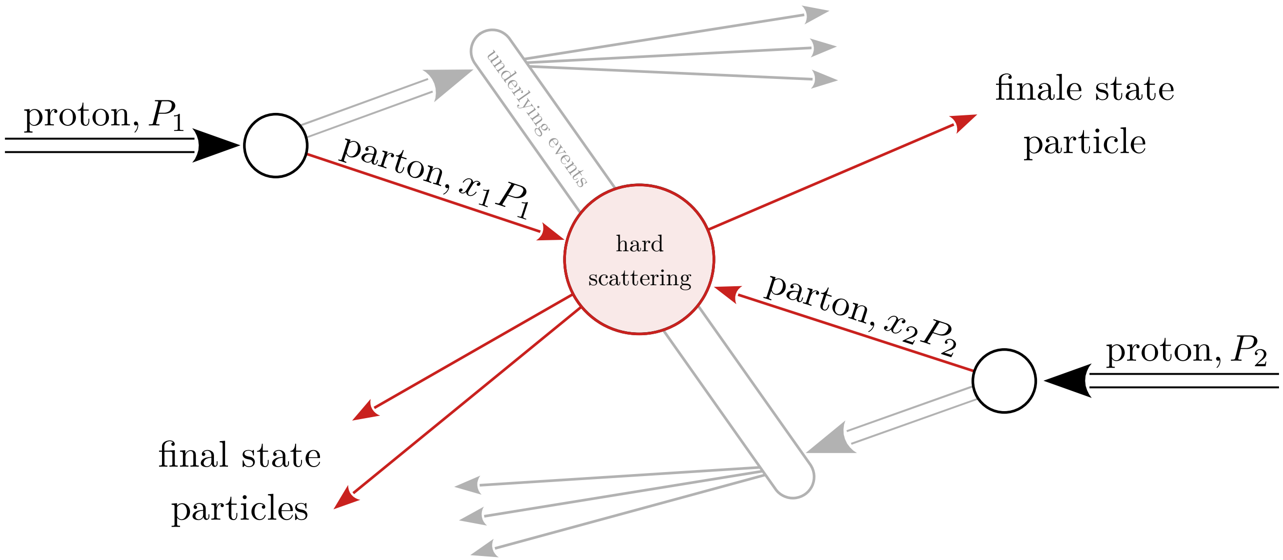
Physics of subtractions

Konstantin Asteriadis | 20.11.2020

HET Lunch Discussions

Precise predictions for hard scattering at hadron colliders



- Expected experimental precision at HL-LHC for many interesting observables $\mathcal{O}(1\%)$
  $\rightarrow$ Enables: precisely measure the Higgs sector of the SM; indirect searches for new physics

- Hadronic cross section [Collins, Soper, Sterman, '89]

$$\mathrm{d}\sigma_{\mathrm{H}} = \sum_{ij} \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \; f_i(x_1) f_j(x_2) \underbrace{\boxed{\mathrm{d}\hat{\sigma}_{ij}(x_1, x_2)}}_{\text{in the following}} \left[1 + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q} \right) \right]$$

- PDFs: non-perturbative but universal, extracted from data, precisely known

- Partonic cross section in perturbative QCD as expansion in the strong coupling constant α_s

$$\mathrm{d}\hat{\sigma}_{ij}(x_1, x_2) = \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{lo}}(x_1, x_2) + \underbrace{\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{nlo}}(x_1, x_2)}_{\mathcal{O}(10\%)} + \boxed{\underbrace{\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{nnlo}}(x_1, x_2)}_{\mathcal{O}(1\%)}} + \mathcal{O}(\alpha_s^3)$$

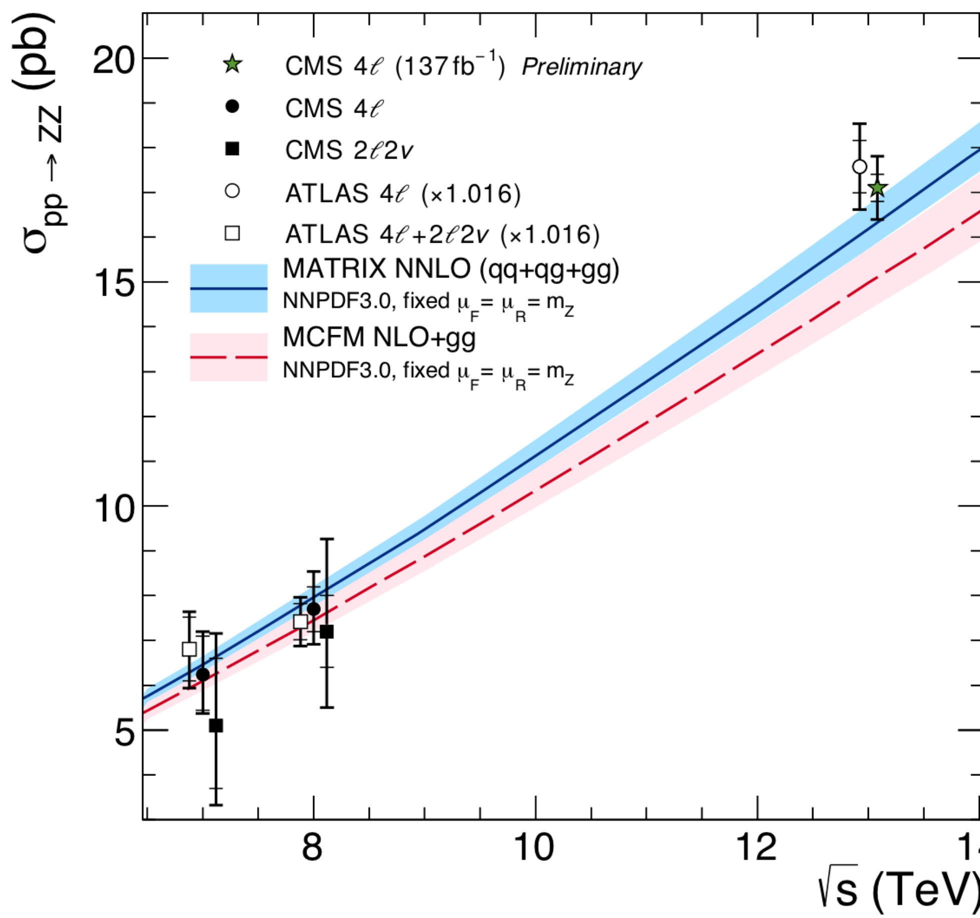
Precise predictions for hard scattering at hadron colliders

pr

NLO vs. NNLO

in pp $\to$ ZZ

[CMS-PAS-SMP-19-001]



- Expected experime ... %)
  $\to$ Enables: prec ... h for new physics
- Hadronic cross sec

  dσ

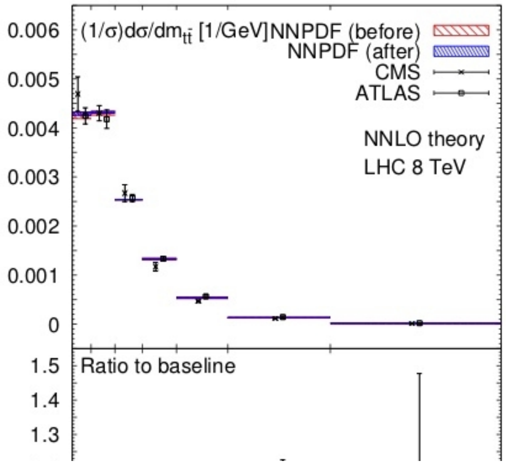
- PDFs: non-perturb
- Partonic cross sect ... nstant α_s

  d$\hat{\sigma}_i$ ... $_s^3$)

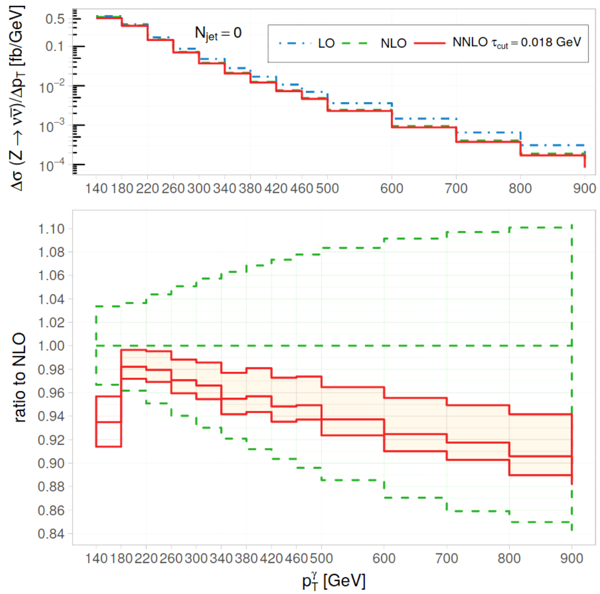
$\mathcal{O}(10\%)$ $\boxed{\mathcal{O}(1\%)}$

Many processes known at NNLO QCD (2004 - today)

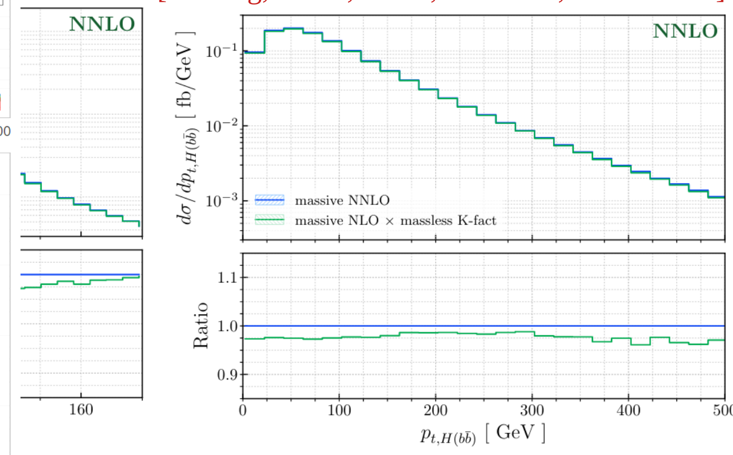
[Czakon, Hartland, Mitov, Nocera, Rojo]



