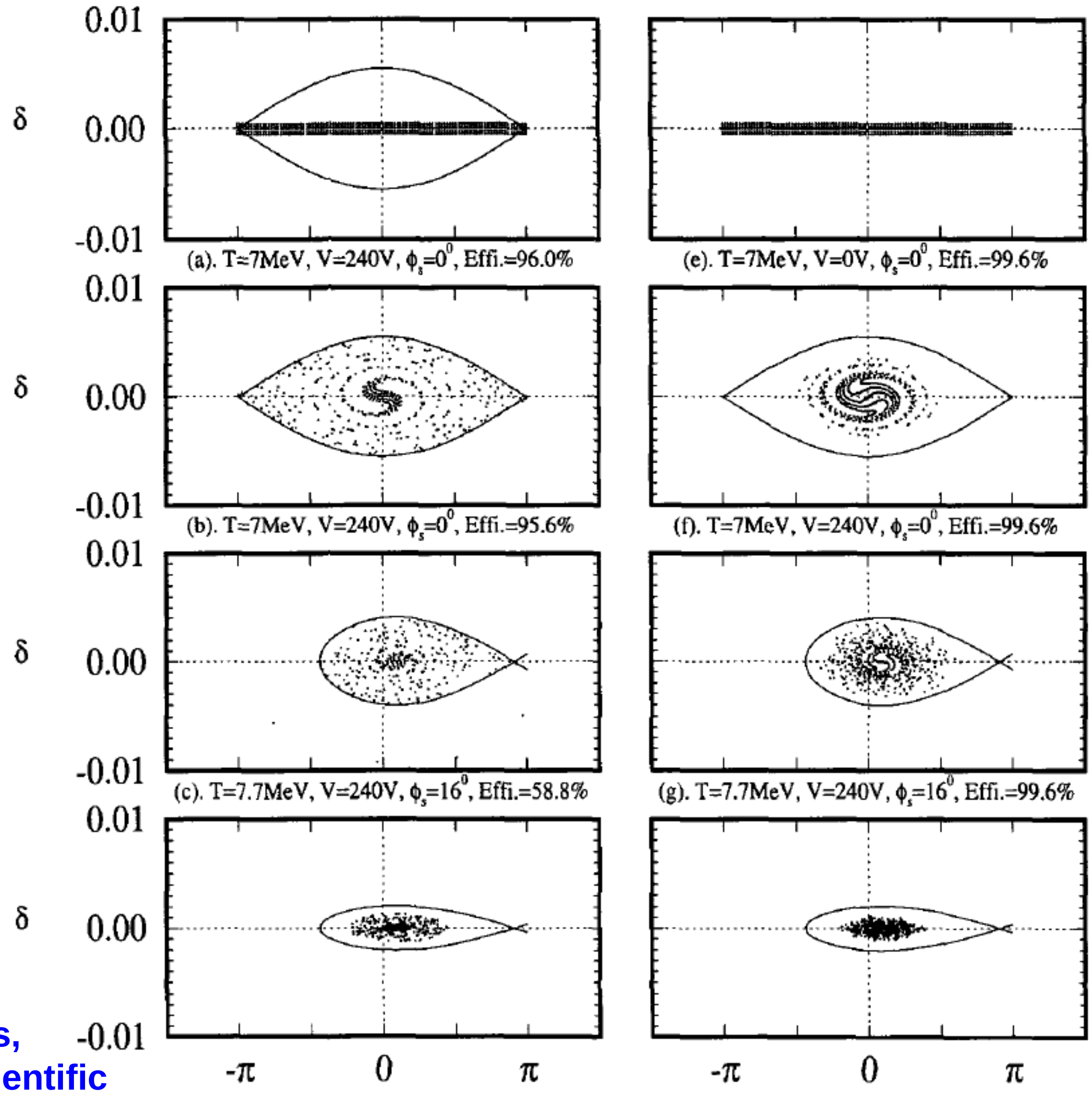
    at seperatrix, the synchrotron mo becomes very slow



Beam manipulation: RF capture

non-adiabatic capture

Adiabatic capture



Accelerator Physics,
S. Y: Lee, World Scientific

Types of magnets in a synchrotron

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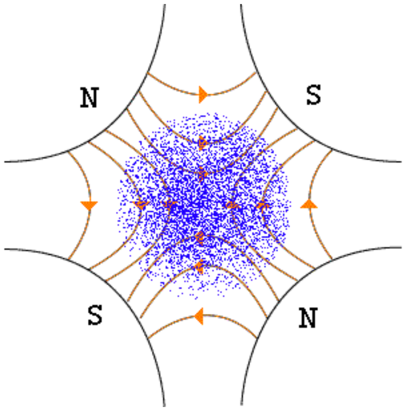
- Dipoles: uniform magnetic field in the gap
  - Bending dipoles
  - Orbit steering
- Quadrupoles
  - Providing focusing field to keep beam from being diverged



- Sextupoles:
  - Provide corrections of chromatic effect of beam dynamics
- Higher order multipoles

Focusing from quadrupole

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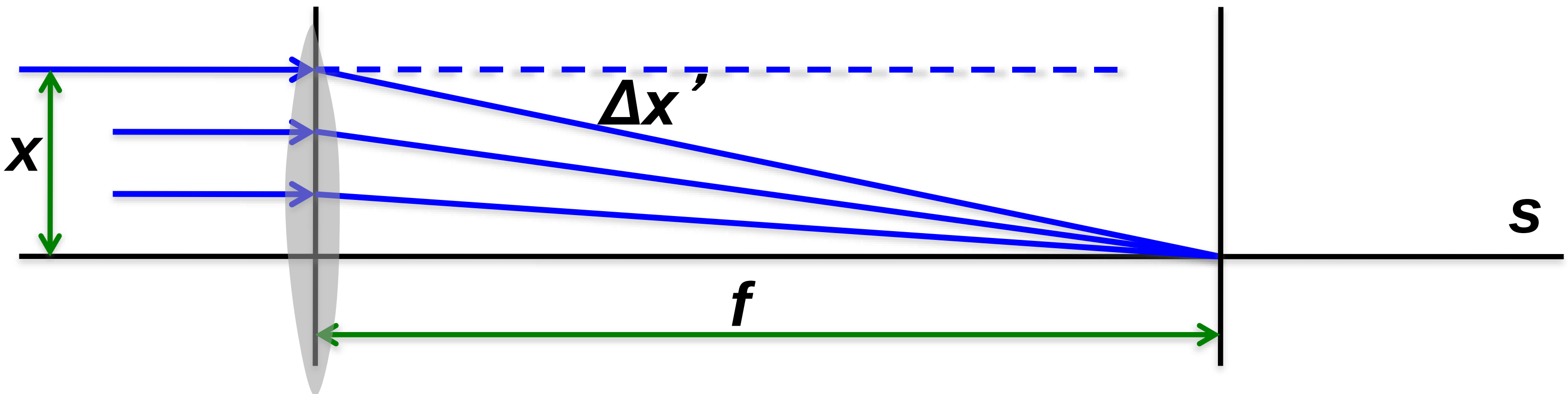


$$\frac{x}{f} = \frac{l}{r} = l\frac{qB_y}{\gamma mv} = l\frac{qB'}{\gamma mv}x \Rightarrow \frac{1}{f} = \frac{qB'l}{\gamma mv} = kl$$

Required by Maxwell equation, a single quadrupole has to provide focusing in one plane and defocusing in the other plane

$$\nabla \times \vec{B} = 0 \quad B_x = B'y \text{ and } B_y = B'x$$

Transfer matrix of a qudruploe

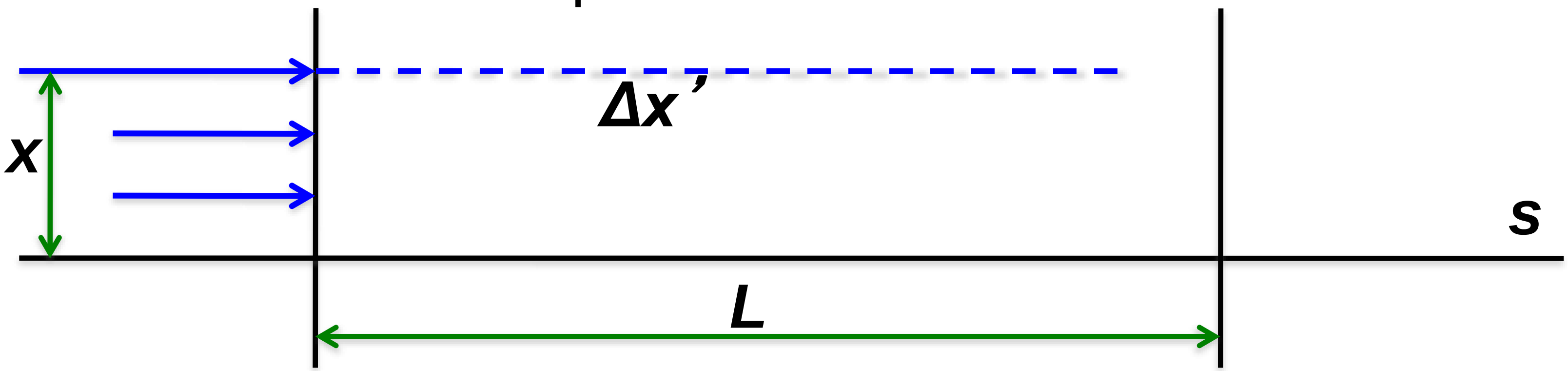
Thin lens: length of quadrupole is negligible to the displacement relative to the center of the magnet

$$\Delta x' = -\frac{l}{\rho} = -l\frac{qB_y}{\gamma m v} = -\frac{qB'l}{\gamma m v}x = -klx$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

