Spin dynamics in electron storage rings: A stochastic differential equations approach

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July 1, 2020

1This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award Numbers DE-SC0018008 and DE-SC0018370
Outline and motivation

Outline
- Spin-orbit dynamics in Lab frame
- Spin-orbit dynamics in beam frame
- Reduced linear orbital dynamics and nonlinear spin
- Derbenev-Kondratenko formula and Bloch equation (BE)
- Effective Bloch equation via averaging

Motivation
We use 3 approaches
- Derbenev-Kondratenko (DK) formula for depolarization time
- Bloch equation for polarization density
- Monte-Carlo spin tracking

We are able to base all 3 approaches on stochastic differential equations of Itô type
Spin-orbit dynamics in Lab frame

SDEs in Lab frame (Cartesian coordinates)

\[
\dot{\tilde{Y}} = \tilde{f}(t, \tilde{Y}) + \tilde{g}(t, \tilde{Y})\xi(t),
\]

\[
\dot{\tilde{S}} = \tilde{\Omega}(t, \tilde{Y})\tilde{S} + \tilde{M}(t, \tilde{Y})\tilde{S} + \tilde{G}(t, \tilde{Y}) + \tilde{H}(t, \tilde{Y})\xi(t),
\]

where $\tilde{Y} \in \mathbb{R}^n$, $\tilde{S} \in \mathbb{R}^3$ and $\xi$ is scalar white noise

QoI: Lab-frame polarization vector

\[
\tilde{P}(t) = \langle \tilde{S}(t) \rangle = \int \int \tilde{s}\tilde{p}_{ys}(t, \tilde{y}, \tilde{s}) \, d\tilde{s} \, d\tilde{y} \equiv \int \tilde{\eta}(t, \tilde{y}) \, d\tilde{y}
\]

where $\tilde{p}_{ys}$ = joint probability density of $\tilde{Y}$ and $\tilde{S}$ and $\tilde{\eta}$ = polarization density

- Bloch equation for polarization density $\tilde{\eta}$ discovered by Derbenev and Kondratenko (DK) (1975)
- The complete form of SDE (2) obtained from DK Bloch equation via reverse engineering (2019)
Spin-orbit dynamics in beam frame

**SDEs in beam frame**

\[
Y' = f(\theta, Y) + g(\theta, Y)\xi(\theta), \\
S' = \underbrace{\Omega(\theta, Y)}_{\text{BMT}}S + \underbrace{M(\theta, Y)S + G(\theta, Y) + H(\theta, Y)\xi(\theta)}_{\text{ST effect, BK correction, kinetic polarization}},
\]

where \( Y \in \mathbb{R}^n, S \in \mathbb{R}^3 \), where coefficients are 2\( \pi \)-periodic in \( \theta \) and \( \xi \) is vector white noise

**QoI: Beam-frame polarization vector**

\[
P(\theta) = \langle S(\theta) \rangle = \int \int s_p y_s(\theta, y, s) \, ds \, dy \equiv \int \eta(\theta, y) \, dy
\]

where \( \eta \) is polarization density \( \propto \) spin angular momentum density

- \( P(\theta) \approx \tilde{P}(t_r(\theta)), \ t_r(\theta) \) is time of reference particle at azimuth \( \theta \)
Reduced spin-orbit dynamics in beam frame to study spin diffusion

Reduced SDEs in beam frame

\[ Y' = f(\theta, Y) + g(\theta, Y)\xi(\theta), \]
\[ S' = \Omega(\theta, Y)S \]

where \( Y \in \mathbb{R}^n, S \in \mathbb{R}^3 \), coefficients are \( 2\pi \)-periodic in \( \theta \) and \( \xi \) is vector white noise

- Quantity of interest: Beam-frame polarization vector \( P(\theta) \)
- Reduced SDEs ignore self polarization effect
- Goal: Quantify decay of \( P(\theta) \), i.e., compute depolarization time
- Next: we linearize equation for the orbit (7) and linearize \( \Omega(\theta, Y) \) in \( Y \) in (8).
Linearized model in beam frame

Reduced orbit linearized SDEs

\[ Y' = [A(\theta) + \varepsilon_1 \delta A(\theta)] Y + \sqrt{\varepsilon_1} B(\theta) \xi(\theta), \quad (9) \]

\[ S' = [\Omega_0(\theta) + \varepsilon_2 \sum_{j=1}^{n} \Omega_j(\theta) Y_j] S \quad (10) \]

where \( Y \in \mathbb{R}^n, S \in \mathbb{R}^3 \), coefficients are \( 2\pi \)-periodic in \( \theta \), \( B(\theta) \) is diagonal matrix and \( \xi \) is vector white noise.