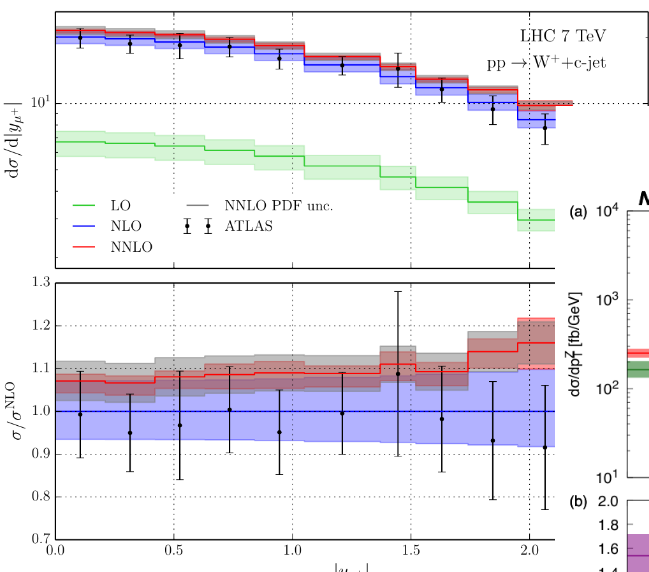
[Campbell, Neumann, Williams '17]



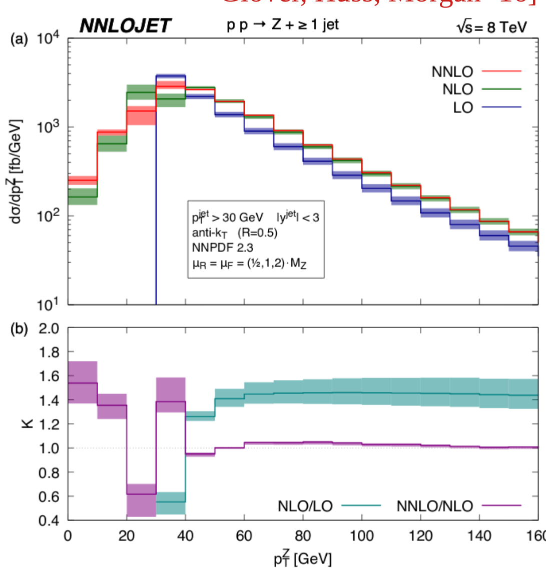
[Behring, Bizoń, Caola, Melnikov, Röntsch '20]



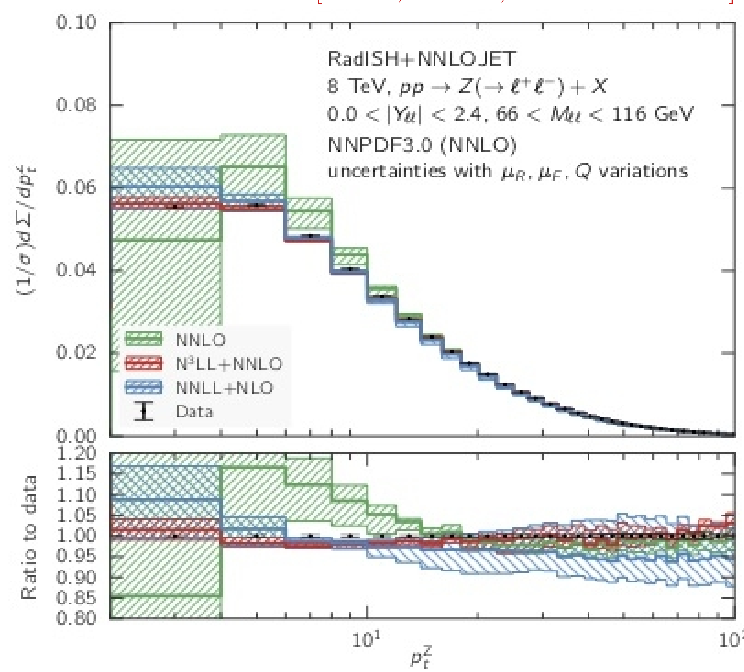
[Czakon, Mitov, Pellen, Poncelet '20]



[Gehrmann-De Ridder, Gehrmann, Glover, Huss, Morgan '16]

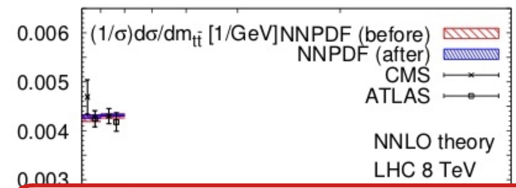


[Bizon, Glover, Gehrmann et al.]



Many processes known at NNLO QCD (2004 - today)

[Czakon, Hartland, Mitov, Nocera, Rojo]



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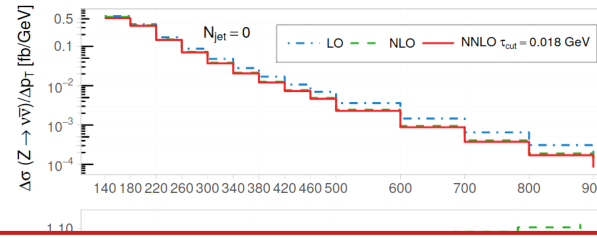
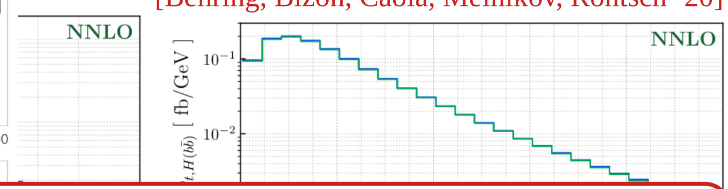


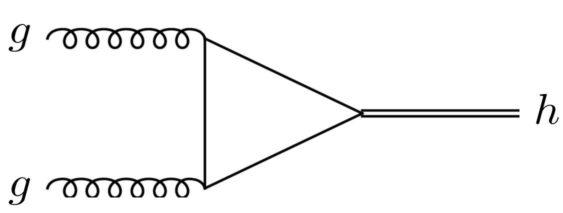
[Behring, Bizoń, Caola, Melnikov, Röntsch '20]

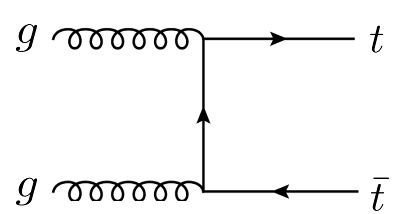


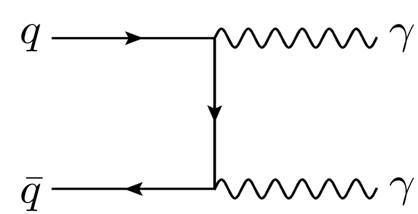
- Current methods are CPU intense

  ... but physics that can be extracted is inversely proportional to required CPU hours

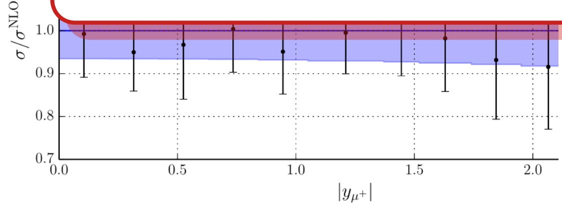
- Mainly $2 \to 1$ and $2 \to 2$ processes

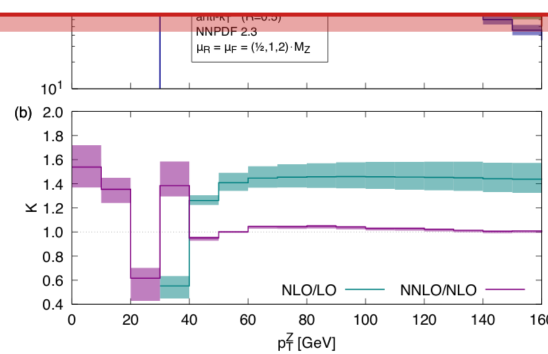


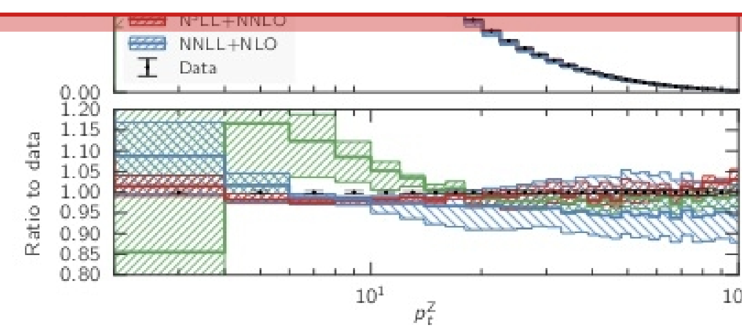




- $2 \to 3$ processes with current methods will be challenging







Higher orders in perturbative QCD

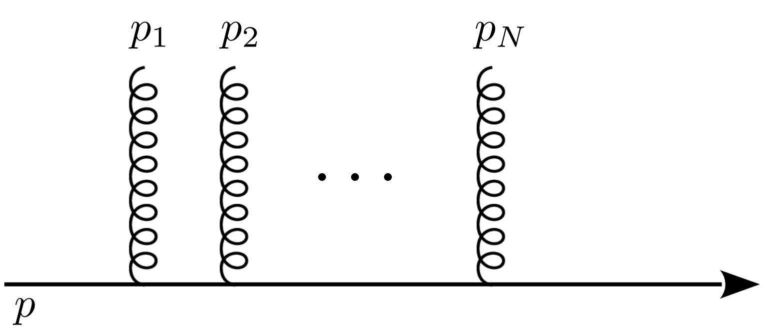
- Computation of higher orders in perturbative QCD non-trivial due to ...

