Spin dynamics in electron storage rings:
A stochastic differential equations approach

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Outline and motivation

Outline

- Spin-orbit dynamics in Lab frame
- Spin-orbit dynamics in beam frame
- Reduced linear orbital dynamics and nonlinear spin
- Derbenev-Kondratenko formula and Bloch equation (BE)
- Effective Bloch equation via averaging

Motivation

We use 3 approaches

- Derbenev-Kondratenko (DK) formula for depolarization time
- Bloch equation for polarization density
- Monte-Carlo spin tracking

We are able to base all 3 approaches on stochastic differential equations of Itô type

Spin-orbit dynamics in Lab frame

SDEs in Lab frame (Cartesian coordinates)

$$\dot{\tilde{Y}} = \tilde{f}(t, \tilde{Y}) + \tilde{g}(t, \tilde{Y})\xi(t), \tag{1}$$

$$\dot{\tilde{S}} = \underbrace{\tilde{\Omega}(t, \tilde{Y})\tilde{S}}_{\text{BMT}} + \underbrace{\tilde{M}(t, \tilde{Y})\tilde{S} + \tilde{G}(t, \tilde{Y}) + \tilde{H}(t, \tilde{Y})\xi(t)}_{\text{ST effect, BK correction, kinetic polarization}} \tag{2}$$

where $\tilde{Y} \in \mathbb{R}^n, \tilde{S} \in \mathbb{R}^3$ and ξ is scalar white noise

QoI: Lab-frame polarization vector

$$\tilde{P}(t) = \langle \tilde{S}(t) \rangle = \iint \tilde{s}\tilde{p}_{ys}(t, \tilde{y}, \tilde{s})\, d\tilde{s}\, d\tilde{y} \equiv \int \tilde{\eta}(t, \tilde{y})\, d\tilde{y} \tag{3}$$

where $\tilde{p}_{ys}$= joint probability density of $\tilde{Y}$ and $\tilde{S}$ and $\tilde{\eta}$ = polarization density

- Bloch equation for polarization density $\tilde{\eta}$ discovered by Derbenev and Kondratenko (DK) (1975)
- The complete form of SDE (2) obtained from DK Bloch equation via reverse engineering (2019)

Spin-orbit dynamics in beam frame

SDEs in beam frame

$$Y' = f(\theta, Y) + g(\theta, Y)\xi(\theta), \tag{4}$$

$$S' = \underbrace{\Omega(\theta, Y)S}_{\text{BMT}} + \underbrace{M(\theta, Y)S + G(\theta, Y) + H(\theta, Y)\xi(\theta)}_{\text{ST effect, BK correction, kinetic polarization}} \tag{5}$$

where $Y \in \mathbb{R}^n, S \in \mathbb{R}^3$, where coefficients are 2π-periodic in θ and ξ is vector white noise

QoI: Beam-frame polarization vector

$$P(\theta) = \langle S(\theta) \rangle = \iint s p_{ys}(\theta, y, s)\ ds\ dy \equiv \int \eta(\theta, y)\ dy \tag{6}$$

where η is polarization density $\propto$ spin angular momentum density

- $P(\theta) \approx \tilde{P}(t_r(\theta))$, $t_r(\theta)$ is time of reference particle at azimuth θ

Reduced spin-orbit dynamics in beam frame to study spin diffusion

Reduced SDEs in beam frame

$$Y' = f(\theta, Y) + g(\theta, Y)\xi(\theta), \tag{7}$$
$$S' = \Omega(\theta, Y)S \tag{8}$$

where $Y \in \mathbb{R}^n, S \in \mathbb{R}^3$, coefficients are 2π-periodic in θ and ξ is vector white noise

- Quantity of interest: Beam-frame polarization vector $P(\theta)$
- Reduced SDEs ignore self polarization effect
- Goal: Quantify decay of $P(\theta)$, i.e., compute depolarization time
- Next: we linearize equation for the orbit (7) and linearize $\Omega(\theta, Y)$ in Y in (8).