Transfer matrix of a drift space

Transfer matrix of a drift space



$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

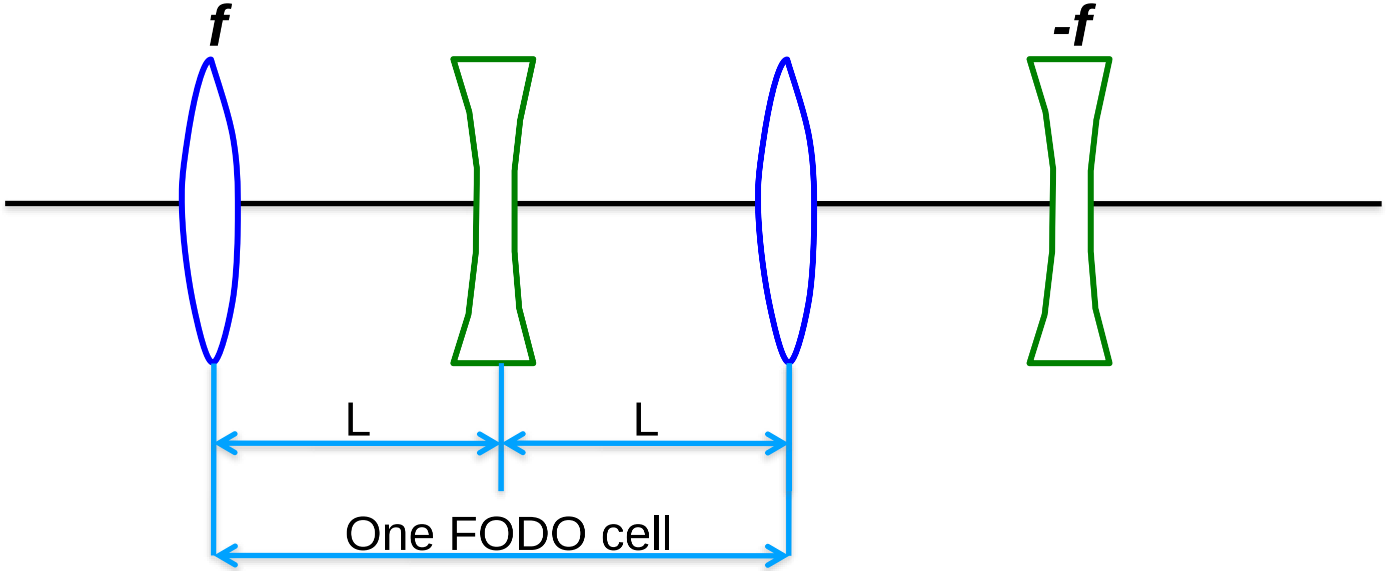
Lattice

Arrangement of magnets: structure of beam line

- Bending dipoles, Quadrupoles, Steering dipoles, Drift space and Other insertion elements

Example:

- FODO cell: alternating arrangement between focusing and defocusing quadrupoles



FODO lattice

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

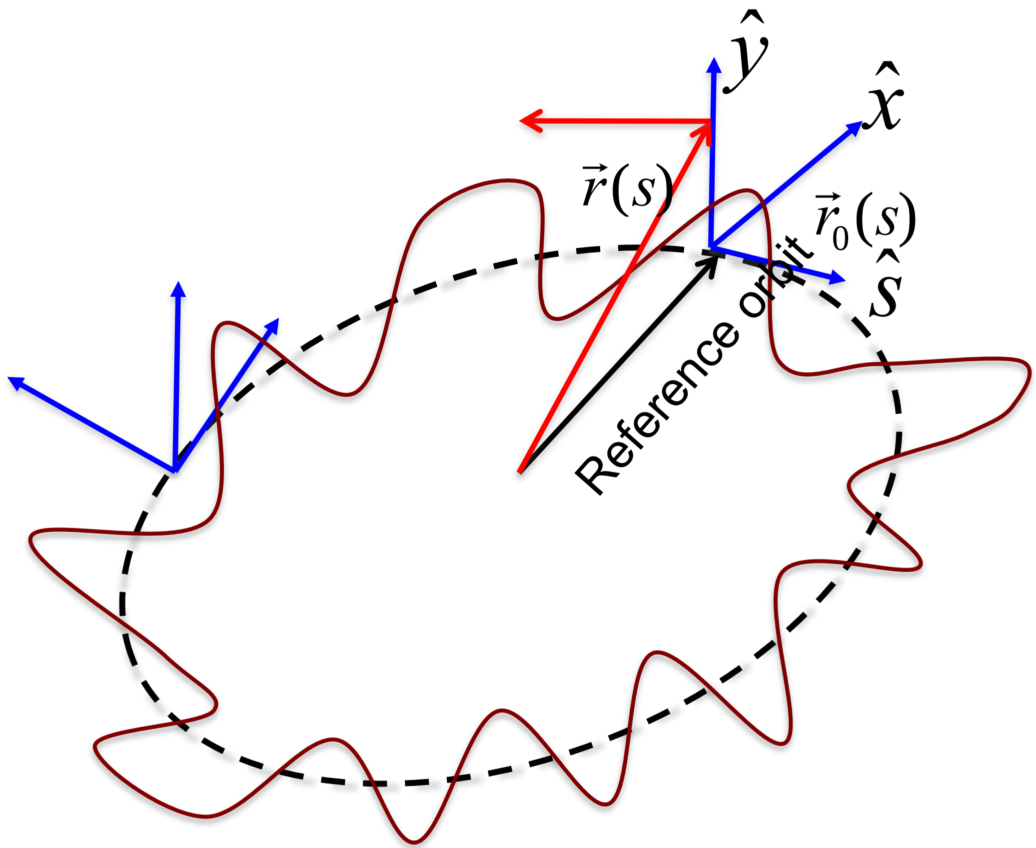
$$= \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 - \frac{L}{2f}) \\ -\frac{L}{2f^2} & 1 - \frac{L^2}{2f^2} \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

Net effect is focusing

Provide focusing in both planes!

Curverlinear coordinate system

- Coordinate system to describe particle motion in an accelerator
- Moves with the particle



Set of unit vectors:

$$\hat{s}(s) = \frac{d\vec{r}_0(s)}{ds}$$

$$\hat{x}(s) = -r\,\frac{d\hat{s}(s)}{ds}$$

$$\hat{y}(s) = \hat{x}(s)\ ´\ \hat{s}(s)$$

Equation of motion

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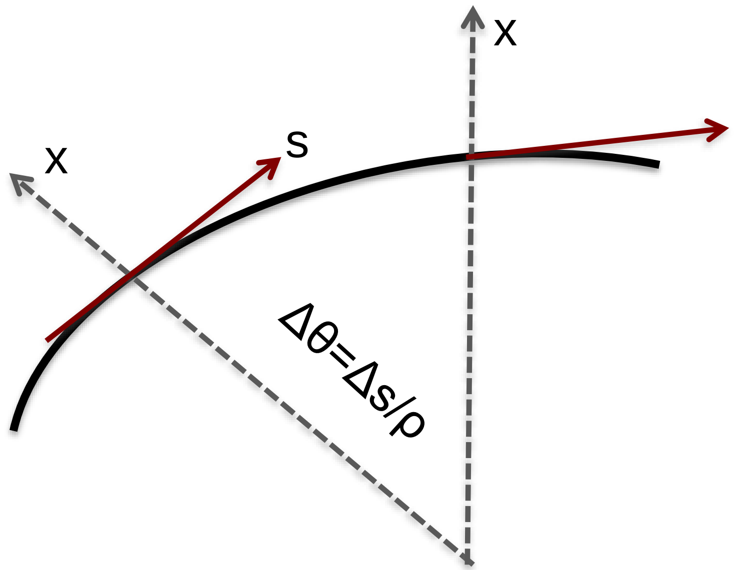

$$\frac{d\hat{s}(s)}{ds} = -\frac{1}{r}\hat{x}(s)$$

$$\frac{d\hat{x}(s)}{ds} = \frac{1}{r}\hat{s}(s)$$

$$\frac{d\hat{y}(s)}{ds} = 0$$

Equation of motion in transverse plane

$$\vec{r}(s) = \vec{r}_0(s) + x\hat{x}(s) + y\hat{y}(s)$$

Equation of motion

$$\frac{d\vec{r}(s)}{dt} = \frac{ds}{dt}\left[\frac{d\vec{r}_0}{ds} + x'\hat{x} + x\frac{d\hat{x}}{ds} + y'\hat{y}\right] = \frac{ds}{dt}\left[\left(1+\frac{x}{r}\right)\hat{s} + x'\hat{x} + y'\hat{y}\right]$$

$$\vec{v} = \frac{ds}{dt}\left[\left(1+\frac{x}{r}\right)\hat{s} + x'\hat{x} + y'\hat{y}\right] = v_s\hat{s} + v_x\hat{x} + v_y\hat{y}$$

$$v^2 = \left|\vec{v}\right|^2 = \left(\frac{ds}{dt}\right)^2\left[\left(1+\frac{x}{r}\right)^2 + x'^2 + y'^2\right]$$

$$\frac{d^2\vec{r}(s)}{dt^2} = \frac{ds}{dt}\frac{d\vec{v}}{ds} \approx \frac{v^2}{\left(1+\frac{x}{r}\right)^2}\left[\left(x'' - \frac{r+x}{r}\right)\hat{x} + \frac{x'}{r}\hat{s} + y''\hat{y}\right]$$

Equation of motion

$$\frac{d^2\vec{r}(s)}{dt^2} \approx \frac{v^2}{(1+\frac{x}{\rho})^2}\left[\left(x'' - \frac{\rho + x}{\rho}\right)\hat{x} + \frac{x'}{\rho}\hat{s} + y''\hat{y}\right] = \frac{q\vec{v} \times \vec{B}}{\gamma m}$$

$$x'' - \frac{\rho + x}{\rho^2} = -\frac{qB_y}{\gamma m v}\left(1+\frac{x}{\rho}\right)^2 \quad \Longrightarrow \quad x'' + \frac{qB'}{\gamma m v}x = 0$$

$$y'' = \frac{qB_x}{\gamma m v}\left(1+\frac{x}{\rho}\right)^2 \quad \Longrightarrow \quad y'' - \frac{qB'}{\gamma m v}y = 0$$

Transverse motion: betatron oscillation

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The general case of equation of motion in an accelerator

$$x'' + kx = 0$$ Where *k* can also be negative

- For k > 0

$$x(s) = A\cos(\sqrt{k}s + c) \qquad x'(s) = -A\sqrt{k}\sin(\sqrt{k}s + c)$$

- For k < 0

$$x(s) = A\cosh(\sqrt{k}s + c) \qquad x'(s) = -A\sqrt{k}\sinh(\sqrt{k}s + c)$$

Transfer matrix of a quadrupole

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- For a focusing quadrupole

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} \cos\sqrt{k}l & \frac{1}{\sqrt{k}}\sin\sqrt{k}l \\ -\sqrt{k}\sin\sqrt{k}l & \cos\sqrt{k}l \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

- For a de-focusing quadrupole

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} \cosh\sqrt{k}l & \frac{1}{\sqrt{k}}\sinh\sqrt{k}l \\ -\sqrt{k}\sinh\sqrt{k}l & \cosh\sqrt{k}l \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

Hill's equation

In an accelerator which consists individual magnets, the equation of motion can be expressed as,

$$x'' + k(s)x = 0 \quad k(s + L_p) = k(s)$$

Here, $k(s)$ is an periodic function of L_p , which is the length of the periodicity of the lattice, i.e. the magnet arrangement. It can be the circumference of machine or part of it.