  $\cdots$ loop integrals (multi-loop integrals are work-in-progress);

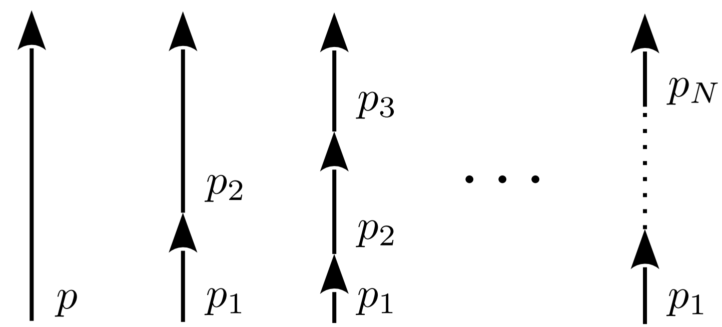
  $\cdots$ **infrared singularities (in the following).**

The KLN theorem [Kinoshita '62; Lee, Nauenberg '64]

- Non-degenerate perturbation theory with fixed number of external particles is not valid
- States are degenerate in energy: $X \to Y$, $X \to Y + (N$ gluons with 0 energy$)$ ...



$p_1 = p_2 = \cdots = p_N = 0$



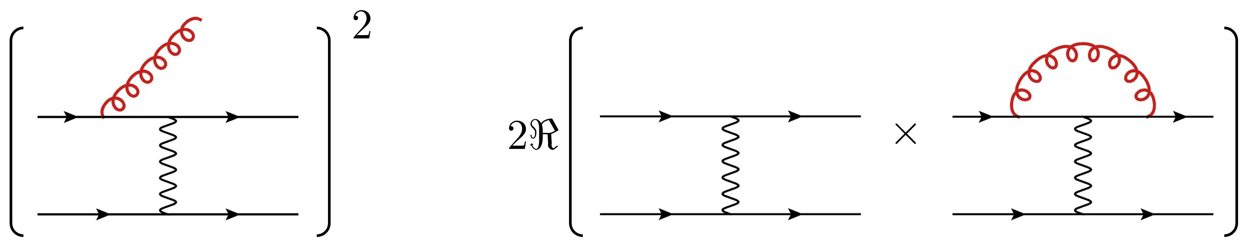
$p_1 \parallel p_2 \parallel \cdots \parallel p_N$; $p = p_1 + p_2 + \cdots + p_N$

- All singularities vanish if all degenerate states with arbitrary multiplicities in the initial and final states are properly combined.
- For collider processes: remove final state singularities by considering processes with different multiplicities and initial state collinear singularities by PDF redefinition.

IR finite differential cross section @NLO QCD

$$\mathrm{d}\hat{\sigma}_{\mathrm{nlo}} = \mathrm{d}\hat{\sigma}_{\mathrm{r}} + \mathrm{d}\hat{\sigma}_{\mathrm{v}} + \mathrm{d}\hat{\sigma}_{\mathrm{pdf}}$$

contains **explicit** infrared poles in $1/\epsilon$



real contribution

virtual correction

contains infrared **singularities** that become poles in $1/\epsilon$ **only** upon phase space integration

- In dimensional regularization ($d = 4 - 2\epsilon$) the explicit poles of 1-loop and 2-loop amplitudes are known **independent of the hard matrix element** [Catani '98; Becher, Neubert '09]

$$\mathcal{M}_{\text{1-loop}}(\{p\}) = \left[\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i \left(\frac{1}{\epsilon^2} + \frac{g_i}{\mathbf{T}_i^2} \frac{1}{\epsilon} \right) \sum_{j \neq i} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \left(\frac{\mu^2}{-s_{ij}} \right)^{\epsilon} \right] \mathcal{M}_{\text{tree}}(\{p\}) + \mathcal{M}^{\text{fin}}_{\text{1-loop}}(\{p\})$$

To get a physical answer we need to ...

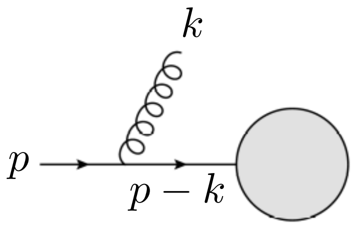
1) Regulate infrared singularities of the real emission contributions;
2) Extract infrared $1/\epsilon$ poles in d-dimensions explicitly **without integrating over the resolved phase** space to keep description fully differential;
3) Cancel $1/\epsilon$ poles against explicit poles in loop and collinear renormalization contributions;
4) Take physical $\epsilon \to 0$ limit.

Solved problem at NLO QCD (20 years ago)

- **FKS subtraction** [Frixione, Kunszt, Signer '96], Dipole subtraction [Catani, Seymour, '97], ...
- **Process-independent** description of $1/\epsilon$ poles that originate from real emission contributions without integrating over resolved phase-space
- Cancellation of $1/\epsilon$ infrared poles between real and virtual contributions demonstrated **in a general case**

Singularities of real emission contributions

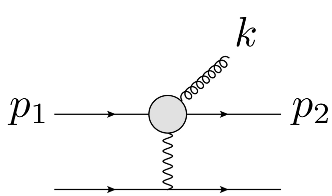
- Singularities of QCD amplitudes come in two varieties: soft ($E \to 0$) and collinear ($\vec{p}_i \parallel \vec{p}_j$)



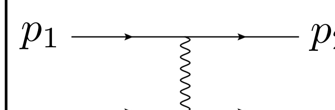
$$\sim \frac{1}{(p-k)^2} \sim \frac{1}{E_p \times \boxed{E_k} \times \boxed{(1-\vec{n}_p\cdot\vec{n}_k)}} \longrightarrow \infty \quad \begin{cases} \text{for} & E_k \to 0 \\ \text{for} & \vec{n}_p \parallel \vec{n}_k \end{cases}$$

Soft singularity   Collinear singularity

- The corresponding limits of amplitudes are **generic** and **independent** of a hard process
- For example, the soft $(E_k \to 0\,)$ limit of a single real emission DIS amplitude is

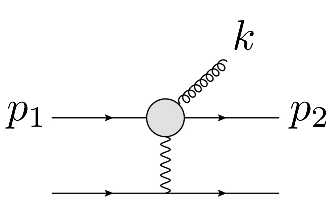


$$\left| \cdots \right|^2 \underset{E_k\to 0}{\approx} 2C_F\; g_{s,b}^2 \times \boxed{\frac{p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}} \times \left| \cdots \right|^2$$

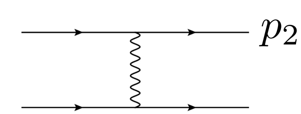


Eikonal function

... whereas the collinear $\vec{k} \parallel \vec{p}_1$ limit is



$$\left| \cdots \right|^2 \underset{k\parallel p_1}{\approx} -g_{s,b}^2 \times \frac{1}{p_1\cdot p_k} \boxed{P_{qq}\left(\frac{E_1}{E_1-E_k}\right)} \times \left| \left(\frac{E_1-E_k}{E_1}\right)\cdot p_1 \cdots \right|^2$$



Splitting function

How to regulate and extract singularities without integration?

- Soft and collinear singularities turn into $1/\epsilon$ poles upon phase space integration.

$$\int \frac{\mathrm{d}^{d-1}k}{2E}|M(\{p\},k)|^2 \sim \int \frac{\mathrm{d}E}{E^{1+\epsilon}}\ \frac{\mathrm{d}\theta}{\theta^{1+2\epsilon}} \times |M(\{p\})|^2 \sim \frac{1}{\epsilon^2}$$

- We would like to extract singularities without integration over resolved phase space. Currently two approaches used: **slicing** and **subtraction**.
- To illustrate the basic idea of **subtraction**, consider an integral

$$I = \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}}\mathrm{F}(x)$$

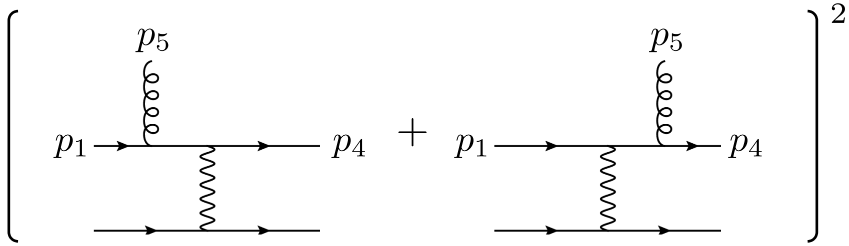
where $\mathrm{F}(0)$ is finite. We then write

$$I = \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}}\left[\mathrm{F}(x)-\mathrm{F}(0)\right] + \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}}\mathrm{F}(0) = \overbrace{\int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}}\left[\mathrm{F}(x)-\mathrm{F}(0)\right]}^{\text{regulated, finite in the } \varepsilon \to 0 \text{ limit}} - \underbrace{\frac{1}{\epsilon}\mathrm{F}(0)}_{\text{extracted } 1/\varepsilon \text{ pole}}$$

FKS subtraction @NLO [Frixione, Kunszt, Signer '96]

- Example: quark channel of deep inelastic scattering

$$|M^{\mathrm{nlo}}(\{p\},p_5)|^2 =$$



- Differential cross section

$$2s \cdot \mathrm{d}\sigma_{\mathrm{r}} = \int [\mathrm{d}g_5]\; F_{\mathrm{LM}}(1,4,5) \equiv \langle F_{\mathrm{LM}}(1,4,5)\rangle$$

with

$$F_{\mathrm{LM}}(1,4,5) = \mathcal{N}\int \mathrm{dLips}\;(2\pi)^d \delta^{(d)}(p_1+p_2-p_3-p_4-p_5) \times |M^{\mathrm{nlo}}(\{p\}),p_5|^2 \times \mathcal{O}(p_3,p_4,p_5)$$

$$[\mathrm{d}g_i] = \frac{\mathrm{d}^{d-1}p_i}{(2\pi)^{d-1}2E_i}\boxed{\theta(E_{\max}-E_i)}$$

Needs to be sufficiently large but otherwise arbitrary!

