CFNS SUMMER SCHOOL 2021

TUTORIAL

3D structure @ EIC

1) Phase-space representation

Wave-packet
$$\hat{\psi}(\vec{p}) = \langle \vec{p} | \psi \rangle$$
$$\psi(\vec{r}) = \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{r}} \hat{\psi}(\vec{p})$$

Normalization
$$\langle \vec{p}\,' | \vec{p} \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}\,' - \vec{p})$$

A. Show that
$$\langle \psi | j^\mu(\vec{r}) | \psi \rangle = \int \frac{d^3P}{(2\pi)^3} d^3R \; \rho_\psi(\vec{R},\vec{P}) \langle j^\mu \rangle_{\vec{R},\vec{P}}(\vec{r})$$

with
$$\rho_\psi(\vec{R},\vec{P}) = \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{R}} \hat{\psi}^*(\vec{P}+\tfrac{\vec{q}}{2}) \hat{\psi}(\vec{P}-\tfrac{\vec{q}}{2})$$

- $$\langle \psi | j^\mu(\vec{r}) | \psi \rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \hat{\psi}^*(\vec{p}\,') \hat{\psi}(\vec{p}) \langle \vec{p}\,' | j^\mu(\vec{r}) | \vec{p} \rangle$$

$$\vec{p}\,' = \vec{P} + \frac{\vec{\Delta}}{2}, \quad \vec{p} = \vec{P} - \frac{\vec{\Delta}}{2} \quad \Rightarrow \quad d^3p'\, d^3p = d^3P\, d^3\Delta$$

Thus
$$\langle \psi | j^\mu(\vec{r}) | \psi \rangle = \int \frac{d^3P}{(2\pi)^3} \frac{d^3\Delta}{(2\pi)^3} d^3R \; e^{i\vec{\Delta}\cdot\vec{R}} \rho_\psi(\vec{R},\vec{P}) \langle \vec{P}+\tfrac{\vec{\Delta}}{2} | j^\mu(\vec{r}) | \vec{P}-\tfrac{\vec{\Delta}}{2} \rangle$$
$$= \int \frac{d^3P}{(2\pi)^3} d^3R \; \rho_\psi(\vec{R},\vec{P}) \langle j^\mu \rangle_{\vec{R},\vec{P}}(\vec{r})$$

with $\quad \langle j^\mu \rangle_{\vec{R},\vec{p}}(\vec{r}) = \int \frac{d^3\Delta}{(2\pi)^3} \, e^{i\vec{\Delta}\cdot\vec{R}} \langle \vec{p}+\frac{\vec{\Delta}}{2} | j^\mu(\vec{r}) | \vec{p}-\frac{\vec{\Delta}}{2} \rangle$

$$= \int \frac{d^3\Delta}{(2\pi)^3} \, e^{-i\vec{\Delta}\cdot(\vec{r}-\vec{R})} \langle \vec{p}+\frac{\vec{\Delta}}{2} | j^\mu(\vec{0}) | \vec{p}-\frac{\vec{\Delta}}{2} \rangle$$

using $\quad j^\mu(\vec{r}) = e^{-i\hat{\vec{P}}\cdot\vec{r}} \, j^\mu(\vec{0}) \, e^{i\hat{\vec{P}}\cdot\vec{r}}$

$$\hat{\vec{P}} |\vec{p}\rangle = \vec{p} \, |\vec{p}\rangle$$

(B) Show that

$$\int \frac{d^3p}{(2\pi)^3} d^3R \, \vec{R} \, \rho_\psi(\vec{R},\vec{p}) = \int d^3R \, \vec{R} \, |\psi(\vec{R})|^2$$
$$= \int \frac{d^3p}{(2\pi)^3} \, \hat{\psi}^*(\vec{p}) \, \frac{i}{2}\overleftrightarrow{\nabla}_p \, \hat{\psi}(\vec{p})$$