Similar to harmonic oscillator, expect solution as

$$x(s) = A(s)\cos(\psi(s) + \chi)$$

or:

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s) + \chi) \quad \beta_x(s + L_p) = \beta_x(s)$$

Hill's equation: cont'd

$$x'(s) = -A\sqrt{\beta_x(s)}\psi'(s)\sin(\psi(s)+\varphi) + \frac{\beta'_x(s)}{2}A\sqrt{1/\beta_x(s)}\cos(\psi(s)+\varphi)$$

with

$$\psi'(s) = \frac{1}{\beta_x(s)} \qquad \frac{\beta_x''}{2}\beta_x - \frac{\beta_x'^2}{4} + k\beta_x^2 = 1$$

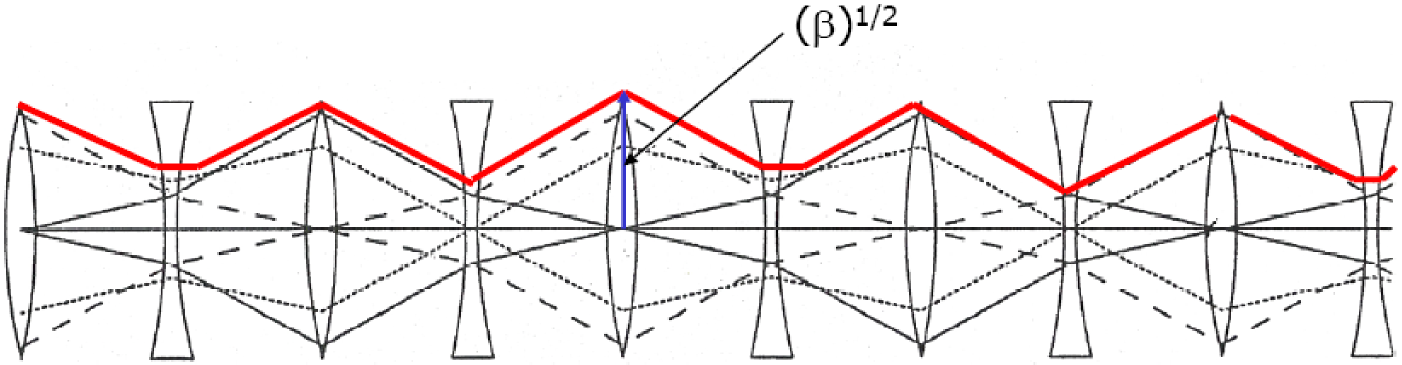
- Hill's equation $x'' + k(s)x = 0$ is satisfied

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s)+\varphi)$$

$$x'(s) = -A\sqrt{1/\beta_x(s)}\sin(\psi(s)+\varphi) + \frac{\beta'_x(s)}{2}A\sqrt{1/\beta_x(s)}\cos(\psi(s)+\varphi)$$

Betatron oscillation

- Beta function $\beta_x(s)$:
  - Describes the envelope of the betatron oscillation in an accelerator



  - Phase advance: $$\psi(s) = \int_0^s \frac{1}{\beta_x(s)} ds$$
  - Betatron tune: number of betatron oscillations in one orbital turn

$$Q_x = \frac{\psi(0 \mid C)}{2\pi} = \oint \frac{ds}{\beta_x(s)} / 2\pi = \frac{R}{\langle \beta_x \rangle}$$

Phase space

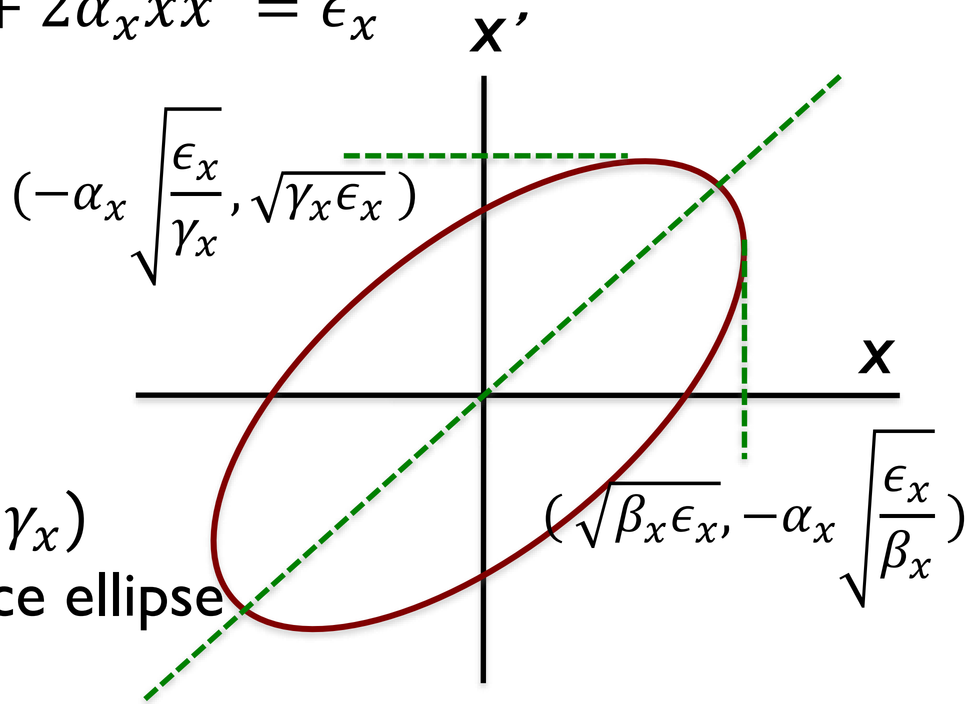
- In a space of x-x’ , the betatron oscillation projects an ellipse

$$\beta_x x'^2 + \gamma_x x^2 + 2\alpha_x x x' = \epsilon_x$$

where

$$\alpha_x = -\frac{1}{2}\beta'_x$$

$$\beta_x \gamma_x = 1 + \alpha_x^2$$



- The set of parameter $(\beta_x, \alpha_x, \gamma_x)$ which describe the phase space ellipse

- Courant-Snyder invariant ϵ_x: the area of the ellipse in unit of π

Transfer Matrix of beam transport

Proof the transport matrix from point 1 to point 2 is

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{s_2s_1} + \alpha_1 \sin\psi_{s_2s_1}) & \sqrt{\beta_1\beta_2}\sin\psi_{s_2s_1} \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\psi_{s_2s_1} + \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1\beta_2}}\cos\psi_{s_2s_1} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{s_2s_1} - \alpha_2 \sin\psi_{s_2s_1}) \end{pmatrix} \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

- Hint:

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s) + c)$$

$$x'(s) = -A\sqrt{1/\beta_x(s)}\sin(\psi(s) + c) + \frac{\beta'_x(s)}{2}A\sqrt{1/\beta_x(s)}\cos(\psi(s) + c)$$

One Turn Map

Transfer matrix of one orbital turn

$$\begin{pmatrix} x(s_0+C) \\ x'(s_0+C) \end{pmatrix} = \begin{pmatrix} (\cos 2\pi Q_x + \alpha_{x,s_0}\sin 2\pi Q_x) & \beta_{x,s_0}\sin 2\pi Q_x \\ -\frac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}}\sin 2\pi Q_x & (\cos 2\pi Q_x - \alpha_{x,s_0}\sin 2\pi Q_x) \end{pmatrix} \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

$$Tr(M_{s,s+C}) = 2\cos 2\pi Q_x$$

Stable condition

$$\left|\frac{1}{2} Tr(M_{s,s+C})\right| \le 1.0$$

- Closed orbit: $\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$

$$\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = M(s+C,s)\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

Stability of transverse motion

SLAC

Matrix from point 1 to point 2

$$M_{s_2|s_1} = M_n \cdots M_2 M_1$$

- Stable motion requires each transfer matrix to be stable, i.e. its eigen values are in form of oscillation

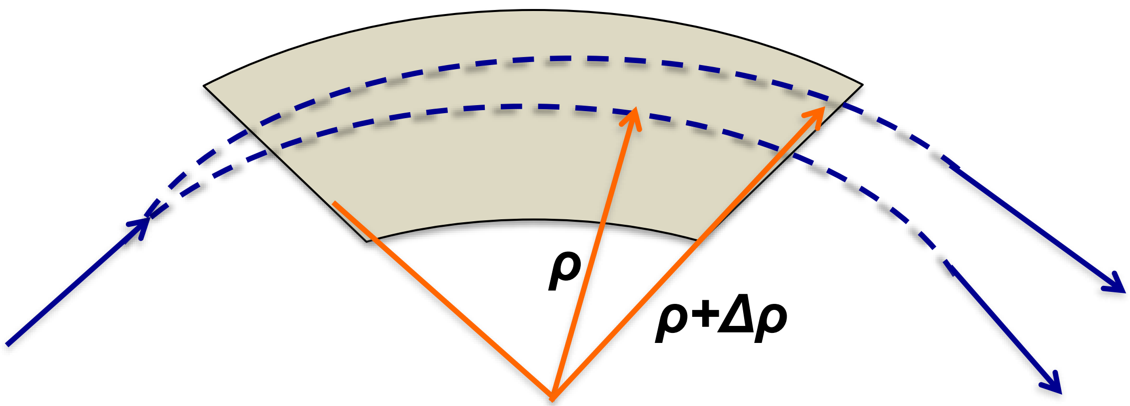
$$|M - \lambda I| = 0 \qquad \text{With } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \det(M) = 1$$

$$\lambda^2 - Tr(M)\lambda + \det(M) = 0$$

$$\lambda = \frac{1}{2} Tr(M) \pm \sqrt{\frac{1}{4}[Tr(M)]^2 - 1} \quad \Rightarrow \quad \left| \frac{1}{2} Tr(M) \right| \le 1.0$$