Reduced Bloch equation (Fokker Planck equation + T–BMT)

\[ \partial_{\theta} \eta = -\sum_{j=1}^{n} \partial_{y_j} \left( ([A(\theta) + \varepsilon_1 \delta A(\theta)] y_j \eta) \right) + \frac{\varepsilon_1}{2} \sum_{j=1}^{n} \left( B(\theta) B^T(\theta) \right)_{jj} \partial_{y_j}^2 \eta + \Omega(\theta, y) \eta. \quad (11) \]

- Linearization in \( Y \) is simplest approximation which captures the main spin effects.
- Unlike SLIM here spin is not linearized (synchrotron sidebands are included).
- Reduced linearized SDEs and the Bloch equation* are key for our current research.

* Bloch equation comes from the condensed matter physics.
Gaussian beam density and equilibrium

Orbit SDE in beam frame

We write (9) more generally as

\[ Y' = A(\theta)Y + B(\theta)\xi(\theta), \quad Y(0) = Y_0, \]  

(12)

with mean and covariance given by

\[ m' = A(\theta)m, \quad m(0) = m_0 \]

(13)

\[ K' = A(\theta)K + K A^T(\theta) + B(\theta)B^T(\theta), \quad K(0) = K_0 \]

(14)

- The PSM for \( A \) is defined by \( \Psi' = A(\theta)\Psi, \quad \Psi(0) = I_{n\times n} \)
- Radiation damping implies \( \Psi(\theta) \rightarrow 0 \) and thus \( m(\theta) \rightarrow 0 \) as \( \theta \rightarrow \infty \) and

\[ K(\theta) = \Psi(\theta) \left( K_0 + \int_0^\theta \psi^{-1}(\theta')B(\theta')B^T(\theta')\psi^{-T}(\theta') \, d\theta' \right) \Psi^T(\theta) \]

There exist unique \( K_0 \) such that \( K_0 = K(2\pi) \) and thus \( K(\theta + 2\pi) = K(\theta) \) therefore we get

\[ K(\theta) = \Psi(\theta) \int_{-\infty}^\theta \psi^{-1}(\theta')B(\theta')B^T(\theta')\psi^{-T}(\theta') \, d\theta' \Psi^T(\theta) =: K_{\text{per}}(\theta) \]

It can be shown that \( (K(\theta) - K_{\text{per}}(\theta)) \rightarrow 0 \) as \( \theta \rightarrow \infty \)

\[ p_Y(\theta, y) \approx p_{eq}(\theta, y) = (2\pi)^{-n/2} \det(K_{eq}(\theta))^{-1/2} \exp\left(-\frac{1}{2}y^T K_{eq}^{-1}(\theta)y\right), \text{ for large } \theta \]
Derbenev–Kondratenko formula for depolarization time

**Invariant spin field (ISF)**

Let $\hat{n}(\theta, y)$ be the unique periodic solution of (11), with $\varepsilon_1 = 0$

\[
\partial_\theta \hat{n} = -\sum_{j=1}^{n} \partial_y \left( [A(\theta)y_j] \hat{n} \right) + \left[ \Omega_0(\theta) + \varepsilon_2 \sum_{j=1}^{n} \Omega_j(\theta) Y_j \right] \hat{n} \tag{15}
\]

- Since $\varepsilon_1$ small, in spirit of DK, we look for a solution of BE in the form

\[
\eta(\theta, y) = c(\theta) p_{eq}(\theta, y) \hat{n}(\theta, y) + \Delta \eta(\theta, y) \tag{16}
\]

- Beam frame polarization vector

\[
P(\theta) \approx c(\theta) \int p_{eq}(\theta, y) \hat{n}(\theta, y) dy \tag{17}
\]