- The function $F_{\mathrm{LM}}(1,4,5)$ possesses three singularities in the gluon phase space: soft ($E_5 \to 0$) and two collinear ($p_5 \parallel p_1$, $p_5 \parallel p_4$)
- **Regulate soft and collinear singularities iteratively**

Subtracting singularities

- Introduce operator S_5 that takes the function $F_{\rm LM}(1,4,5)$ in the soft $E_5 \to 0$ limit

$$S_5 F_{\rm LM}(1,4,5) = S_5 \left[\mathcal{N} \int \mathrm{dLips} \boxed{(2\pi)^d \delta^{(d)}(p_1+p_2-p_3-p_4-p_5)} \boxed{|M^{\rm nlo}(\{p\}), p_5|^2} \boxed{\mathcal{O}(p_3,p_4,p_5)} \right]$$

$$\equiv \boxed{2C_F\ g_{s,b}^2\ \frac{p_1\cdot p_4}{(p_1\cdot p_5)(p_4\cdot p_5)}} \underbrace{\left[\mathcal{N} \int \mathrm{dLips} \boxed{(2\pi)^d \delta^{(d)}(p_1+p_2-p_3-p_4)} \boxed{|M^{\rm lo}(\{p\})|^2} \boxed{\mathcal{O}(p_3,p_4)} \right]}_{\hat{=}\ \text{LO differential cross section}}$$

$$= 2C_F\ g_{s,b}^2\ \frac{p_1\cdot p_4}{(p_1\cdot p_5)(p_4\cdot p_5)} \times F_{\rm LM}(1,4)$$

- S_5 reduces the function $F_{\rm LM}(1,4,5)$ to a function with **lower multiplicity** in the final state
- Soft singularity is regulated by introducing the partition of unity $I = (I - S_5) + S_5$

$$\langle F_{\rm LM}(1,4,5)\rangle = \underbrace{\boxed{\langle (I-S_5)F_{\rm LM}(1,4,5)\rangle}}_{\text{soft singularity regulated}} + \underbrace{\boxed{\langle S_5 F_{\rm LM}(1,4,5)\rangle}}_{\text{contains singularities}}$$

- **Regulated term:** free of soft singularity but still contains collinear singularities
- **Subtraction term:** Contains infrared singularities

Analytic integration of the subtraction terms

- Integration over gluon momentum p_5 **factorizes** and can be performed **analytically**

$$\langle S_5 F_{\mathrm{LM}}(1,4,5)\rangle = \underbrace{\int \frac{\mathrm{d}^{d-1}p_5}{(2\pi)^{d-1}2E_5}\,\theta(E_{\max}-E_5)\left[2C_F\; g_{s,b}^2\,\frac{p_1\cdot p_4}{(p_1\cdot p_5)(p_4\cdot p_5)}\right]}_{} \times \boxed{F_{\mathrm{LM}}(1,4)}$$

$$= \frac{2C_F}{\epsilon^2}\left[\frac{\alpha_s(\mu)}{2\pi}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right]\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\boxed{\left(\frac{4E_{\max}}{\mu^2}\right)^{-2\epsilon}}\frac{\rho_{14}}{2}\;{}_2F_1(1,1;1-\epsilon,1-\rho_{14}/2)$$

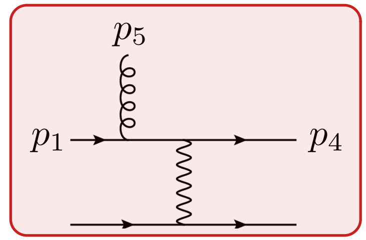
- Upper energy bound $E_{\max}$ needed to avoid artificial divergences at large E_5
- Explicit dependence on $E_{\max}$ cancels with implicit dependence in $\langle (I-S_5)F_{\mathrm{LM}}(1,4,5)\rangle$

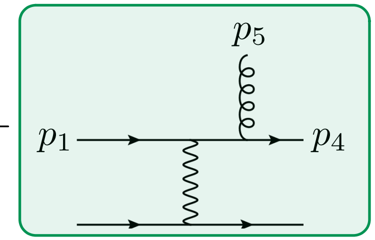
- **Soft and collinear** infrared $1/\epsilon$ poles are extracted explicitly
- Poles multiply the **LO differential cross** section $\langle F_{\mathrm{LM}}(1,4)\rangle$

  $\rightarrow$ contain the **same matrix element and kinematics** as in case of **virtual corrections**

Collinear singularities

$$|M^{\text{tree}}(\{p\}, p_5)|^2 = \left[\;\; \text{(diagram 1)} \;\; + \;\; \text{(diagram 2)} \;\; \right]^2$$





- Two collinear singularities present: $(p_5 \parallel p_1)$ and $(p_5 \parallel p_4)$
- The different configurations are separated by introducing partition functions in the phase space

$$1 = \boxed{w^{51}} + \boxed{w^{54}} \qquad \text{with} \qquad \lim_{5\|i} w^{5j} \sim \delta_{ij}$$

- One possible choice

$$w^{51} = \frac{\rho_{45}}{\rho_{15} + \rho_{45}}, \; w^{54} = \frac{\rho_{15}}{\rho_{15} + \rho_{45}} \qquad \text{with} \quad \rho_{ij} = 1 - \cos\theta_{ij}$$

Note that
$p_i \cdot p_j = E_i E_j \rho_{ij}$

- Then

$$\boxed{w^{51} F_{\text{LM}}(1,4,5)} \quad \begin{cases} \text{singular when } (5||1) \\ \text{finite when } (5||4) \end{cases} \qquad \boxed{w^{54} F_{\text{LM}}(1,4,5)} \quad \begin{cases} \text{singular when } (5||4) \\ \text{finite when } (5||1) \end{cases}$$

Regulating collinear singularities

- Introducing partition functions in the phase space

$$\langle (I-S_5)F_{\mathrm{LM}}(1,4,5)\rangle = \boxed{\langle (I-S_5)w^{51}F_{\mathrm{LM}}(1,4,5)\rangle} + \langle (I-S_5)w^{54}F_{\mathrm{LM}}(1,4,5)\rangle$$

- Regulate collinear singularities iterativeely, e.g. partition w^{51}

$$\boxed{\langle (I-S_5)w^{51}F_{\mathrm{LM}}(1,4,5)\rangle} = \underbrace{\langle (I-C_{51})(I-S_5)w^{51}F_{\mathrm{LM}}(1,4,5)\rangle}_{\text{fully regulated} \,\to\, \text{finite}} + \underbrace{\langle C_{51}(I-S_5)w^{51}F_{\mathrm{LM}}(1,4,5)\rangle}_{\text{subtraction term}}$$

- Integrate subtraction term analytically over unresolved phase space

$$\Big\langle C_{51}\left[I-S_5\right]w^{51}F_{\mathrm{LM}}(1,4,5)\Big\rangle = -\frac{1}{\epsilon}\left[\frac{\alpha_s(\mu)}{2\pi}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right]\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left(\frac{4E_1^2}{\mu^2}\right)^{-\epsilon}$$

$$\boxed{\times\int_0^1 \mathrm{d}z\,\left(2C_F\left[\frac{(1-z)^{-2\epsilon}}{1-z}\right]_+ - C_F(1-z)^{-2\epsilon}[(1+z)+\epsilon(1z)]\right)\left\langle\frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\right\rangle}$$

"boosted" LO cross section

$$-\,2C_F\,\frac{1}{\epsilon}\left[\frac{\alpha_s(\mu)}{2\pi}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right]\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\left[\frac{(4^2/\mu^2)^{-\epsilon}-(4E_1^2/\mu^2)^{-\epsilon}}{2\epsilon}\right]\big\langle F_{\mathrm{LM}}(1,4)\big\rangle$$

- Since the soft singularity is already regulated, the subtraction term is of order $\mathcal{O}(\epsilon^{-1})$
- The second partition w^{54} is treated similarly.