Dispersion function

Transverse trajectory is function of particle momentum



$$\Delta\theta = \theta \frac{\Delta p}{p}$$

Momentum spread

Define $x = D(s)\dfrac{\Delta p}{p}$

Dispersion function

Dispersion function: cont'd

In drift space

$$\frac{1}{r} = 0 \quad \text{and} \quad B' = 0 \Rightarrow \qquad D'' = 0$$

dispersion function has a constant slope

- In dipoles,

$$\frac{1}{r} \neq 0 \quad \text{and} \quad B' = 0 \qquad D'' + [\frac{1}{r^2}\frac{2p_0 - p}{p}]D = \frac{1}{r}$$

Dispersion function: cont'd

- For a focusing quad,

$$\frac{1}{r} = 0 \quad \text{and} \quad B' > 0 \qquad \text{Þ} \quad D'' + B'\frac{p_0}{p}D = 0$$

dispersion function oscillates sinusoidally

- For a defocusing quad,

$$\frac{1}{r} = 0 \quad \text{and} \quad B' < 0 \qquad \text{Þ} \quad D'' - B'\frac{p_0}{p}D = 0$$

dispersion function evolves exponentially

Chromatic effect

Comes from the fact the the focusing effect of an quadrupole is momentum dependent

$$\frac{1}{f} = kl$$ → **Particles with different momentum have different betatron tune**

- Higher energy particle has less focusing

Chromaticity: tune spread due to momentum spread

$$\xi_{x,y} = \frac{\Delta Q_{x,y}}{\Delta p / p}$$

$\Delta Q_{x,y}$ → Tune spread

$\Delta p / p$ → momentum spread

Chromaticity

SLAC

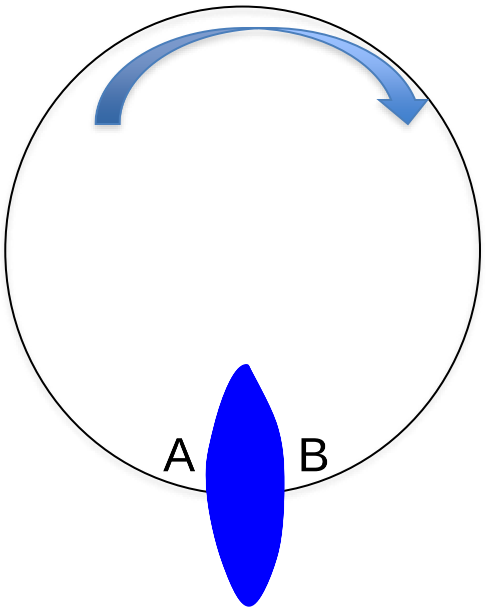
- Transfer matrix of a thin quadrupole

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ -\frac{1}{f}(1-\frac{\Delta p}{p}) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f}\frac{\Delta p}{p} & 1 \end{pmatrix}$$

Transfer matrix

$$M(s+C,s) = M(B,A) \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$= M(B,A) \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f}\frac{\Delta p}{p} & 1 \end{pmatrix}$$



Chromaticity

SLAC

$$M(s+C,s)=\begin{pmatrix}(\cos 2\pi Q_x+\alpha_{x,s_0}\sin 2\pi Q_x) & \beta_{x,s_0}\sin 2\pi Q_x \\ -\dfrac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}}\sin 2\pi Q_x & (\cos 2\pi Q_x-\alpha_{x,s_0}\sin 2\pi Q_x)\end{pmatrix}\begin{pmatrix}1 & 0 \\ \dfrac{1}{f}\dfrac{\Delta p}{p} & 1\end{pmatrix}$$

$$=\begin{pmatrix}(\cos 2\pi Q_x+\alpha_{x,s_0}\sin 2\pi Q_x)+\dfrac{1}{f}\dfrac{\Delta p}{p}\beta_{x,s_0}\sin 2\pi Q_x & \beta_{x,s_0}\sin 2\pi Q_x \\ -\dfrac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}}\sin 2\pi Q_x+(\cos 2\pi Q_x-\alpha_{x,s_0}\sin 2\pi Q_x)\dfrac{1}{f}\dfrac{\Delta p}{p} & (\cos 2\pi Q_x-\alpha_{x,s_0}\sin 2\pi Q_x)\end{pmatrix}$$

$$\cos[2\pi(Q_x+\Delta Q_x)]=\frac{1}{2}Tr(M(s+C,s))$$

$$\cos[2\pi(Q_x+\Delta Q_x)]=\cos 2\pi Q_x+\frac{1}{2}\beta_{x,s_0}\sin 2\pi Q_x\frac{1}{f}\frac{\Delta p}{p}$$

Chromaticity

$$\cos[2\pi(Q_x + \Delta Q_x)] = \cos 2\pi Q_x + \frac{1}{2}\beta_{x,s_0} \sin 2\pi Q_x \frac{1}{f}\frac{\Delta p}{p}$$

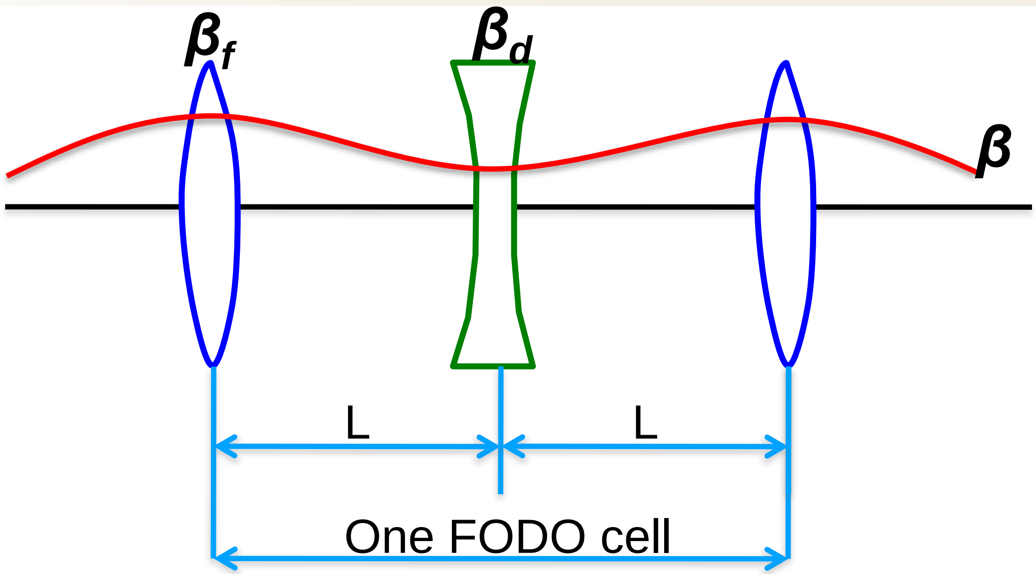
Assuming the tune change due to momentum difference is small

$$\cos 2\pi Q_x - 2\pi \Delta Q_x \sin 2\pi Q_x = \cos 2\pi Q_x + \frac{1}{2}\beta_{x,s_0} \sin 2\pi Q_x \frac{1}{f}\frac{\Delta p}{p}$$

$$\Delta Q_x = -\frac{1}{4\pi}\beta_{x,s_0}\frac{1}{f}\frac{\Delta p}{p} \qquad \xi_x = \frac{\Delta Q_x}{\Delta p / p} = -\frac{1}{4\pi}\frac{1}{f}\beta(s)$$

$$x_x = \frac{DQ_x}{Dp / p} = - \frac{1}{4p}\sum_i k_i l_i b_{x,i}$$

Chromaticity of a FODO cell



$$\beta_{f,d} = \frac{2L(1 \pm \sin[\Delta\psi/2])}{\sin[\Delta\psi]}$$

$$\sin[\Delta\psi/2] = \frac{L}{f}$$

$$\xi_x = -\frac{1}{4\pi}\left(\beta_f \frac{1}{f} - \beta_d \frac{1}{f}\right) \Rightarrow \xi_x = -\frac{1}{\pi}\frac{L/f}{\sin\Delta\psi}$$

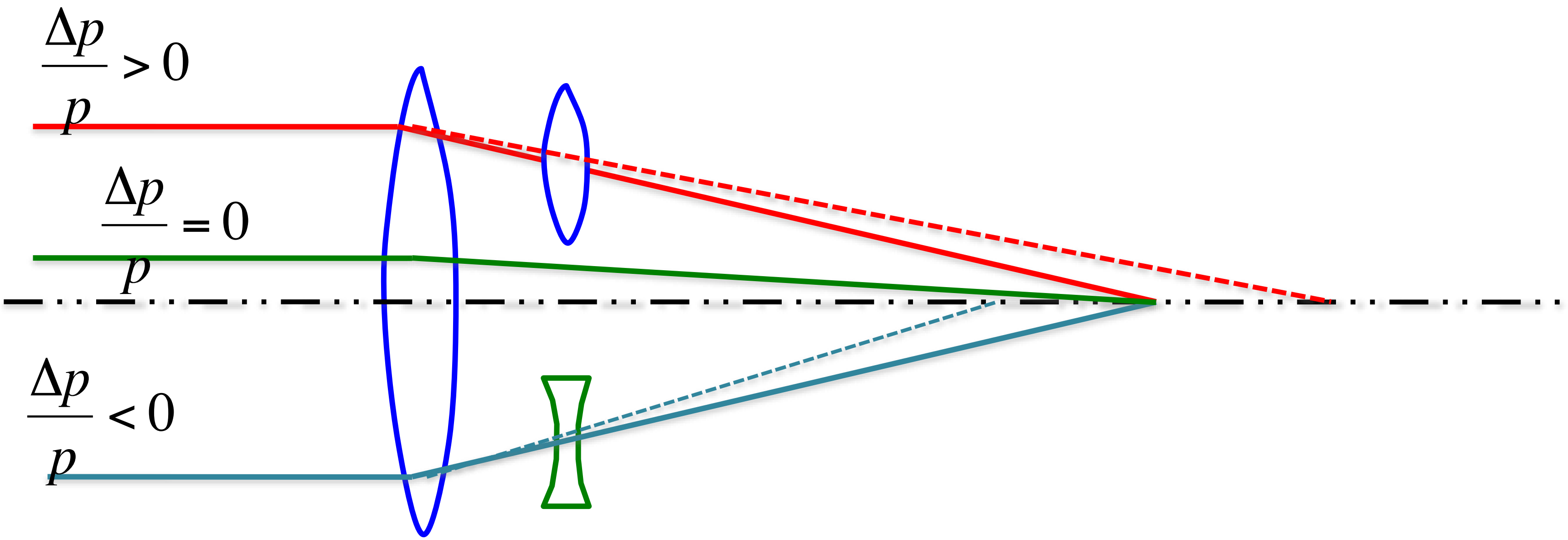
$$\xi_x = -\frac{1}{\pi}\tan\frac{\Delta\psi}{2}$$