- Bloch equation for $\eta$ gives ODE for $c$ and PDE for $\Delta \eta$ coupled to $c$

\[
c'(\theta) = -\varepsilon_1 q(\theta) c(\theta), \tag{18}
\]

\[
q(\theta) \equiv \frac{1}{2} \sum_{j=1}^{n} B_{jj}(\theta) \int p_{eq}(\theta, y) \left| \frac{\partial \hat{n}}{\partial y_j}(\theta, y) \right|^2 dy \tag{19}
\]
Unsolved questions

1. How does $\Delta \eta$ affect the depolarization time?
2. When is $\Delta \eta$ negligible?

We have a simple model where question 1 is easy to answer
Toy model

Model SDEs

\[ Y' = [A + \varepsilon_1 \delta A]Y + \sqrt{\varepsilon_1}B(\xi_1(\theta), \xi_2(\theta))^T, \quad (20) \]

\[ S' = [\Omega_0 + \varepsilon_2 \sum_{j=1}^2 \Omega_j Y_j]S \quad (21) \]

where \( Y \in \mathbb{R}^2 \), \( S \in \mathbb{R}^3 \) and \( B \) is diagonal matrix with \( \xi_1(\theta), \xi_2(\theta) \) statistically independent white noise processes.

\[
A = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}, \quad \delta A = \begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix}, \quad \Omega_0 = \begin{pmatrix} 0 & -\sigma_1 & 0 \\ \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\Omega_1 = \begin{pmatrix} 0 & 0 & \sigma_2 \\ 0 & 0 & 0 \\ -\sigma_2 & 0 & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sigma_2 \\ 0 & \sigma_2 & 0 \end{pmatrix}
\]

Goal: Compute depolarization time by integrating the BE and comparing with DK formula from previous slides.
Toy model: Bloch equation

- Goal: Compute depolarization time
- Tool 1:

**Bloch equation for polarization density**

\[
\partial_\theta \eta = \varepsilon_1 a (\partial_{y_1} (y_1 \eta) + \partial_{y_2} (y_2 \eta)) + b (\partial_{y_1} (y_2 \eta) - \partial_{y_2} (y_1 \eta)) \\
+ \frac{\varepsilon_1}{2} \left( B_{11}^2 \partial_{y_1}^2 + B_{22}^2 \partial_{y_2}^2 \right) \eta + [\Omega_0 + \varepsilon_2 \sum_{j=1}^{2} \Omega_j y_j] \eta
\]  

(22)

Numerical method* (for long time simulations)

- Spectral Chebyshev-Fourier discretization in phase space
- Embedded high order additive Runge-Kutta time evolution

Toy model: Derbenev-Kondratenko formula

- Goal: Compute depolarization time
- Tool 2:

Derbenev-Kondratenko formula for depolarization time

- Invariant spin field:

\[
\hat{n}(y) = \sqrt{\frac{1}{(\sigma_1 - b)^2 + \sigma_2^2 (y_1^2 + y_2^2)}} (\sigma_2 y_1, \sigma_2 y_2, \sigma_1 - b)^T \tag{23}
\]

- Write polarization density \(\eta\) as

\[
\eta(\theta, y) = c(\theta) p_{eq}(y) \hat{n}(y) + \Delta \eta(\theta, y) \tag{24}
\]

\[
c'(\theta) = -\varepsilon_1 q c(\theta), \quad q = \frac{1}{2} \sum_{j=1}^{2} B_{jj} \int p_{eq}(y) \left| \frac{\partial \hat{n}}{\partial y_j}(y) \right|^2 dy
\]

\[
p_{eq}(y) = \frac{a}{\pi \Gamma^2} \exp \left( -\frac{a}{\Gamma^2} (y_1^2 + y_2^2) \right), \quad B_{11} = B_{22} = \Gamma
\]
Numerical results

- Via Bloch equation

\[ P(\theta) = \int \eta(\theta, y) \, dy \]

- Via DK

\[ P(\theta) \approx c(30) e^{-\varepsilon_1 q(\theta-30)} \int p_{eq}(y) \hat{n}(y) \, dy \]

- Damping time is \(1/\varepsilon_1 a = 10\)
Orbital dynamics: Averaging approximation - 1