Finite result in 4-dimensions

- Combining real, virtual and collinear renormalization contributions

$$
\begin{aligned}
2s\cdot \mathrm{d}\sigma_{\mathrm{nlo}} = & \boxed{\sum_{i=1,4}\langle (I-C_{5i})(I-S_5)w^{5i}\ F_{\mathrm{LM}}(1,4,5)\rangle} + \boxed{\langle F_{\mathrm{LV}}^{\mathrm{fin}}(1,4)\rangle} \\
& \boxed{\begin{aligned} &+\frac{\alpha_s(\mu)}{2\pi}\int_0^1 \left\{\mathcal{P}'_{qq}(z)+\ln\left(\frac{4E_1^2}{\mu^2}\right)\hat{P}^{(0)}_{qq}(z)\right\}\left\langle \frac{F_{\mathrm{LM}}\left(z\cdot 1,4\right)}{z}\right\rangle \\ &+\frac{\alpha_s(\mu)}{2\pi}\left\langle \left\{2C_F S_{14}^{E_{\max}}+\gamma'_q\right\} F_{\mathrm{LM}}(1,4)\right\rangle \end{aligned}}
\end{aligned}
$$

Structure of the result:

- Subtracted NLO matrix element
- Process dependent finite part of the 1-loop amplitude
- Finite parts of the d-dimensional subtraction terms that multiply LO matrix elements

- This function can be used to calculate **arbitrary infra-red safe observables** numerically in **4-dimensions.**
- Note that the cancellation of divergences has been achieved without specifying any of the matrix elements of the hard process.

Finite result in 4-dimensions

$$
\begin{aligned}
2s \cdot \mathrm{d}\sigma_{\mathrm{nlo}} = & \boxed{\sum_{i=1,4} \langle (I - C_{5i})(I - S_5) w^{5i} \; F_{\mathrm{LM}}(1,4,5) \rangle} + \langle F_{\mathrm{LV}}^{\mathrm{fin}}(1,4) \rangle \\
& \boxed{\begin{aligned} & + \frac{\alpha_s(\mu)}{2\pi} \int_0^1 \left\{ \mathcal{P}'_{qq}(z) + \ln\left(\frac{4E_1^2}{\mu^2}\right) \hat{P}_{qq}^{(0)}(z) \right\} \left\langle \frac{F_{\mathrm{LM}}(z \cdot 1, 4)}{z} \right\rangle \\
& + \frac{\alpha_s(\mu)}{2\pi} \left\langle \left\{ 2C_F S_{14}^{E_{\max}} + \gamma'_q \right\} F_{\mathrm{LM}}(1,4) \right\rangle \end{aligned}}
\end{aligned}
$$

Subtractions @NLO QCD are ...

... physically transparent "physical" singularities and clear mechanism of cancellation

... local subtracted matrix elements are finite at any point in the phase-space

... analytic analytic formulas for integrated subtraction terms

Finite result in 4-dimensions

$$
\begin{aligned}
2s\cdot \mathrm{d}\sigma_{\mathrm{nlo}} = &\sum_{i=1,4} \langle (I-C_{5i})(I-S_5)w^{5i}\, \boxed{F_{\mathrm{LM}}(1,4,5)\rangle} + \langle F_{\mathrm{LV}}^{\mathrm{fin}}(1,4)\rangle \\
&+ \frac{\alpha_s(\mu)}{2\pi}\int_0^1 \left\{\mathcal{P}'_{qq}(z) + \ln\left(\frac{4E_1^2}{\mu^2}\right)\hat{P}_{qq}^{(0)}(z)\right\} \boxed{\left\langle \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\right\rangle} \\
&+ \frac{\alpha_s(\mu)}{2\pi}\left\langle \left\{2C_F S_{14}^{E_{\max}} + \gamma'_q\right\} \boxed{F_{\mathrm{LM}}(1,4)\right\rangle}
\end{aligned}
$$

Subtractions @NLO QCD are ...

... modular — subtractions for complex processes are built from subtraction terms established in analyses of simpler processes (soft singularities are sensitive to pairs of emittors; collinear singularities factorize on external lines)

... efficient — efficient numerical evaluation (as result of local and analytic)

Finite result in 4-dimensions

$$2s\cdot \mathrm{d}\sigma_{\mathrm{nlo}} = \sum_{i=1,4}\langle (I-C_{5i})(I-S_5)w^{5i}\;F_{\mathrm{LM}}(1,4,5)\rangle + \langle F_{\mathrm{LV}}^{\mathrm{fin}}(1,4)\rangle$$
$$+\frac{\alpha_s(\mu)}{2\pi}\int_0^1 \left\{\mathcal{P}'_{qq}(z)+\ln\left(\frac{4E_1^2}{\mu^2}\right)\hat{P}^{(0)}_{qq}(z)\right\}\left\langle \frac{F_{\mathrm{LM}}\,(z\cdot 1,4)}{z}\right\rangle$$
$$+\frac{\alpha_s(\mu)}{2\pi}\left\langle\left\{2C_F S_{14}^{E_{\max}}+\gamma'_q\right\}F_{\mathrm{LM}}(1,4)\right\rangle$$

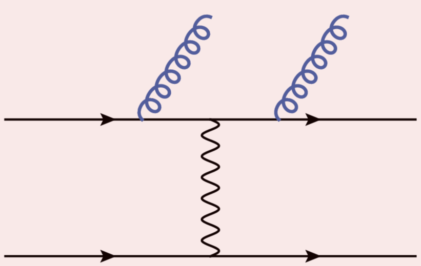
Can we do something similar @NNLO?

- Many subtraction schemes at NNLO [Gehrmann-de Ridder, Gehrmann, Glover ‘05; Czakon ‘10, ‘11; Cacciari et al ‘15; Somogyi, Trócsányi, Del Duca ‘05; Caola, Melnikov, Röntsch ‘17; Herzog ‘18; Magnea et al ‘18; …]
- None of the existing subtraction schemes satisfies all of the above criteria (up to now this was not a problem for phenomenology)
- For more complex processes, better subtraction schemes may become a necessity

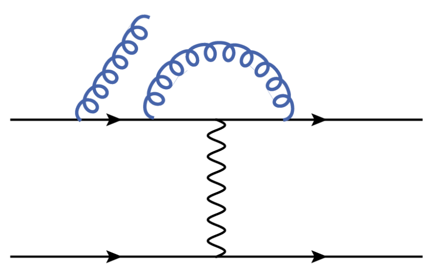
Partonic cross section @NNLO QCD

- Extension of FKS subtraction to NNLO proved to be non-trivial
- Contributions to the partonic cross section

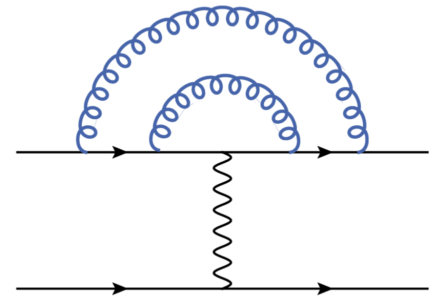
$$\mathrm{d}\hat{\sigma}_{\mathrm{nnlo}} = \mathrm{d}\hat{\sigma}_{\mathrm{rr}} + \mathrm{d}\hat{\sigma}_{\mathrm{rv}} + \mathrm{d}\hat{\sigma}_{\mathrm{vv}} + \mathrm{d}\hat{\sigma}_{\mathrm{pdf}}$$



double real emission contributions



real-virtual contributions

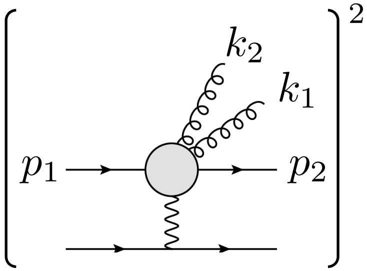


double virtual corrections

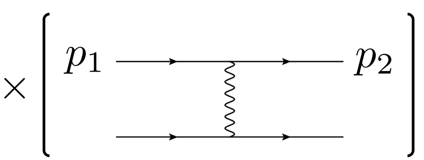
- **In the following: double real emission of two gluons**

Factorization formulas @NNLO QCD

- Two new genuine NNLO singularities: double soft and triple collinear
- Factorization formulas for double soft singularities are known [Catani, Grazzini '99; ...]



$$\underset{E_{k_1}\sim E_{k_2}\to 0}{\approx} \; g_{s,b}^4 \times \boxed{\text{Eikonal}(\{p_1,p_2\},k_1,k_2)} \times$$



double soft eikonal function

... the same holds true for triple collinear singularities [Catani, Grazzini '99; ...]

$$|M(\{p\},k_1,k_2)|^2$$

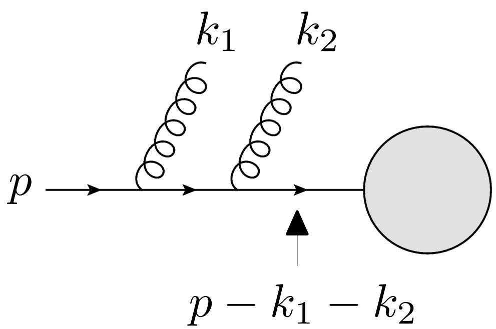
$$\underset{k_1||k_2||p_1}{\approx} \frac{1}{(p_1-k_1-k_2)^2} \times \boxed{P(s_{1k_1},s_{1k_2},s_{k_1k_2})} \times \left| M\left(\left\{\frac{E_1-E_{k_1}-E_{k_2}}{E_1}\cdot p_1,\ldots\right\}\right)\right|^2$$

triple collinear splitting function

- They are structurally similar to the NLO case

Entangled soft and collinear limits

- Many entangled limits: soft/soft, collinear/collinear and soft/collinear



$$\sim \frac{1}{(p-k_1-k_2)^2} \sim \frac{1}{2p\cdot k_1 + 2p\cdot k_2 - 2k_1\cdot k_2} \underset{\substack{k_1 \parallel p \\ k_2 \to 0}}{\longrightarrow} \infty$$

- For a **given amplitude** it can be checked explicitly that entangled **soft/collinear** singularities do not occur

- This observation is general thanks to a phenomenon known as **colour coherence** (a soft gluon does not resolve details of a collinear splitting) [Caola, Melnikov, Röntsch, '17]

As a result ...