Chromaticity correction

SLAC

- Nature chromaticity is always negative and can be large and can result to large tune spread and get close to resonance condition

- Solution:
  - A special magnet which provides stronger focusing for particles with higher energy: sextupole



Sextupole

$$B_x = mxy \qquad B_y = \frac{1}{2} m(x^2 - y^2)$$

Focusing strength in horizontal plane:

$$B'_y = mx$$

where $m = \dfrac{\partial^2 B_y}{\partial x^2}$ and $k_{sx} = \dfrac{ml}{B\rho}$, l is the magnet length

Tune change due to a sextupole:

$$\Delta Q_x = \frac{1}{4\pi} \beta_{x,s_0} k_{sx}\, x \qquad let\; x = D\frac{\Delta p}{p}$$

$$\Delta Q_x / \frac{\Delta p}{p} = \frac{1}{4\pi} \beta_{x,s_0} k_{sx} D_x$$

Chromaticity Correction

$$\Delta Q_x / \frac{\Delta p}{p} = \frac{1}{4\pi} \beta_{x,s_0} k_{sx} D_x$$

Sextupole produces a chromaticity with the opposite sign of the quadrupole!

It prefers to be placed after a bending dipole where dispersion function is non zero

Chromaticity after correction

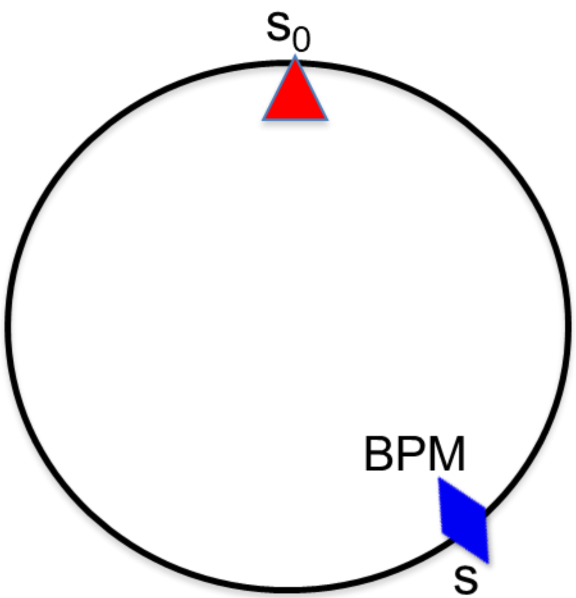
$$\xi_x = \frac{\Delta Q_x}{\Delta p / p} = -\frac{1}{4\pi} \sum_i k_i \beta_{x,i} + \frac{1}{4\pi} \sum_i k_{sx,i} \beta_{x,i} D_x$$

Effect of Errors

Closed orbit distortion

closed orbit is defined as

$$\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s+C,s) \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$



This yields

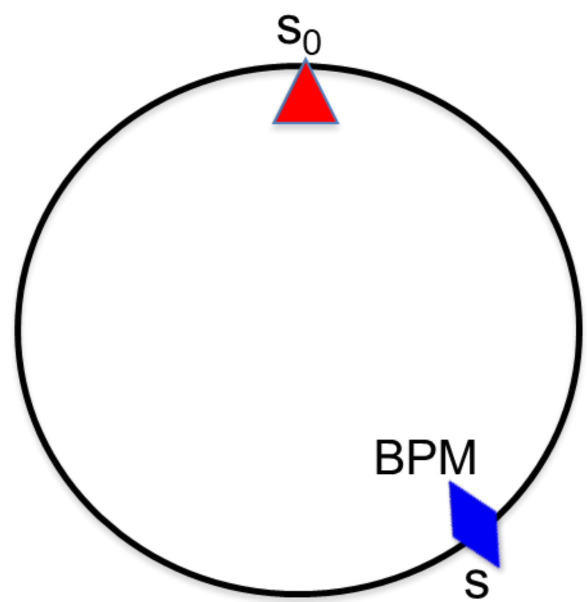
$$(M(s+C,s)-I)\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = 0$$

In a circular accelerator without error, one can see $x = 0 = x'$ is the closed orbit, i.e the orbit through the center of the quadrupoles is the closed orbit

Closed orbit distortion

Dipole kicks can cause particle’s trajectory deviate away from the designed orbit

- Dipole error
- Quadrupole misalignment



Assuming a circular ring with a single dipole error, closed orbit then becomes:

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s,s_0)[M(s_0,s)\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}]$$

Closed orbit: single dipole error

- Let's first solve the closed orbit at the location where the dipole error is

$$\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} = M(s_0 + C, s_0)\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$x(s_0) = \beta_x(s_0)\frac{\theta}{2\sin\pi Q_x}\cos\pi Q_x$$

$$\boxed{x(s) = \sqrt{\beta_x(s_0)\beta_x(s)}\frac{\theta}{2\sin\pi Q_x}\cos[\psi(s,s_0) - \pi Q_x]}$$

- The closed orbit distortion reaches its maximum at the opposite side of the dipole error location

Closed orbit distortion

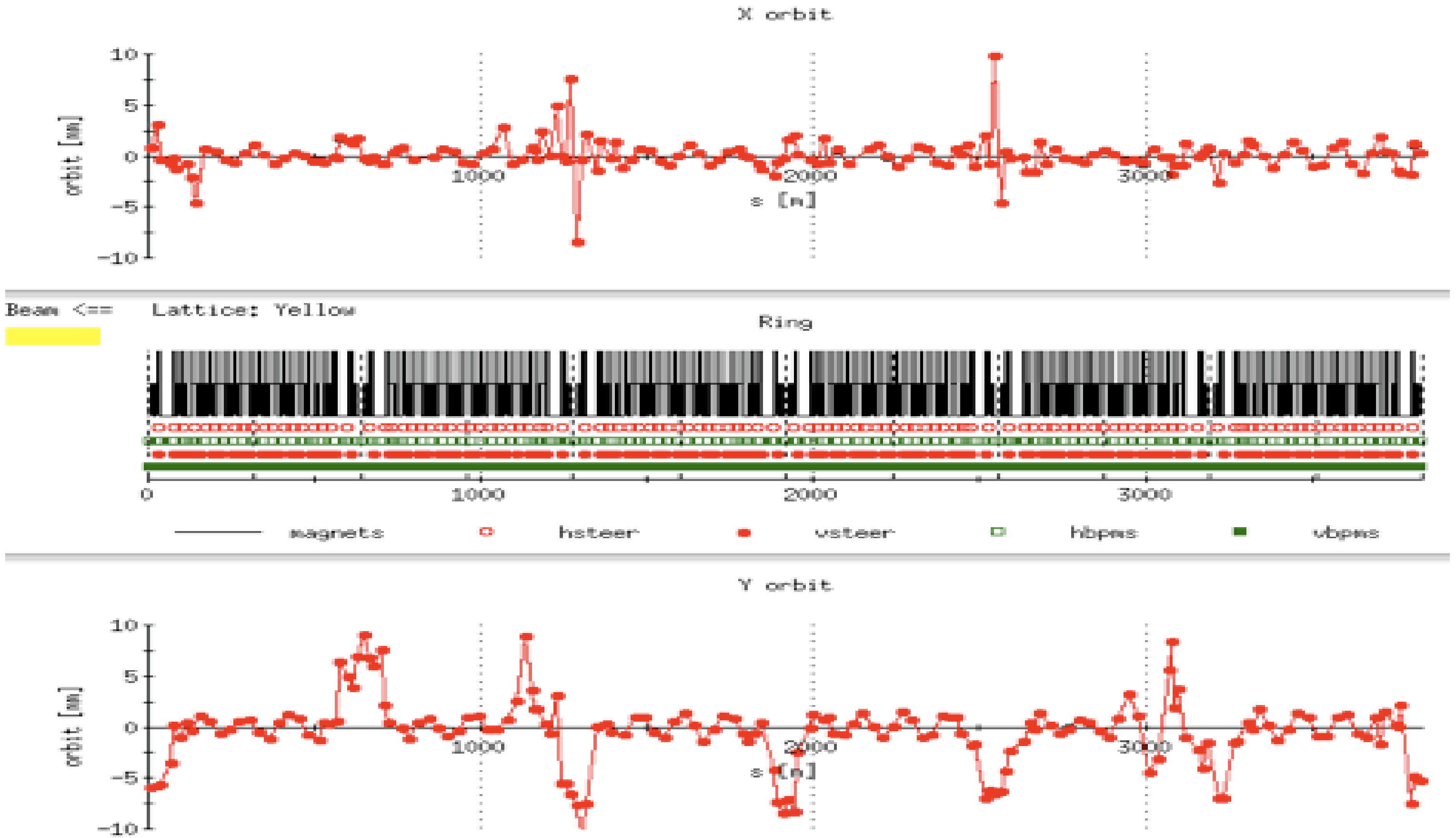
- In the case of multiple dipole errors distributed around the ring. The closed orbit is

$$x(s) = \sqrt{\beta_x(s)} \sum_i \sqrt{\beta_x(s_i)} \frac{\theta_i}{2\sin \pi Q_x} \cos[\psi(s_i, s_0) - \pi Q_x]$$

- Amplitude of the closed orbit distortion is inversely proportion to $\sin \pi Q_{x,y}$
  - **No stable orbit if tune is integer!**

Measure closed orbit

- Distribute beam position monitors around ring.