- $\int \frac{d^3p}{(2\pi)^3} d^3R \, \vec{R} \int d^3z \, e^{-i\vec{p}\cdot\vec{z}} \, \psi^*(\vec{R}-\frac{\vec{z}}{2}) \psi(\vec{R}+\frac{\vec{z}}{2})$

$$= \int d^3R \, \vec{R} \int d^3z \, \delta^{(3)}(\vec{z}) \, \psi^*(\vec{R}-\frac{\vec{z}}{2}) \psi(\vec{R}+\frac{\vec{z}}{2})$$

$$= \int d^3R \, \vec{R} \, |\psi(\vec{R})|^2$$

- $\int \frac{d^3p}{(2\pi)^3} d^3R \, \vec{R} \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{R}} \, \hat{\psi}^*(\vec{p}+\frac{\vec{q}}{2}) \hat{\psi}(\vec{p}-\frac{\vec{q}}{2})$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} d^3R \left[i\vec{\nabla}_q \, e^{-i\vec{q}\cdot\vec{R}} \right] \hat{\psi}^*(\vec{p}+\frac{\vec{q}}{2}) \hat{\psi}(\vec{p}-\frac{\vec{q}}{2})$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} d^3R \, e^{-i\vec{q}\cdot\vec{R}} (-i)\vec{\nabla}_q \left[\hat{\psi}^*(\vec{p}+\frac{\vec{q}}{2}) \hat{\psi}(\vec{p}-\frac{\vec{q}}{2}) \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(\vec{q}) \, \hat{\psi}^*(\vec{p}+\frac{\vec{q}}{2}) \left(\frac{i}{2}\overrightarrow{\nabla}_p - \frac{i}{2}\overleftarrow{\nabla}_p \right) \hat{\psi}(\vec{p}-\frac{\vec{q}}{2})$$

$$= \int \frac{d^3p}{(2\pi)^3} \, \hat{\psi}^*(\vec{p}) \, \frac{i}{2}\overleftrightarrow{\nabla}_p \, \hat{\psi}(\vec{p})$$

2) Onshell conditions

(A) Show that $P^2 = M^2 - \frac{\Delta^2}{4}$ & $P \cdot \Delta = 0$

- $p'^2 = p^2 = M^2$

$\Rightarrow M^2 = (P \pm \frac{\Delta}{2})^2 = P^2 + \frac{\Delta^2}{4} \pm P \cdot \Delta$

(B) Show that $\Delta = p' - p$ is spacelike and $P = \frac{p' + p}{2}$ is timelike

- $\Delta^2 = (p' - p)^2 = 2M^2 - 2p' \cdot p$

$= 2M(M - p'^0)$ in the initial rest frame defined by $p^0 = M$

< 0

$\Rightarrow P^2 = M^2 - \frac{\Delta^2}{4} > 0$

3) Mechanical equilibrium

$$\langle T^{ij} \rangle_{\vec{0},\vec{0}}(\vec{r}) = \delta^{ij} p(r) + \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij} \right) s(r)$$

isotropic pressure $p(r) = \frac{p_r(r) + 2p_t(r)}{3}$

pressure anisotropy $s(r) = p_r(r) - p_t(r)$

(A) Show that $\nabla^i T^{ij} = 0$ implies $\frac{dp_r(r)}{dr} = -\frac{2s(r)}{r}$

- $0 = \nabla^i \langle T^{ij} \rangle_{\vec{0},\vec{0}}(\vec{r}) = \nabla^j p(r) + \left(\frac{r^j \vec{r} \cdot \vec{\nabla}}{r^2} - \frac{1}{3}\nabla^j \right) s(r) + \left(\frac{3r^j}{r^2} + \frac{r^i \delta^{ij}}{r^2} - 2\frac{r^2 r^j}{r^4} \right) s(r)$

$\times \frac{r^j}{r} \Rightarrow 0 = \frac{d}{dr}\left[p(r) + \frac{2}{3} s(r) \right] + \frac{2}{r} s(r) \Rightarrow \frac{dp_r(r)}{dr} = -\frac{2s(r)}{r}$

(B) Derive the von Laue condition $\int_0^\infty dr\, r^2 p(r) = 0$

- $$0 = \int d^3r\, r^3 \left[\frac{dp_r(r)}{dr} + \frac{2S(r)}{r}\right]$$
$$= \int d^3r\, r^2 \left[-3p_r(r) + 2S(r)\right]$$
$$= -3 \int d^3r\, r^2 p(r)$$