... known soft and collinear limits of amplitudes are sufficient to construct all relevant subtraction terms;

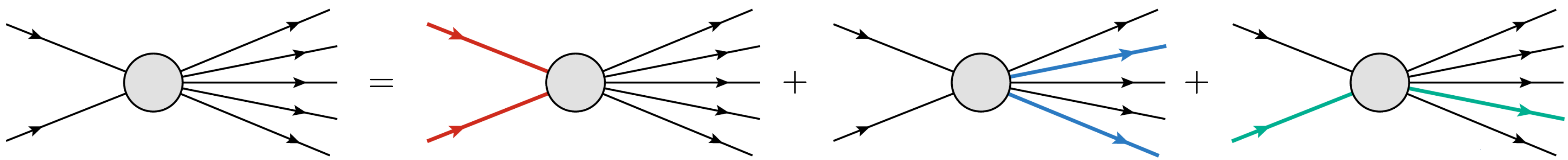
... soft and collinear limits can be treated independently.

→ Nested soft-collinear subtraction scheme

[Caola, Melnikov, Röntsch, '17]

Building blocks for the description of arbitrary LHC processes

- Most complex singular contributions (both soft and collinear) only depend on the properties of two external partons
- Separation of complex $pp \to N$ processes into simpler building blocks

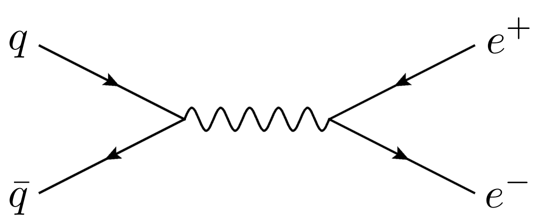


- Building blocks can be obtained from studying simple processes

... and checked extensively agains already existing results

Drell-Yan process

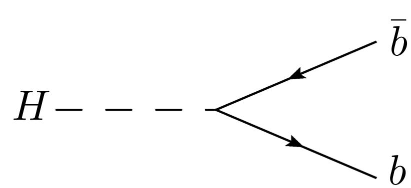
both momenta are **initial states**



[Caola, Melnikov, Röntsch '19]

$H \to b\bar{b}$ **decay**

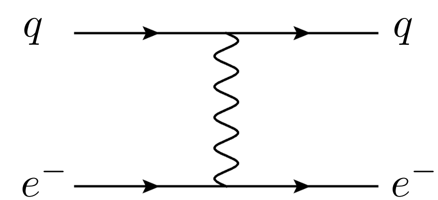
both momenta are **finale states**



[Caola, Melnikov, Röntsch '19]

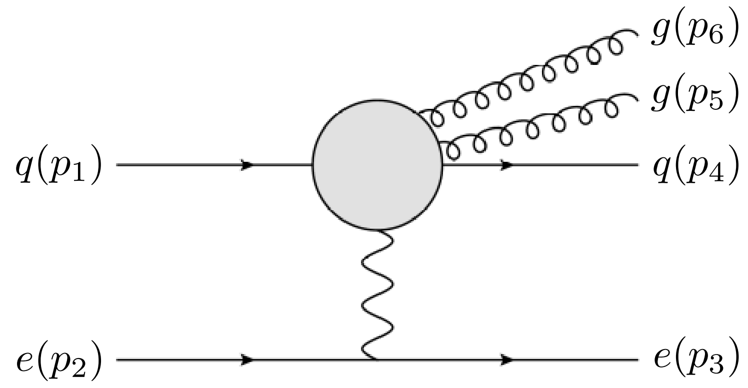
Deep inelastic scattering

one momenta is an **initial** and one a **finale state**



[KA, Caola, Melnikov, Röntsch '19]

Deep inelastic scattering @NNLO QCD



- We write the differential cross section as

Energy ordering

$$2s \cdot \mathrm{d}\sigma_{\mathrm{rr}} = \int [\mathrm{d}g_5][\mathrm{d}g_6] \boxed{\theta(E_5 - E_6)}\, F_{\mathrm{LM}}(1,4,5,6) \equiv \langle F_{\mathrm{LM}}(1,4,5,6)\rangle$$

with

$$F_{\mathrm{LM}}(1,4,5,6) = \mathcal{N} \int \mathrm{dLips}\ (2\pi)^d \delta^{(d)}(p_1 + p_2 - p_3 - p_4 - p_5 - p_6)$$
$$\times\, |M^{\mathrm{tree}}(\{p\}), p_5, p_6|^2 \times \mathcal{O}(p_3, p_4, p_5, p_6)$$

$$[\mathrm{d}g_i] = \frac{\mathrm{d}^{d-1}p_i}{(2\pi)^{d-1} 2E_i}\ \theta(E_{\max} - E_i)$$

- The integral diverges and needs to be regulated. Due to the absence of entangled soft and collinear singularities all singularities can be subtracted **iteratively**

Soft singularities

- We begin with the double-soft singularity. Introduce operator $\mathbb{S}$ that extracts the leading double soft singularity ($E_5 \sim E_6 \to 0$) and insert unity decomposed as $I = (I - \mathbb{S}) + \mathbb{S}$ into the phase space

Double-soft singularity regularized but still contains single soft and collinear singularities.

$$\langle F_{\mathrm{LM}}(1,4,5,6)\rangle = \boxed{\langle (I-\mathbb{S})F_{\mathrm{LM}}(1,4,5,6)\rangle} + \boxed{\langle \mathbb{S}F_{\mathrm{LM}}(1,4,5,6)\rangle}$$

- Soft gluons decouple from the matrix element and the observable. Hence we can integrate the subtraction term analytically over the phase space of gluons 5 and 6 [Caola, Delto, Frellesvig, Melnikov '18]

- Thanks to **energy ordering** ($E_6 < E_5$) only one single soft singularity for $E_6 \to 0$ needs to be regulated

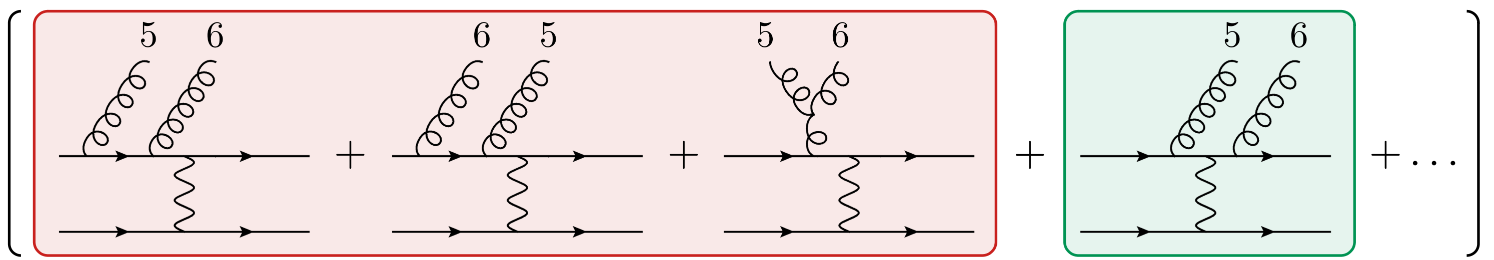
All soft singularities regularized but still contains collinear singularities

$$\langle (I-\mathbb{S})F_{\mathrm{LM}}(1,4,5,6)\rangle = \boxed{\langle (I-S_6)(I-\mathbb{S})F_{\mathrm{LM}}(1,4,5,6)\rangle} + \boxed{\langle S_6(I-\mathbb{S})F_{\mathrm{LM}}(1,4,5,6)\rangle}$$

Since gluon 6 decouples, this term reduces to NLO corrections to DIS

Collinear singularities

$$\left|M^{\mathrm{nnlo}}(\{p\},p_5,p_6)\right|^2 =$$



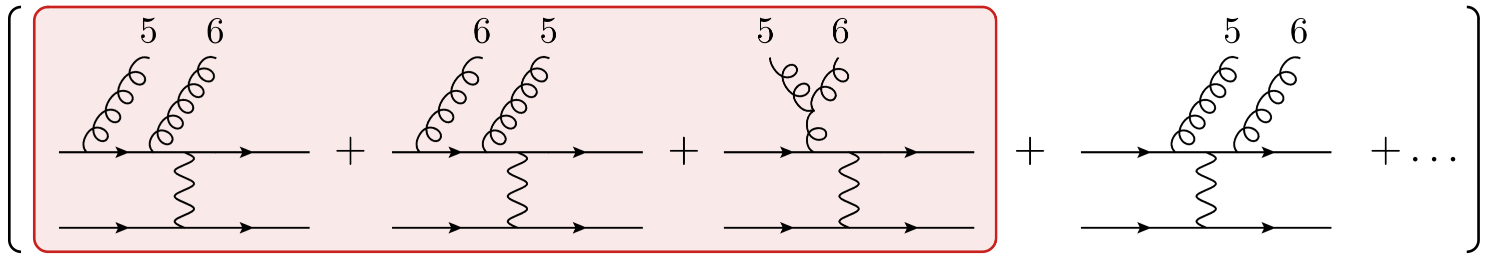
- In the collinear limits, many different singular configurations exist, but collinear singularities factorize on external legs, therefore either **three partons** become collinear or **two pairs of partons** become collinear at once.
- To control which partons these are, the different configurations are separated by **introducing partition functions** (similarly to NLO)

$$1 = \boxed{w^{51,61}} + w^{54,64} + \boxed{w^{51,64}} + w^{54,61}$$

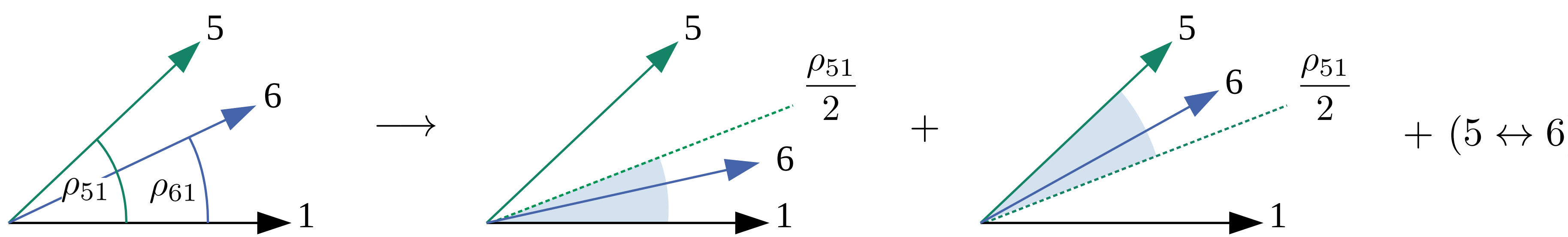
- Singularities in **double collinear sectors** are separated.
- Different collinear singularities in **triple collinear partitions** are isolated in the angular phase space.
- We separate them by **splitting the phase space** into different sectors.