Control closed orbit

- Minimized the closed orbit distortion.
  - Large closed orbit distortions cause limitation on the physical aperture
  - Need dipole correctors and beam position monitors distributed around the ring
    - Assuming we have m beam position monitors and n dipole correctors, the response at each beam position monitor from the n correctors is:

$$x_k = \sqrt{\beta_{x,k}} \sum_{k=1}^{n} \sqrt{\beta_{x,i}} \frac{\theta_i}{2\sin \pi Q_x} \cos\left[\psi(s_i, s_0) - \pi Q_x\right]$$

Control closed orbit

- Or,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = (M) \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix}$$

- To cancel the closed orbit measured at all the bpms, the correctors are then

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix} = (M^{-1}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

Quadrupole errors

- Misalignment of quadrupoles
  - dipole-like error: kx
  - results in closed orbit distortion

- Gradient error:
  - Cause betatron tune shift
  - induce beta function deviation: beta beat

Tune change due to a single gradient error

Suppose a quadrupole has an error in its gradient, i.e.

$$M = \begin{pmatrix} 1 & 0 \\ -kl & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ -(kl + \Delta kl) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -kl & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Delta kl & 1 \end{pmatrix}$$

$$M(s+C,s) = \begin{pmatrix} (\cos 2\pi Q_{x0} + \alpha_{x,s_0} \sin 2\pi Q_{x0}) & \beta_{x,s0} \sin 2\pi Q_{x0} \\ -\dfrac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}} \sin 2\pi Q_{x0} & (\cos 2\pi Q_{x0} - \alpha_{x,s_0} \sin 2\pi Q_{x0}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Delta kl & 1 \end{pmatrix}$$

$$\cos 2\pi (Q_{x0} + \delta Q_x) = \frac{1}{2} Tr(M(s+C,s)) \qquad \delta Q_x = \frac{1}{4\pi} \beta_{x,s_0} \Delta kl$$

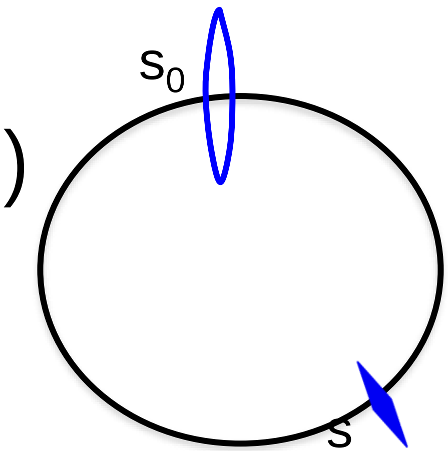
With multiple errors, the corresponding tune shift is

$$\delta Q_x = \frac{1}{4\pi} \sum_i \beta_{x,i} \Delta k_i l \;\; \text{, and } \delta Q_y = \frac{1}{4\pi} \sum_i \beta_{y,i} \Delta k_i l$$

Beta beat

In a circular ring with a gradient error at s0, the tune shift is

$$M(s+C,s) = M(s,s_0)\begin{pmatrix} 1 & 0 \\ -\Delta kl & 1 \end{pmatrix} M(s_0,s)$$



$$\beta_x(s)\sin 2\pi Q_x = \beta_{x0}(s)\sin 2\pi Q_{x0} +$$
$$\Delta kl\,\frac{\beta_{x0}(s)\beta_{x0}(s_0)}{2}[\cos(2\pi Q_{x0} + 2|\Delta\psi_{s,s0}|)]$$

$$\frac{\Delta\beta}{\beta} = \Delta kl\,\frac{\beta_{x0}(s_0)}{2\sin 2\pi Q_{x0}}\cos(2\pi Q_{x0} + 2|\Delta\psi_{s,s0}|)$$

Unstable betatron motion if tune is half integer!

Beta beat

In a circular ring with multiple gradient errors,

$$\frac{\Delta\beta}{\beta}(s) = \frac{\sqrt{\beta_{x0}(s)}}{2\sin 2\pi Q_{x0}} \sum_i \sqrt{\beta_{x0}(s_i)}\,\Delta k_i l \cos(2\pi Q_{x0} + 2\,|\,\Delta\psi_{s,si}\,|)$$

Unstable betatron motion if tune is half integer!

Beta beat wave varies twice of betatron tune around the ring

Resonance condition

Tune change due to a single quadrupole error

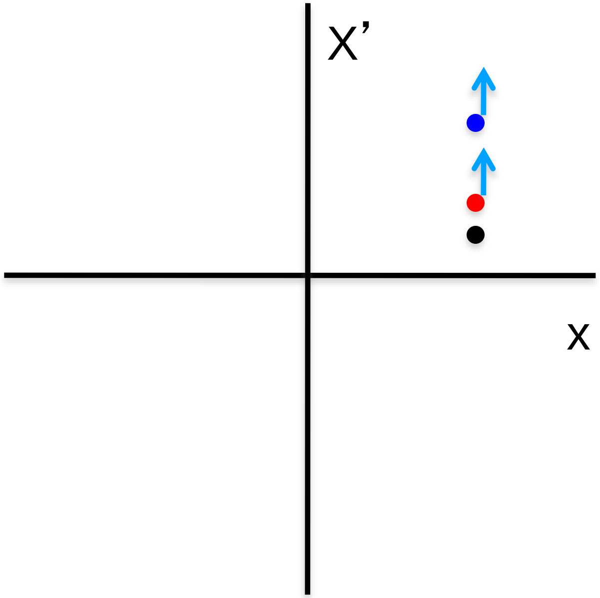
$$\cos[2\pi(Q_{x0}+\delta Q_x)] = \cos 2\pi Q_{x0} - \frac{1}{2}\beta_{x,s_0}\Delta kl \sin 2\pi Q_{x0}$$

If $Q_{x0} = (2k+1)\frac{1}{2} + \epsilon$, the above equation becomes

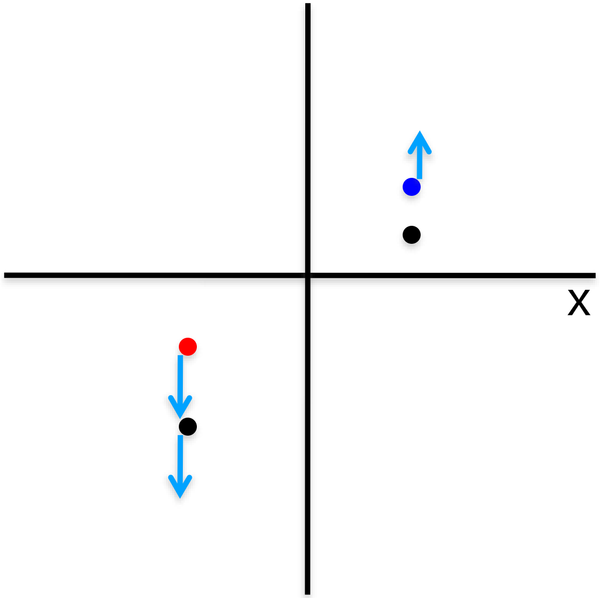
$$\cos[2\pi(Q_{x0}+\delta Q_x)] \approx 1 + \frac{1}{2}\beta_{x,s_0}\Delta kl\epsilon$$

and Qx can become a complex number which means the betatron motion can become unstable

Resonance



Integer resonance



Half Integer resonance

Resonance condition

SLAC

- In the absence of coupling between horizontal and vertical

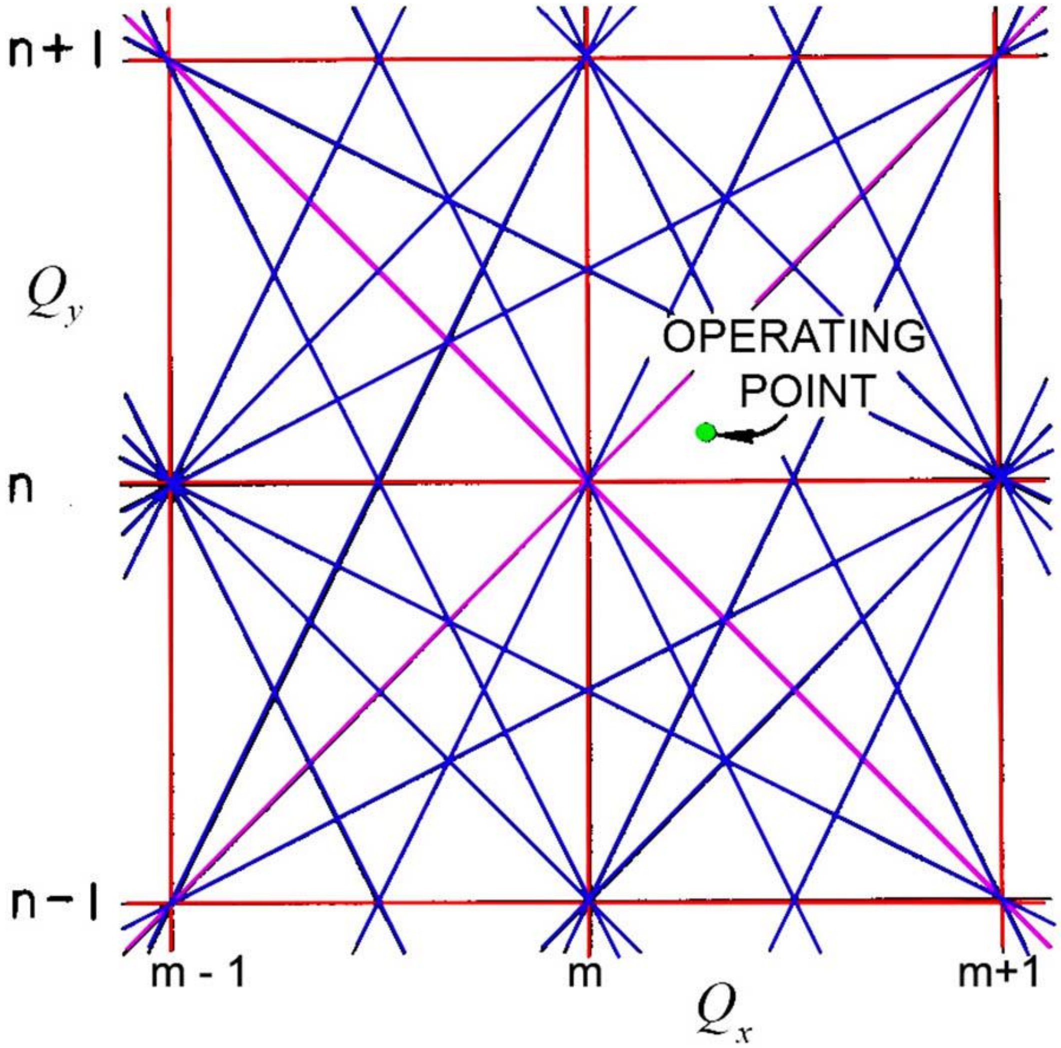
$$k = (n \pm 1)Q_{x,y}$$

error	n	
dipole	0	Qx,y=integer
quadrupole	1	2Qx,y=integer
Sextupole	2	3Qx,y=integer
Octupole	3	4Qx,y=integer