4) GPDs & gravitational form factors

$$\langle p'|T^{\mu\nu}(0)|p\rangle = \bar{u}(p')\Big[P^{\{\mu}\gamma^{\nu\}} A(t) + \frac{P^{\{\mu} i\sigma^{\nu\}\lambda}\Delta_\lambda}{2M} B(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} C(t) + M g^{\mu\nu}\bar{C}(t) + \frac{P^{[\mu} i\sigma^{\nu]\lambda}\Delta_\lambda}{2M} D(t)\Big] u(p)$$

$$a^{\{\mu}b^{\nu\}} = \frac{1}{2}(a^\mu b^\nu + a^\nu b^\mu)$$
$$a^{[\mu}b^{\nu]} = \frac{1}{2}(a^\mu b^\nu - a^\nu b^\mu)$$

Twist-2 vector GPDs (quarks)

$$\frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p'|\bar{\psi}(-\tfrac{z}{2})\gamma^+\psi(\tfrac{z}{2})|p\rangle\Big|_{z^+=z_\perp=0}$$
$$= \frac{1}{2P^+}\bar{u}(p')\Big[\gamma^+ H(x,\xi,t) + \frac{i\sigma^{+\lambda}\Delta_\lambda}{2M} E(x,\xi,t)\Big] u(p)$$

$$t = \Delta^2 = (p'-p)^2 \qquad \xi = -\frac{\Delta^+}{2P^+}$$

(A) Show that
$$\int dx\; x H(x,\xi,t) = A(t) + 4\xi^2 C(t)$$
$$\int dx\; x E(x,\xi,t) = B(t) - 4\xi^2 C(t)$$

$$\int dx \; x \; \frac{1}{2}\int \frac{dz^-}{2\pi}\, e^{ixP^+z^-}\, \bar\psi(-\tfrac{z}{2})\gamma^+\psi(\tfrac{z}{2})\Big|_{z^+=z_\perp=0} \qquad \boxed{A^+=0}$$

$$= \frac{1}{2(P^+)^2}\int d(xP^+)\,\frac{dz^-}{2\pi}\left[-i\frac{\partial}{\partial z^-}e^{ixP^+z^-}\right]\bar\psi(-\tfrac{z}{2})\gamma^+\psi(\tfrac{z}{2})\Big|_{z^+=z_\perp=0}$$

$$= \frac{1}{2(P^+)^2}\int d(xP^+)\,\frac{dz^-}{2\pi}\,e^{ixP^+z^-}\,i\frac{\partial}{\partial z^-}\left[\bar\psi(-\tfrac{z}{2})\gamma^+\psi(\tfrac{z}{2})\right]_{z^+=z_\perp=0}$$

$$= \frac{1}{2(P^+)^2}\int \frac{dz^-}{2\pi}\,2\pi\delta(z^-)\left[\bar\psi(r-\tfrac{z}{2})\gamma^+\tfrac{i}{2}\overleftrightarrow{\partial}^+_r\psi(r+\tfrac{z}{2})\right]_{\substack{r=0\\ z^+=z_\perp=0}}$$

$$= \frac{1}{2(P^+)^2}\left(\bar\psi\gamma^+\tfrac{i}{2}\overleftrightarrow{D}^+\psi\right)(0)$$

$$\Rightarrow \frac{1}{2P^+}\,\bar u(p')\left[\gamma^+\int dx\, x\, H + \frac{i\sigma^{+\perp}\Delta_\perp}{2M}\int dx\, x\, E\right]u(p)$$

$$= \frac{1}{2(P^+)^2}\,\bar u(p')\left[P^+\gamma^+ A + \frac{P^+\, i\sigma^{+\perp}\Delta_\perp}{2M}B + \frac{(\Delta^+)^2}{M}C\right]u(p)$$

NB: $g^{++} = \frac{1}{2}(g^{00}+g^{03}+g^{30}+g^{33}) = 0$

Gordon identity:

$$\bar u(p')\gamma^\mu u(p) = \bar u(p')\left(\frac{P^\mu}{M} + \frac{i\sigma^{\mu\perp}\Delta_\perp}{2M}\right)u(p)$$

$$\Rightarrow \bar u(p')u(p) = \frac{M}{P^+}\,\bar u(p')\left(\gamma^+ - \frac{i\sigma^{+\perp}\Delta_\perp}{2M}\right)u(p)$$

$$\Rightarrow \frac{1}{2P^+}\bar u(p')\gamma^+u(p):\ \int dx\, x\, H = A + 4\xi^2 C \qquad \Delta^+ = -2\xi P^+$$

$$\frac{1}{2P^+}\bar u(p')\frac{i\sigma^{+\perp}\Delta_\perp}{2M}u(p):\ \int dx\, x\, E = B - 4\xi^2 C$$