Splitting of the angular phase space

$$\left|M^{\mathrm{nnlo}}(\{p\}, p_5, p_6)\right|^2 =$$



- As example consider partition $\boxed{w^{51,61}}$: it is singular when (5||1), (6||1) and (5||6)



double-collinear (6||1)
triple-collinear (6||5||1)

double-collinear (6||5)
triple-collinear (6||5||1)

- In practice this is done by introducing the unity

$$1 = \theta\left(\rho_{61} < \frac{\rho_{51}}{2}\right) + \theta\left(\frac{\rho_{51}}{2} < \rho_{61} < \rho_{51}\right) + \theta\left(\rho_{51} < \frac{\rho_{61}}{2}\right) + \theta\left(\frac{\rho_{61}}{2} < \rho_{51} < \rho_{61}\right)$$

- To integrate singularities analytically it is crucial the the phase space is parameterized in such a way that all singularities are made explicit [Czakon]

Fully regulated double-real contribution

$$
\begin{aligned}
F_{\mathrm{LM}}(1,4,5,6) = &\ \left\langle \mathbb{S} F_{\mathrm{LM}}(1,4,5,6)\right\rangle + \left\langle [I-\mathbb{S}] S_6 F_{\mathrm{LM}}(1,4,5,6)\right\rangle \\
&+ \sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} \Big\langle [I-\mathbb{S}]\,[I-S_6]\Big[C_{5i}w^{5i,6j} + C_{6i}w^{5j,6i} + \Big(\theta_i^{(a)}C_{5i} + \theta_i^{(c)}C_{6i}\Big)w^{5i,6i}\Big] \\
&\qquad\qquad \times [\mathrm{d}p_5][\mathrm{d}p_6]F_{\mathrm{LM}}(1,4,5,6)\Big\rangle \\
&+ \sum_{i\in\{1,4\}} \Big\langle [I-\mathbb{S}]\,[I-S_6]\Big[\theta_i^{(b)}C_{56} + \theta_i^{(d)}C_{56}\Big][\mathrm{d}p_5][\mathrm{d}p_6]w^{5i,6i}F_{\mathrm{LM}}(1,4,5,6)\Big\rangle \\
&- \sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} \Big\langle [I-\mathbb{S}]\,[I-S_6]C_{5i}C_{6j}[\mathrm{d}p_5][\mathrm{d}p_6]w^{5i,6j}F_{\mathrm{LM}}(1,4,5,6)\Big\rangle \\
&+ \sum_{i\in\{1,4\}} \Big\langle [I-\mathbb{S}]\,[I-S_6]\Big[\theta_i^{(a)}\mathbb{C}_i[I-C_{5i}] + \theta_i^{(b)}\mathbb{C}_i[I-C_{56}] + \theta_i^{(c)}\mathbb{C}_i[I-C_{6i}] \\
&\qquad\qquad + \theta_i^{(d)}\mathbb{C}_i[I-C_{56}]\Big][\mathrm{d}p_5][\mathrm{d}p_6]w^{5i,6i}F_{\mathrm{LM}}(1,4,5,6)\Big\rangle
\end{aligned}
$$

subtraction terms

regulated matrix elements

$$
\begin{aligned}
&+ \sum_{\substack{i,j\in\{1,4\}\\ i\neq j}} \Big\langle [1-\mathbb{S}]\,[1-S_6]\,[1-C_{6j}]\,[1-C_{5i}]\ [\mathrm{d}p_5][\mathrm{d}p_6]w^{5i,6j}F_{\mathrm{LM}}(1,4,5,6)\Big\rangle \\
&+ \sum_{i\in\{1,4\}} \Big\langle [1-\mathbb{S}]\,[1-S_6]\,[1-\mathbb{C}_i]\Big(\theta^{(a)}[1-C_{6i}] + \theta^{(b)}[1-C_{56}] \\
&\qquad\qquad + \theta^{(c)}[1-C_{5i}] + \theta^{(d)}[1-C_{56}]\Big)\ [\mathrm{d}p_5][\mathrm{d}p_6]w^{5i,6i}F_{\mathrm{LM}}(1,4,5,6)\Big\rangle
\end{aligned}
$$

- It can be used to **compute arbitrary infra-red safe observables** in 4-dimensions numerically.
- Such formulas can be written straightforwardly for **arbitrary processes**.

Pole structure @NNLO

- Analytic integration of subtraction terms is possible
- Simplifications after recombining subtractions terms

$$\Big\langle [1-\mathbb{S}][1-S_6]\Big[C_{54}w^{54,61}+C_{64}w^{51,64}+\Big(\theta^{(a)}C_{64}+\theta^{(c)}C_{54}\Big)w^{54,64}\Big][\mathrm{d}g_5][\mathrm{d}g_6]F_{LM}(1,4,5,6)\Big\rangle$$

$$=\boxed{\frac{[\alpha_s]C_F}{\epsilon}\Big\langle \sum_{i=1,4}(I-S_5)(I-C_{5i})w^{5i}\Big[\Big(\frac{1}{\epsilon}+Z^{2,2}\Big)(2E_4)^{-2\epsilon}-\frac{1}{\epsilon}(2E_5)^{-2\epsilon}\Big]\Big[w^{51}_{\mathrm{dc}}+w^{54}_{\mathrm{tc}}\Big(\frac{\rho_{54}}{4}\Big)^{-\epsilon}\Big]F_{LM}(1,4,5)\Big\rangle}$$

$$\boxed{\begin{aligned}&+\frac{[\alpha_s]^2C_F^2}{\epsilon^3}\Big\langle\Big[\Big(\frac{1}{\epsilon}+Z^{2,2}\Big)(2E_4)^{-2\epsilon}(2E_{max})^{-2\epsilon}-\frac{1}{2\epsilon}(2E_{max})^{-4\epsilon}\Big]\\&\quad\times\Big[\langle\Delta_{51}\rangle_{S_5}-\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}-\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\Big]F_{LM}(1,4)\Big\rangle\\&+\frac{[\alpha_s]^2C_F^2}{\epsilon^2}\Big[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\Big]\Big[\frac{1}{\epsilon}+Z^{2,2}\Big]\Big[\frac{1}{\epsilon}+Z^{4,2}\Big]\Big\langle(2E_4)^{-4\epsilon}F_{LM}(1,4)\Big\rangle\\&-\frac{[\alpha_s]^2C_F^2}{\epsilon^3}\Big[\frac{1}{2\epsilon}+Z^{2,4}\Big]\Big\langle\Big[\langle\Delta_{51}\rangle_{S_5}+\Big(\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\Big)\Big](2E_4)^{-4\epsilon}F_{LM}(1,4)\Big\rangle\end{aligned}}$$

$$\boxed{\begin{aligned}&-\frac{[\alpha_s]^2C_F^2}{\epsilon^2}\Big[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\Big]\int\mathrm{d}z\,\Big\langle\Big[\Big(\frac{1}{\epsilon}+Z^{2,2}\Big)(2E_4)^{-2\epsilon}-\frac{1}{\epsilon}(2E_1)^{-2\epsilon}(1-z)^{-2\epsilon}\Big]\\&\quad\times(2E_1)^{-2\epsilon}(1-z)^{-2\epsilon}\bar{P}_{qq}(z)\frac{F_{LM}(z\cdot 1,4)}{z}\Big\rangle.\end{aligned}}$$

- The subtraction terms contains the regulated NLO differential cross section (finite remainders need to be computed numerically) $\to$ cancel against similar terms from real virtual contributions
- Regular and "boosted" LO differential cross section $\to$ cancel against double virtual (and collinear renormalization contributions)
- This poles are **universal** and valid for arbitrary processes

Conclusion

- HL-LHC requires high precision theoretical predictions for collider processes.
- Despite progress with developing IR subtraction schemes, the “perfect” subtraction scheme is yet to come.
- The presented nested soft-collinear scheme for NNLO descriptions includes many of the desired properties from FKS @NLO.
- Development status: Complete set of analytic building blocks (obtained from studies of colour singlet production, decay and a DIS process) that can be used as building blocks to design subtractions for arbitrary LHC processes.
- Next steps: Application to more complex processes; in the pipeline: Higgs production in vector boson fusion.