Linearized model in beam frame

Reduced orbit linearized SDEs

$$Y' = [A(\theta) + \varepsilon_1 \delta A(\theta)]Y + \sqrt{\varepsilon_1} B(\theta)\xi(\theta), \tag{9}$$

$$S' = [\Omega_0(\theta) + \varepsilon_2 \sum_{j=1}^{n} \Omega_j(\theta) Y_j]S \tag{10}$$

where $Y \in \mathbb{R}^n$, $S \in \mathbb{R}^3$, coefficients are 2π-periodic in θ, $B(\theta)$ is diagonal matrix and ξ is vector white noise

Reduced Bloch equation (Fokker Planck equation + T–BMT)

$$\partial_\theta \eta = -\sum_{j=1}^{n} \partial_{y_j} \Big(([A(\theta) + \varepsilon_1 \delta A(\theta)]y)_j \, \eta \Big) + \frac{\varepsilon_1}{2} \sum_{j=1}^{n} \Big(B(\theta) B^{\mathrm{T}}(\theta) \Big)_{jj} \partial^2_{y_j} \eta + \Omega(\theta, y)\eta. \tag{11}$$

- Linearization in Y is simplest approximation which captures the main spin effects
- Unlike SLIM here spin is not linearized (synchrotron sidebands are included)
- Reduced linearized SDEs and the Bloch equation* are key for our current research

* Bloch equation comes from the condensed matter physics

Gaussian beam density and equilibrium

Orbit SDE in beam frame

We write (9) more generally as

$$Y' = \mathcal{A}(\theta)Y + \mathcal{B}(\theta)\xi(\theta), \ \ Y(0) = Y_0, \tag{12}$$

with mean and covariance given by

$$m' = \mathcal{A}(\theta)m, \ \ m(0) = m_0 \tag{13}$$

$$K' = \mathcal{A}(\theta)K + K\mathcal{A}^{\mathrm{T}}(\theta) + \mathcal{B}(\theta)\mathcal{B}^{\mathrm{T}}(\theta), \ \ K(0) = K_0 \tag{14}$$

- The PSM for $\mathcal{A}$ is defined by $\Psi' = \mathcal{A}(\theta)\Psi, \ \Psi(0) = I_{n\times n}$
- Radiation damping implies $\Psi(\theta) \to 0$ and thus $m(\theta) \to 0$ as $\theta \to \infty$ and

$$K(\theta) = \Psi(\theta)\left(K_0 + \int_0^\theta \Psi^{-1}(\theta')\mathcal{B}(\theta')\mathcal{B}^{\mathrm{T}}(\theta')\Psi^{-\mathrm{T}}(\theta')\ d\theta'\right)\Psi^{\mathrm{T}}(\theta)$$

There exist unique K_0 such that $K_0 = K(2\pi)$ and thus $K(\theta + 2\pi) = K(\theta)$ therefore we get

$$K(\theta) = \Psi(\theta)\int_{-\infty}^\theta \Psi^{-1}(\theta')\mathcal{B}(\theta')\mathcal{B}^{\mathrm{T}}(\theta')\Psi^{-\mathrm{T}}(\theta')\ d\theta'\ \Psi^{\mathrm{T}}(\theta) =: K_{\mathrm{per}}(\theta)$$

It can be shown that $(K(\theta) - K_{\mathrm{per}}(\theta)) \to 0$ as $\theta \to \infty$

$$p_Y(\theta, y) \approx p_{\mathrm{eq}}(\theta, y) = (2\pi)^{-n/2}\det(K_{\mathrm{eq}}(\theta))^{-1/2}\exp(-\frac{1}{2}y^{\mathrm{T}}K_{\mathrm{eq}}^{-1}(\theta)y), \ \text{for large } \theta$$

Derbenev–Kondratenko formula for depolarization time

Invariant spin field (ISF)

Let $\hat{n}(\theta, y)$ be the unique periodic solution of (11), with $\varepsilon_1 = 0$

$$\partial_\theta \hat{n} = -\sum_{j=1}^{n} \partial_{y_j} ([A(\theta)y]_j \hat{n}) + \left[\Omega_0(\theta) + \varepsilon_2 \sum_{j=1}^{n} \Omega_j(\theta) Y_j\right] \hat{n} \tag{15}$$

- Since ε_1 small, in spirit of DK, we look for a solution of BE in the form

$$\eta(\theta, y) = c(\theta) p_{\mathrm{eq}}(\theta, y) \hat{n}(\theta, y) + \Delta\eta(\theta, y) \tag{16}$$

- Beam frame polarization vector

$$P(\theta) \approx c(\theta) \int p_{\mathrm{eq}}(\theta, y) \hat{n}(\theta, y) dy \tag{17}$$