- In the presence of coupling between horizontal and vertical

$$MQ_x + NQ_y = k$$

Tune diagram



- the resonance strength decreases as the order goes higher
- the working point should be located in an area between resonances there are enough tune space to accommodate tune spread of the beam

3rd order resonance

SLAC

Let's start with linear betatron motion

$$x = A\sqrt{\beta_x}\, cos\, \psi_x$$

$$p_x = \beta_x x' + \alpha_x x = -A\sqrt{\beta_x}\, sin\, \psi_x$$

when $\psi_x = \nu_x \phi + \chi$, where ν_x is the tune, and $\phi = \int \frac{ds}{\nu_x \beta_x}$

Let $a\sqrt{\beta_0}$ be the betatron oscillation amplitude location s_0 and β_0 is the local beta function, we then have

$$x = a\sqrt{\frac{\beta_x(s)}{\beta_0}} \cos \psi_x \text{ and } p_x = -a\sqrt{\frac{\beta_x(s)}{\beta_0}}\, sin\, \psi_x$$

In the presence of a thin lens single sextupole, we have

$$\Delta x = 0,\ \Delta x' = -\frac{\Delta B l}{B\rho} = -\frac{B'' l}{2B\rho} x^2$$

$$\Delta p_x = \beta_x \Delta x' + \alpha_x \Delta x = -\beta_x \frac{\Delta B l}{B\rho} = -\beta_x \frac{B'' l}{2B\rho} x^2$$

3rd order resonance

since

$$\Delta x = \Delta a\sqrt{\frac{\beta_x(s)}{\beta_0}}\,cos\,\psi_x - a\sqrt{\frac{\beta_x(s)}{\beta_0}}\sin\psi_x\,\Delta\psi_x = 0$$

$$\Delta p_x = \Delta a\sqrt{\frac{\beta_x(s)}{\beta_0}}\,sin\,\psi_x + a\sqrt{\frac{\beta_x(s)}{\beta_0}}\cos\psi_x\,\Delta\psi_x = -\beta_x\frac{B''l}{2B\rho}x^2$$

this then leads to

$$\Delta a = \frac{\sqrt{\beta_x\beta_0}}{B\rho}\frac{B''l}{2}x^2\,sin\;\psi_x$$

$$\Delta\psi_x = \frac{\sqrt{\beta_x\beta_0}}{B\rho}\frac{B''l}{2a}x^2\,cos\,\psi_x$$

with $x^2 = a^2\frac{\beta_x}{\beta_0}\cos^2\psi_x$, we then have

$$\frac{da}{dn} = \frac{1}{4}a^2\frac{\beta_0}{B\rho}\left(\frac{\beta_x}{\beta_0}\right)^{\frac{3}{2}}\frac{B''l}{2}[\sin(\nu_x\phi+\chi)+\sin(3\nu_x\phi+3\chi)]$$

3rd order resonance

this then leads to the change of amplitude and phase

$$\Delta a = \frac{\sqrt{\beta_x \beta_0}}{B\rho} \frac{B''l}{2} x^2\, sin[\nu_x \phi + \chi]$$

$$\Delta \psi_x = \frac{\sqrt{\beta_x \beta_0}}{B\rho} \frac{B''l}{2a} x^2\, cos[\nu_x \phi + \chi]$$

with $x^2 = a^2 \frac{\beta_x}{\beta_0} \cos^2[\nu_x \phi + \chi]$, we then have the change per turn

$$\frac{da}{dn} = \frac{1}{4} a^2 \frac{\beta_0}{B\rho} \left(\frac{\beta_x}{\beta_0}\right)^{\frac{3}{2}} \frac{B''l}{2} [\sin \psi_x + \sin 3\psi_x]$$

$$\frac{d\psi_x}{dn} = \frac{1}{4} a \frac{\beta_0}{B\rho} \left(\frac{\beta_x}{\beta_0}\right)^{\frac{3}{2}} \frac{B''l}{2} [3\cos \psi_x + \cos 3\psi_x] + 2\pi \nu_x$$

when $\psi_x = \nu_x \phi + \chi$, where ν_x is the tune, and $\phi = \int \frac{ds}{\nu_x \beta_x}$

3rd order resonance

In the neighborhood of 3rd order resonance, i.e. $3\nu_{x0} = k,\ \nu_x = \nu_{x0} + \delta$, where k is the integer, $\delta \ll 1$, we can rewrite the above as

$$\frac{da}{dn} = \frac{1}{4}a^2 \frac{\beta_0}{B\rho}\left(\frac{\beta_x}{\beta_0}\right)^{\frac{3}{2}} \frac{B''l}{2}\left[\cos 3\nu_{x0}\phi \sin 3\psi + \sin 3\nu_{x0}\phi \cos 3\psi\right]$$

$$\frac{d\psi}{dn} = \frac{1}{4}a \frac{\beta_0}{B\rho}\left(\frac{\beta_x}{\beta_0}\right)^{\frac{3}{2}} \frac{B''l}{2}\left[\cos 3\nu_{x0}\phi \cos 3\psi - \sin 3\nu_{x0}\phi \sin 3\psi\right] + 2\pi\nu_x$$

let $A = \frac{\beta_0}{B\rho}\left(\frac{\beta_x}{\beta_0}\right)^{\frac{3}{2}} \frac{B''l}{2} \cos 3\nu_{x0}\phi$, $B = \frac{\beta_0}{B\rho}\left(\frac{\beta_x}{\beta_0}\right)^{\frac{3}{2}} \frac{B''l}{2} \sin 3\nu_{x0}\phi$, then

$$\frac{da}{dn} = \frac{1}{4}a^2\left[A \sin 3\psi + B \cos 3\psi\right]$$

$$\frac{d\psi}{dn} = \frac{1}{4}a\left[A \cos 3\psi - B \sin 3\psi\right] + 2\pi\nu_x$$

3rd order resonance

let $\tilde{\psi} = \psi - 2\pi n \nu_{x0}$, then

$$\frac{da}{dn} = \frac{1}{4} a^2 \left[A \sin 3\psi + B \cos 3\psi\right]$$

$$\frac{d\tilde{\psi}}{dn} = \frac{1}{4} a \left[A \cos 3\psi \ - B \sin 3\psi\right] + 2\pi\delta$$

at the 3rd order resonance, betatron motion becomes static

$$x = a\sqrt{\frac{\beta_x(s)}{\beta_0}} cos[\frac{k}{3}\phi + \chi], \text{ and } p_x = -a\sqrt{\frac{\beta_x(s)}{\beta_0}} sin[\frac{k}{3}\phi + \chi]$$

In its neighborhood, motion is with an amplitude of $a\sqrt{\frac{\beta_x(s)}{\beta_0}}$ and phase of $\tilde{\psi}$,

hence we have $\tilde{x} = a\sqrt{\frac{\beta_x(s)}{\beta_0}} \cos\tilde{\psi}$ and $\tilde{p}_x = -a\sqrt{\frac{\beta_x(s)}{\beta_0}} \sin\tilde{\psi}$

3rd order resonance

In the neighborhood of 3rd order resonance, the betatron motion is

$$\tilde{x} = a\sqrt{\frac{\beta_x(s)}{\beta_0}}\cos\tilde{\psi},\ \tilde{p}_x = -a\sqrt{\frac{\beta_x(s)}{\beta_0}}\sin\tilde{\psi}$$

and its change per turn is

$$\frac{d\tilde{x}}{dn} = \frac{\tilde{x}}{a}\frac{da}{dn} + \tilde{p}_x\frac{d\tilde{\psi}}{dn},\ \frac{d\tilde{p}_x}{dn} = \frac{\tilde{p}_x}{a}\frac{da}{dn} - \tilde{x}\frac{d\tilde{\psi}}{dn}$$

plug in the amplitude and phase equation, one then gets

$$\frac{d\tilde{x}}{dn} = \frac{a}{4}[\tilde{x}(Asin3\tilde{\psi} + Bcos3\tilde{\psi})] + \frac{a}{4}[\tilde{p}_x(Acos3\tilde{\psi} - Bsin3\tilde{\psi})] + 2\pi\delta\,\tilde{p}_x$$

$$\frac{d\tilde{p}_x}{dn} = \frac{a}{4}[\tilde{p}_x(Asin3\tilde{\psi} + Bcos3\tilde{\psi})] - \frac{a}{4}[\tilde{x}(Acos3\tilde{\psi} - Bsin3\tilde{\psi})] - 2\pi\delta\tilde{x}$$

with $\tilde{\psi}$ is small, so, $sin3\tilde{\psi} \sim 3sin\tilde{\psi}$ and $\cos3\tilde{\psi} \sim cos\tilde{\psi}$

$$\frac{d\tilde{x}}{dn} = \frac{A}{4}[-3\tilde{x}\tilde{p} + \tilde{x}\tilde{p}] + \frac{B}{4}[\tilde{x}^2 + 3\tilde{p}_x^2] + 2\pi\delta\,\tilde{p}_x$$

$$\frac{d\tilde{p}_x}{dn} = \frac{A}{4}[-3\tilde{p}_x^2 - \tilde{x}^2] + \frac{B}{4}[\tilde{p}_x\tilde{x} - 3\tilde{x}\tilde{p}] - 2\pi\delta\tilde{x}$$

3rd order resonance

Assuming *B=0*

$$\frac{d\tilde{x}}{dn} = \frac{A}{4}[-2\tilde{x}\tilde{p}] + 2\pi\delta\,\tilde{p}_x$$

$$\frac{d\tilde{p}_x}{dn} = \frac{A}{4}[-3\tilde{p}_x^2 - \tilde{x}^2] - 2\pi\delta\tilde{x}$$