Belinfante EMT

$$T^{\mu\nu}_{q,\mathrm{Bel}} = T^{\{\mu\nu\}}_q = \bar\psi \gamma^{\{\mu} \tfrac{i}{2} \overleftrightarrow{D}^{\nu\}} \psi$$

Angular momentum operator

$$J^i_q = \varepsilon^{ijk} \int d^3r \; r^j \, T^{0k}_{q,\mathrm{Bel}}(\vec r)$$

(B) Derive Ji's relation $\langle J^z_q \rangle = \frac{A(0)+B(0)}{2}$

valid when both target momentum and polarization are along $\vec e_z$

$\vec s^{\,2} = 1$

- For simplicity we will set $\vec R = \vec 0$

$$\hookrightarrow \langle J^z_q \rangle_{\vec 0, P_z \vec e_z} = \varepsilon^{3jk} \int d^3r \; r^j \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\vec\Delta\cdot\vec r} \frac{\langle p', \vec e_z | T^{\{0k\}}_q(0) | p, \vec e_z \rangle}{\sqrt{2p'^0 \, 2p^0}}$$

⚠ Here we use covariant normalization $|p\rangle = \sqrt{2p^0}\,|\vec p\rangle$

$$= \varepsilon^{3jk} \int d^3r \frac{d^3\Delta}{(2\pi)^3} \left[i \nabla^j_\Delta e^{-i\vec\Delta\cdot\vec r} \right] \frac{\langle p', \vec e_z | T^{\{0k\}}_q(0) | p, \vec e_z \rangle}{\sqrt{2p'^0 \, 2p^0}}$$

$$= \varepsilon^{3jk} \int d^3r \frac{d^3\Delta}{(2\pi)^3} e^{-i\vec\Delta\cdot\vec r} (-i) \nabla^j_\Delta \frac{\langle p', \vec e_z | T^{\{0k\}}_q(0) | p, \vec e_z \rangle}{\sqrt{2p'^0 \, 2p^0}}$$

$$= \varepsilon^{3jk} \int \frac{d^3\Delta}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(\vec\Delta) \, (-i)\nabla^j_\Delta \frac{\langle p', \vec e_z | T^{\{0k\}}_q(0) | p, \vec e_z \rangle}{\sqrt{2p'^0 \, 2p^0}}$$

$$= \varepsilon^{3jk} \left[(-i) \nabla^j_\Delta \frac{\langle p', \vec e_z | T^{\{0k\}}_q(0) | p, \vec e_z \rangle}{\sqrt{2p'^0 \, 2p^0}} \right]_{\vec\Delta = \vec 0}$$

$$P^0\Big|_{\Delta_z=0} = \sqrt{M^2+P_z^2} + \mathcal{O}(\vec{\Delta}_\perp^2)$$

$$\Delta^0\Big|_{\Delta_z=0} = \frac{P_z\Delta_z}{P_0}\Big|_{\Delta_z=0} = 0$$

$$\langle p',\vec{e}_z|\, T_q^{\{0k\}}(0)\,|p,\vec{e}_z\rangle = \bar{u}(p')\Big[P^{\{0}\gamma^{k\}}A(t) + \frac{P^{\{0}i\sigma^{k\}\lambda}\Delta_\lambda}{2M}B(t)\Big]u(p)$$

$$+\mathcal{O}(\vec{\Delta}_\perp^2)$$

$$= \bar{u}(p')\Big[\underbrace{\cancel{\frac{P^0P^k}{M}A(t)}}_{=0 \text{ since } k \text{ is transverse index}} + \frac{P^{\{0}i\sigma^{k\}\lambda}\Delta_\lambda}{2M}(A(t)+B(t))\Big]u(p)$$

$$+\mathcal{O}(\vec{\Delta}_\perp^2)$$

$$= \frac{P^0}{2M}i\Delta_\lambda\,(A(0)+B(0))\;\bar{u}(P)\sigma^{k\lambda}u(P)$$

$$\text{Thus } \langle J_q^z\rangle = \frac{1}{2P^0}\frac{P^0}{2M}(A(0)+B(0))\;\underbrace{\varepsilon^{3jk}\,\bar{u}(P)\sigma^{jk}u(P)}_{=2M\,\underbrace{S_z}_{=1}}$$

for longitudinal polarization

$$= \frac{A(0)+B(0)}{2}$$