- Bloch equation for η gives ODE for c and PDE for $\Delta\eta$ coupled to c

$$c'(\theta) = -\varepsilon_1 q(\theta) c(\theta), \tag{18}$$

$$q(\theta) \equiv \frac{1}{2} \sum_{j=1}^{n} B_{jj}(\theta) \int p_{\mathrm{eq}}(\theta, y) \left| \frac{\partial \hat{n}}{\partial y_j}(\theta, y) \right|^2 dy \tag{19}$$

Unsolved questions

1. How does $\Delta\eta$ affect the depolarization time?
2. When is $\Delta\eta$ negligible?

We have a simple model where question 1 is easy to answer

Toy model

Model SDEs

$$Y' = [A + \varepsilon_1 \delta A] Y + \sqrt{\varepsilon_1} B(\xi_1(\theta), \xi_2(\theta))^T, \tag{20}$$

$$S' = [\Omega_0 + \varepsilon_2 \sum_{j=1}^{2} \Omega_j Y_j] S \tag{21}$$

where $Y \in \mathbb{R}^2, S \in \mathbb{R}^3$ and B is diagonal matrix with $\xi_1(\theta), \xi_2(\theta)$ statistically independent white noise processes

$$A = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}, \quad \delta A = \begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix}, \quad \Omega_0 = \begin{pmatrix} 0 & -\sigma_1 & 0 \\ \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Omega_1 = \begin{pmatrix} 0 & 0 & \sigma_2 \\ 0 & 0 & 0 \\ -\sigma_2 & 0 & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sigma_2 \\ 0 & \sigma_2 & 0 \end{pmatrix}$$

Goal: Compute depolarization time by integrating the BE and comparing with DK formula from previous slides

Toy model: Bloch equation

- Goal: Compute depolarization time
- Tool 1:

Bloch equation for polarization density

$$\partial_\theta \eta = \varepsilon_1 a \left(\partial_{y_1}(y_1\eta) + \partial_{y_2}(y_2\eta)\right) + b\left(\partial_{y_1}(y_2\eta) - \partial_{y_2}(y_1\eta)\right)$$
$$+ \frac{\varepsilon_1}{2}\left(B_{11}^2\partial_{y_1}^2 + B_{22}^2\partial_{y_2}^2\right)\eta + [\Omega_0 + \varepsilon_2 \sum_{j=1}^{2} \Omega_j y_j]\eta \qquad (22)$$

Numerical method* (for long time simulations)

- Spectral Chebyshev-Fourier discretization in phase spase
- Embedded high order additive Runge-Kutta time evolution

* O. Beznosov, K. Heinemann, J.A. Ellison, D. Appelö, D.P. Barber, Spin Dynamics in Modern Electron Storage Rings: Computational Aspects, Proceedings of ICAP18, Key West, October 2018.

Toy model: Derbenev-Kondratenko formula

- Goal: Compute depolarization time
- Tool 2:

Derbenev-Kondratenko formula for depolarization time

- Invariant spin field:

$$\hat{n}(y) = \sqrt{\frac{1}{(\sigma_1 - b)^2 + \sigma_2^2(y_1^2 + y_2^2)}}(\sigma_2 y_1, \sigma_2 y_2, \sigma_1 - b)^T \tag{23}$$

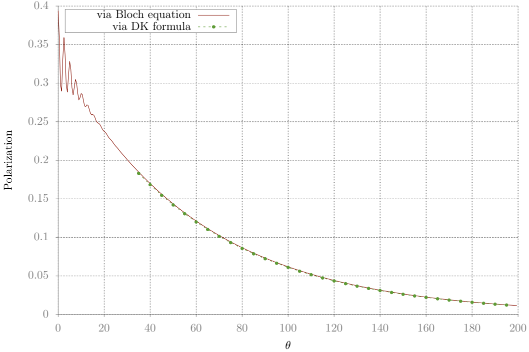
- Write polarization density η as

$$\eta(\theta, y) = c(\theta) p_{\mathrm{eq}}(y) \hat{n}(y) + \Delta\eta(\theta, y) \tag{24}$$

$$\boxed{c'(\theta) = -\varepsilon_1 q\, c(\theta), \;\; q = \frac{1}{2}\sum_{j=1}^{2} B_{jj} \int p_{\mathrm{eq}}(y) \left| \frac{\partial \hat{n}}{\partial y_j}(y) \right|^2 dy}$$

$$p_{\mathrm{eq}}(y) = \frac{a}{\pi \Gamma^2} \exp\left(-\frac{a}{\Gamma^2}(y_1^2 + y_2^2) \right), B_{11} = B_{22} = \Gamma$$