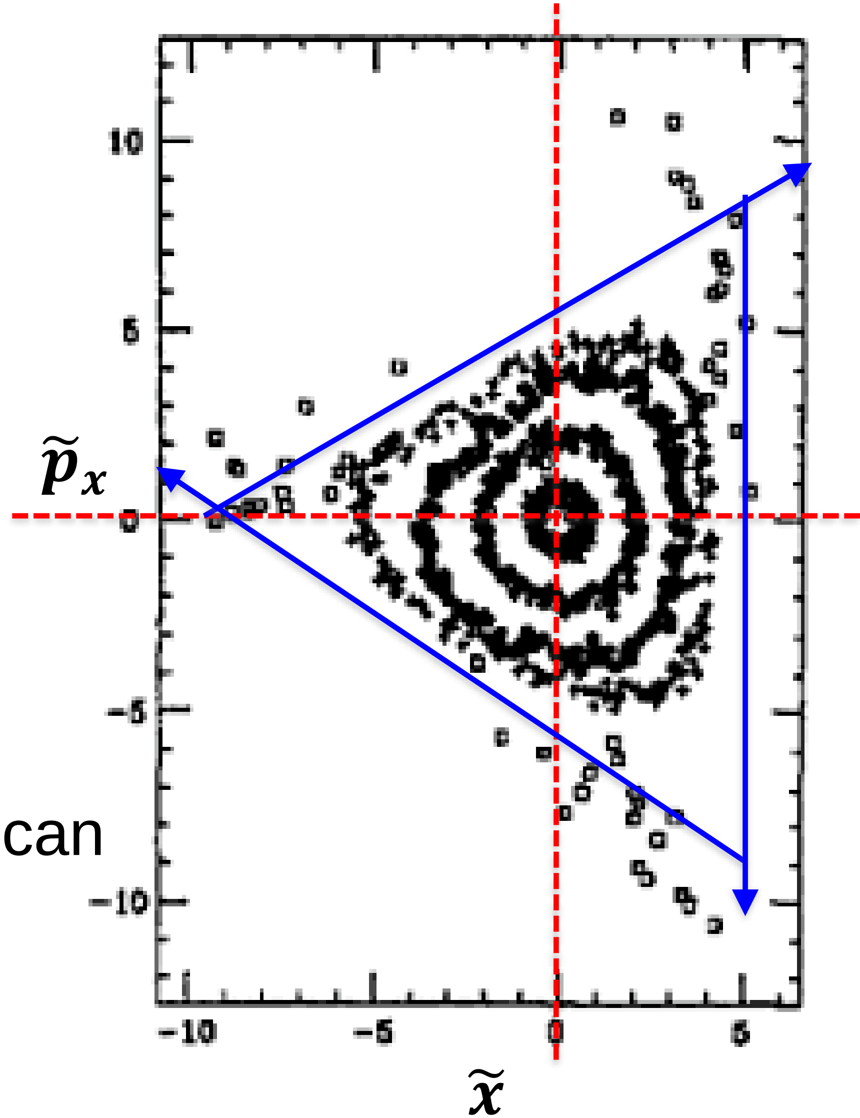
The fixed points $\frac{d\tilde{x}}{dn} = \frac{d\tilde{p}_x}{dn} = 0$ are

$$\left(\tilde{x} = -\frac{8\pi\delta}{A}, \tilde{p}_x = 0\right)$$

$$\left(\tilde{x} = \frac{4\pi\delta}{A}, \tilde{p}_x = \pm\sqrt{3}\,\frac{4\pi\delta}{A}\right)$$

The separatrix, i.e. the boundary between stable and unstable region can be given

$$\left(\tilde{x} - \frac{4\pi\delta}{A}\right)\left[\tilde{p}_x^2 - \frac{1}{3}\left(x + \frac{8\pi\delta}{A}\right)^2\right] = 0$$



3rd order resonance

The fixed points $\frac{d\tilde{x}}{dn} = \frac{d\tilde{p}_x}{dn} = \mathbf{0}$ are

$$\left(\tilde{x} = -\frac{8\pi\delta}{A}, \tilde{p}_x = 0\right)$$

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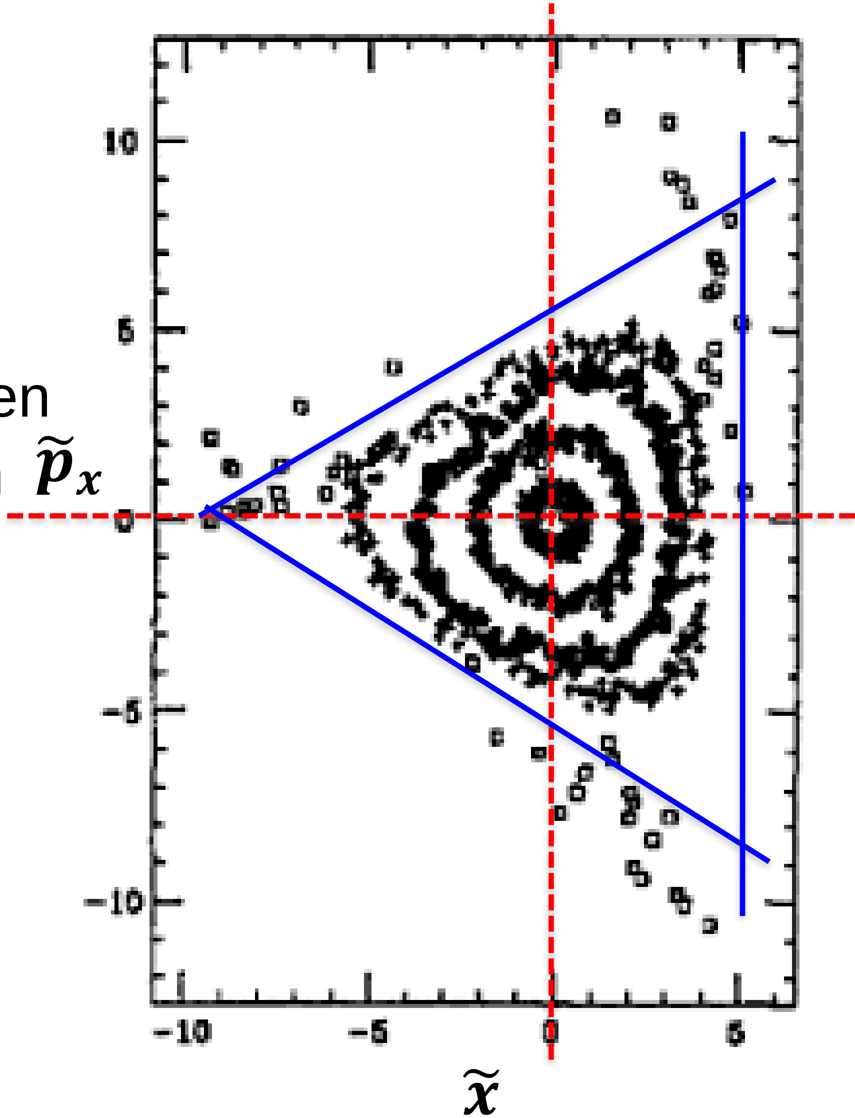
The separatrix, i.e. the boundary between stable and unstable region can be given

$$\left(\tilde{x} - \frac{4\pi\delta}{A}\right)\left[\tilde{p}_x^2 - \frac{1}{3}\left(x + \frac{8\pi\delta}{A}\right)^2\right] = 0$$

And, the area of separatrix is

$$\epsilon_{sp} = \frac{48\sqrt{3}\pi}{A^2}\delta^2, \text{ where}$$

$$A = \frac{\beta_0}{B\rho}\left(\frac{\beta_x}{\beta_0}\right)^{\frac{3}{2}}\frac{B''l}{2} \text{ on resonance}$$



3rd order resonance

Stable condition

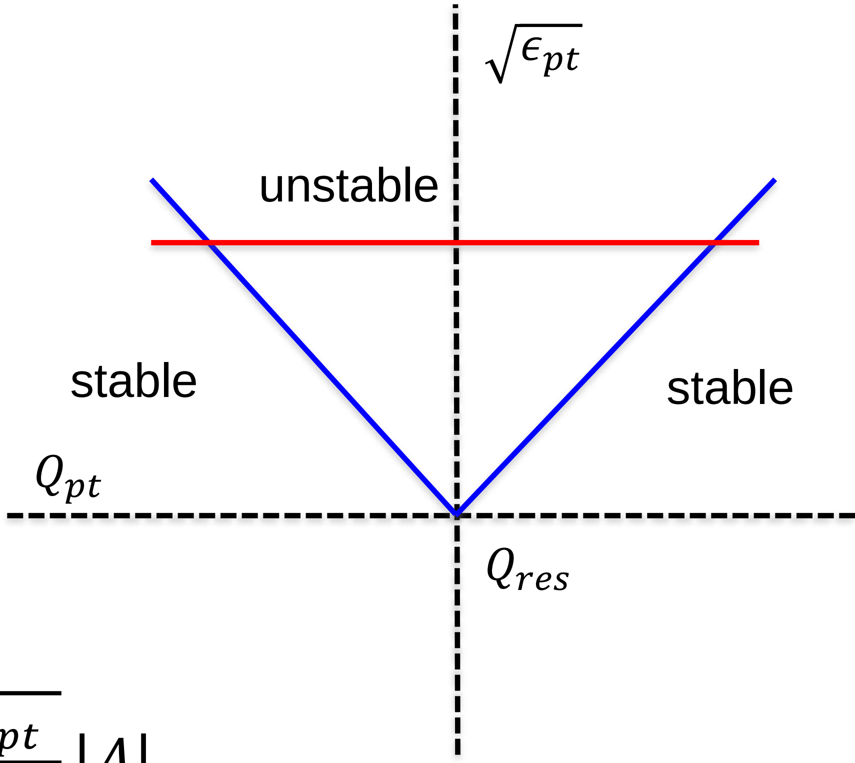
$$\epsilon_{pt} \leq \epsilon_{sp} = \frac{48\sqrt{3}}{A^2}\delta^2$$

where

$$A = \frac{\beta_0}{B\rho}\left(\frac{\beta_x}{\beta_0}\right)^{\frac{3}{2}}\frac{B''l}{2} \text{ on resonance}$$

or:

$$Q_{res} - \sqrt{\frac{\epsilon_{pt}}{48\sqrt{3}}}|A| < Q_{pt} < Q_{res} + \sqrt{\frac{\epsilon_{pt}}{48\sqrt{3}}}|A|$$



Phase space: High order resonances