Goal: Find effective Bloch equation

Reduced linearized orbit & nonlinear spin SDEs

\[
Y' = \left[ A(\theta) + \varepsilon_1 \delta A(\theta) \right] Y + \sqrt{\varepsilon_1} B(\theta) \xi(\theta), \tag{25}
\]

\[
S' = \left[ \Omega_0(\theta) + \varepsilon_2 \sum_{j=1}^n \Omega_j(\theta) Y_j \right] S \tag{26}
\]

There are two different versions of averaging approximation:

- (i) $\varepsilon_2 = 1$
- (ii) $\varepsilon_1 = \varepsilon_2$

We are here doing (i)

- Fundamental solution matrix $\Phi$ of Hamiltonian part of SDEs is defined by:

\[
\Phi'(\theta) = A(\theta) \Phi(\theta) \tag{27}
\]

where $\Phi(\theta)$ is quasiperiodic

- Transform $Y$ to $U$ to get standard form for averaging:

\[
U(\theta) = \Phi^{-1}(\theta) Y(\theta) \tag{28}
\]
Orbital dynamics: Averaging approximation - 2

SDEs in slowly varying form

\[ U' = \varepsilon_1 D(\theta)U + \sqrt{\varepsilon_1} \Phi^{-1}(\theta)B(\theta)\xi(\theta), \] (29)

\[ S' = [\Omega_0(\theta) + \sum_{j=1}^{n} \Omega_j(\theta)(\Phi(\theta)U)_j]S \] (30)

where \( D(\theta) \) is quasiperiodic

- ODE for \( m_U \) and ODE for \( K_U \):

\[ m'_U = \varepsilon_1 D(\theta)m_U \]

\[ K'_U = \varepsilon_1 [D(\theta)K_U + K_U D^T(\theta)] + \varepsilon_1 \Phi^{-1}(\theta)B(\theta)B^T(\theta)\Phi^{-T}(\theta) \]
Orbital dynamics: Averaging approximation - 3

Averaged SDEs

Averaging gives us $V \approx U$ with the SDEs

$$V' = \varepsilon_1 \bar{D} V + \sqrt{\varepsilon_1} \bar{B} \xi(\theta), \quad (31)$$

$$S' = [\Omega_0(\theta) + \sum_{j=1}^n \Omega_j(\theta)(\Phi(\theta)V)_j]S \quad (32)$$

- First-moment vector $m_V$ of $U$ and covariance matrix $K_V$ of $V$ satisfy ODEs

$$m'_V = \bar{D} m_V \quad (33)$$

$$K'_V = \varepsilon_1[\bar{D}K_V + K_V \bar{D}^T] + \varepsilon_1 \Phi^{-1} B B^T \Phi^{-T} \quad (34)$$

- Note that the SDEs for $V$ are obtained from the ODEs for $m_V, K_V$ via reverse engineering.
Effective Bloch equation

The Bloch equation for the polarization density $\eta$ corresponding to averaged SDEs reads as

$$
\partial_\theta \eta_\nu = -\varepsilon_1 \sum_{j=1}^n \partial_{v_j}(\bar{D}v)_j \eta_\nu + \frac{1}{2} \varepsilon_1 \sum_{i,j=1}^n (\Phi^{-1}BB^T\Phi^{-T})_{ij} \partial_{v_i} \partial_{v_j} \eta_\nu
$$

$$
+ [\Omega_0(\theta) + \sum_{j=1}^n \Omega_j(\theta)(\Phi(\theta)v)_j] \eta_\nu
$$

(35)

- To give $\bar{D}$ its simplest form we choose the fundamental solution matrix $\Phi$ along the lines of A.W. Chao (see handbook)
- Numerical scheme follows the same approach as for the toy model (Derived in collaboration with Daniel Appelö and Stephen Lau)
Future work

- Use SDEs to guide the Monte-Carlo spin–orbit tracking
- Wrap up the numerical scheme for the Effective Bloch equation for realistic machine
- Full Bloch equation simulations for models and realistic machines (including ST self polarization)
- Investigate white noise assumption

References

- [https://math.unm.edu/~ellison/](https://math.unm.edu/~ellison/)