Numerical results



- Via Bloch equation

$$P(\theta) = \int \eta(\theta, y)\ dy$$

- Via DK

$$P(\theta) \approx c(30)e^{-\varepsilon_1 q(\theta-30)} \int p_{\mathrm{eq}}(y)\hat{n}(y)\ dy$$

- Damping time is $1/\varepsilon_1 a = 10$

Orbital dynamics: Averaging approximation - 1

Goal: Find effective Bloch equation

Reduced linearized orbit & nonlinear spin SDEs

$$Y' = [A(\theta) + \varepsilon_1 \delta A(\theta)]Y + \sqrt{\varepsilon}_1 B(\theta)\xi(\theta), \quad (25)$$

$$S' = [\Omega_0(\theta) + \varepsilon_2 \sum_{j=1}^{n} \Omega_j(\theta) Y_j]S \quad (26)$$

There are two different versions of averaging approximation:

- (i) $\varepsilon_2 = 1$ (ii) $\varepsilon_1 = \varepsilon_2$

We are here doing (i)

- Fundamental solution matrix Φ of Hamiltonian part of SDEs is defined by:

$$\Phi'(\theta) = A(\theta)\Phi(\theta) \quad (27)$$

  where $\Phi(\theta)$ is quasiperiodic
- Transform Y to U to get standard form for averaging:

$$U(\theta) = \Phi^{-1}(\theta) Y(\theta) \quad (28)$$

Orbital dynamics: Averaging approximation - 2

SDEs in slowly varying form

$$U' = \varepsilon_1 D(\theta)U + \sqrt{\varepsilon_1}\Phi^{-1}(\theta)B(\theta)\xi(\theta), \tag{29}$$

$$S' = [\Omega_0(\theta) + \sum_{j=1}^{n} \Omega_j(\theta)(\Phi(\theta)U)_j]S \tag{30}$$

where $D(\theta)$ is quasiperiodic

- ODE for m_U and ODE for K_U:

$$m_U' = \varepsilon_1 D(\theta) m_U$$
$$K_U' = \varepsilon_1[D(\theta)K_U + K_U D^T(\theta)] + \varepsilon_1 \Phi^{-1}(\theta)B(\theta)B^T(\theta)\Phi^{-T}(\theta)$$

Averaged SDEs

Averaging gives us $V \approx U$ with the SDEs

$$V' = \varepsilon_1 \bar{D} V + \sqrt{\varepsilon_1}\bar{B}\xi(\theta), \tag{31}$$

$$S' = [\Omega_0(\theta) + \sum_{j=1}^{n} \Omega_j(\theta)(\Phi(\theta)V)_j]S \tag{32}$$

- First-moment vector m_V of U and covariance matrix K_V of V satisfy ODEs

$$m_V' = \bar{D} m_V \tag{33}$$

$$K_V' = \varepsilon_1[\bar{D}K_V + K_V\bar{D}^T] + \varepsilon_1\overline{\Phi^{-1}BB^T\Phi^{-T}} \tag{34}$$

- Note that the SDEs for V are obtained from the ODEs for m_V, K_V via reverse engineering

Orbital dynamics: Averaging approximation - 4

Effective Bloch equation

The Bloch equation for the polarization density η corresponding to averaged SDEs reads as

$$\partial_\theta \eta_v = -\varepsilon_1 \sum_{j=1}^{n} \partial_{v_j} (\bar{D} v)_j \eta_v + \frac{1}{2} \varepsilon_1 \sum_{i,j=1}^{n} (\overline{\Phi^{-1} B B^T \Phi^{-T}})_{ij} \partial_{v_i} \partial_{v_j} \eta_v + [\Omega_0(\theta) + \sum_{j=1}^{n} \Omega_j(\theta)(\Phi(\theta) v)_j] \eta_v \tag{35}$$

- To give $\bar{D}$ its simplest form we choose the fundamental solution matrix Φ along the lines of A.W. Chao (see handbook)
- Numerical scheme follows the same approach as for the toy model (Derived in collaboration with Daniel Appelö and Stephen Lau)

Future work

- Use SDEs to guide the Monte-Carlo spin–orbit tracking
- Wrap up the numerical scheme for the Effective Bloch equation for realistic machine
- Full Bloch equation simulations for models and realistic machines (including ST self polarization)
- Investigate white noise assumption

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