Construction of the nested soft-collinear subtraction scheme is based on …

… iterative extraction of soft and collinear singularities;

… partitioning of angular phase space into sectors to obtain well-defined sets of collinear limits;

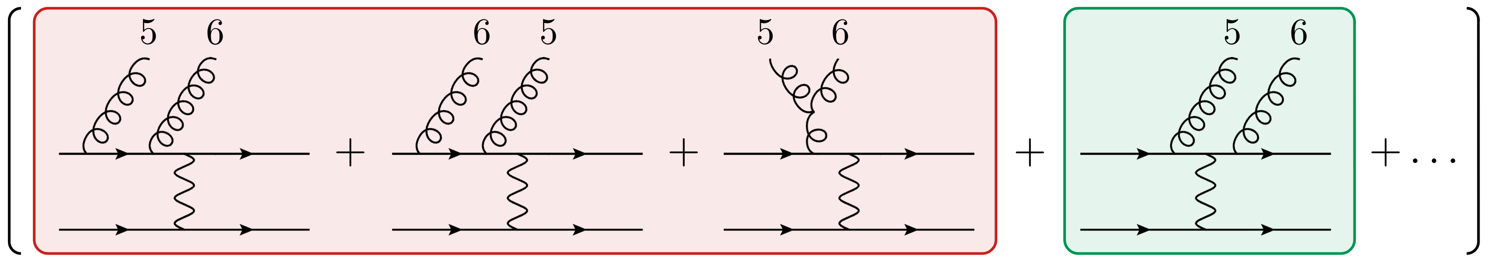
… (not shown) the possibility to parametrize phase space in a way that makes analytic integration of subtraction terms possible.

Backup

Notes

Partition functions

$$\left|M^{\rm nnlo}(\{p\},p_5,p_6)\right|^2 =$$



- The different configurations are separated by **introducing partition functions** in the phase space

$$1 = \boxed{w^{51,61}} + w^{54,64} + \boxed{w^{51,64}} + w^{54,61}$$

with

$$\lim_{5\|l} w^{5i,6j} \sim \delta_{li}\,, \quad \lim_{6\|l} w^{5i,6j} \sim \delta_{lj} \quad \text{and} \quad \lim_{5\|i}\lim_{6\|j} w^{5i,6j} = 1\,.$$

- One possible choice

$$w^{51,61} = \frac{\rho_{54}\rho_{64}}{d_5 d_6}\left(1 + \frac{\rho_{51}}{d_{5641}} + \frac{\rho_{61}}{d_{5614}}\right), \quad w^{51,64} = \frac{\rho_{54}\rho_{61}\rho_{56}}{d_5 d_6 d_{5614}}\,,$$
$$w^{54,64} = \frac{\rho_{51}\rho_{61}}{d_5 d_6}\left(1 + \frac{\rho_{64}}{d_{5641}} + \frac{\rho_{54}}{d_{5614}}\right), \quad w^{54,61} = \frac{\rho_{51}\rho_{64}\rho_{56}}{d_5 d_6 d_{5641}}\,,$$

where

$$d_{i=5,6} \equiv \rho_{1i} + \rho_{4i}\,, \quad d_{5614} \equiv \rho_{56} + \rho_{51} + \rho_{64}\,, \quad d_{5641} \equiv \rho_{56} + \rho_{54} + \rho_{61}\,.$$

Subtraction terms before NLO regulation

- Single collinear finial state emission

$$
\begin{aligned}
&\left\langle [I-\mathbb{S}][I-S_6]\Big[C_{54}w^{54,61}+C_{64}w^{51,64}+\left(\theta^{(a)}C_{64}+\theta^{(c)}C_{54}\right)w^{54,64}\Big][\mathrm{d}g_5][\mathrm{d}g_6]F_{LM}(1,4,5,6)\right\rangle \\
&= \frac{[\alpha_s]C_F}{\epsilon}\left\langle \left[\left(\frac{1}{\epsilon}+Z^{2,2}\right)(2E_4)^{-2\epsilon}-(2E_5)^{-2\epsilon}\right]\left(w_{\mathrm{DC}}^{51}+w_{\mathrm{TC}}^{54}\left(\frac{\rho_{54}}{4}\right)^{-\epsilon}\right)\boxed{F_{LM}(1,4,5)}\right\rangle \\
&\quad - \frac{[\alpha_s]^2C_F^2}{\epsilon^3}\left(\frac{1}{2\epsilon}+Z^{2,4}\right)\left\langle \langle\Delta_{51}\rangle_{S_5}(2E_4)^{-4\epsilon}F_{LM}(1,4)\right\rangle .
\end{aligned}
$$

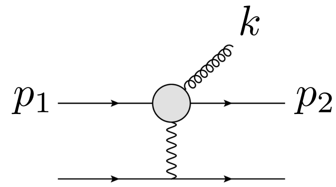
with

$$
\begin{aligned}
Z^{n,m} &= -\frac{2}{m\epsilon}-\int_0^1 \mathrm{d}z\; z^{-n\epsilon}(1-z)^{-m\epsilon}P_{qq}(z)=\frac{3}{2}+\frac{1}{12}\left[6+21m+15n-4n\pi^2\right]\epsilon+\mathcal{O}(\epsilon^2)\,, \\
\langle\Delta_{51}\rangle_{S_5} &= \left(-\frac{1}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]2^{-2\epsilon}\right)^{-1}\int \mathrm{d}\Omega_5^{(d-1)}\;\frac{\rho_{14}}{\rho_{15}\rho_{45}}\left[w_{\mathrm{DC}}^{51}+w_{\mathrm{TC}}^{54}\left(\frac{\rho_{54}}{4}\right)^{-\epsilon}\right]=\frac{3}{2}+\epsilon\left(\frac{\ln 2}{2}-2\ln\eta_{14}\right)+\mathcal{O}(\epsilon^2)\,, \\
w_{\mathrm{DC}}^{51} &= C_{64}w^{51,64}\,, \\
w_{\mathrm{TC}}^{54} &= C_{64}w^{54,64}\,.
\end{aligned}
$$

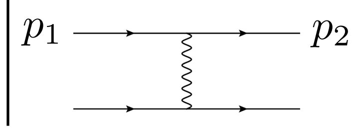
- The subtraction terms contains the **NLO differential cross-section** with **NLO singularities**

Single and double soft limit

- Single soft at NLO



$$\Big|\cdots\Big|^2 \underset{E_k\to 0}{\approx} 2C_F\; g_{s,b}^2 \times \underbrace{\boxed{\frac{p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}}}_{\text{Eikonal function}} \times \Big|\cdots\Big|^2$$



- Single soft at NNLO

$$S_6 F_{\rm LM}(1,4,5,6) = g_{s,b}^2 \times \frac{1}{E_6^2}\left[(2C_F - C_A)\frac{\rho_{14}}{\rho_{16}\rho_{46}} + C_A\left(\frac{\rho_{15}}{\rho_{16}\rho_{56}} + \frac{\rho_{45}}{\rho_{46}\rho_{56}}\right)\right] \times F_{\rm LM}(1,4,5)$$

- Double soft eikonal

$$\begin{aligned}
\text{Eikonal}(1,4,6,7) &= 4C_F^2 S_{14}(6)S_{14}(7) + C_A C_F\left[2S_{12}(6,7) - S_{11}(6,7) - S_{22}(6,7)\right], \\
S_{ij}(k) &= \frac{p_i\cdot p_j}{[p_i\cdot p_k][p_j\cdot p_k]}, \\
S_{ij}(k,l) &= S_{ij}^{\rm so}(k,l) - \frac{2[p_i\cdot p_j]}{[p_k\cdot p_l][p_i\cdot(p_k+p_l)][p_j\cdot(p_k+p_l)]} \\
&\quad + \frac{[p_i\cdot p_k][p_j\cdot p_l] + [p_i\cdot p_l][p_j\cdot p_k]}{[p_i\cdot(p_k+p_l)][p_j\cdot(p_k+p_l)]}\left(\frac{1-\epsilon}{[p_k\cdot p_l]^2} - \frac{1}{2}S_{ij}^{\rm so}(k,l)\right), \\
S_{ij}^{\rm so}(k,l) &= \frac{p_i\cdot p_j}{p_k\cdot p_l}\left(\frac{1}{[p_i\cdot p_k][p_j\cdot p_l]} + \frac{1}{[p_i\cdot p_l][p_j\cdot p_k]}\right) - \frac{[p_i\cdot p_j]^2}{[p_i\cdot p_k][p_j\cdot p_k][p_i\cdot p_l][p_j\cdot p_l]}.
\end{aligned}$$

Phase space parametrization [Czakon]

- We parametrize the directions of gluons 5 and 6 as

$$
\begin{aligned}
n_5^\mu &= t^\mu + \cos\theta_5 \epsilon_3^\mu + \sin\theta_5 b^\mu \,, \\
n_6^\mu &= t^\mu + \cos\theta_6 \epsilon_3^\mu + \sin\theta_6 (\cos\varphi_6 b^\mu + \sin\varphi_6 a^\mu) \,,
\end{aligned}
$$

and write the angular phase space as

$$
\mathrm{d}\Omega_5 \mathrm{d}\Omega_6 = \mathrm{d}\Omega_{56} = \frac{\mathrm{d}\Omega_b^{(d-2)} \mathrm{d}\Omega_a^{(d-3)}}{2^{6\epsilon}(2\pi)^{2d-2}} [\eta_5(1-\eta_5)]^{-\epsilon} [\eta_6(1-\eta_6)]^{-\epsilon} \frac{|\eta_5-\eta_6|^{1-2\epsilon}}{D^{1-2\epsilon}} \frac{\mathrm{d}\eta_5 \mathrm{d}\eta_6 \mathrm{d}\lambda}{[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}}
$$

where

$$
D = \eta_5\eta_6 - 2\eta_5\eta_6 + 2(2-1)\sqrt{\eta_5\eta_6(1-\eta_5)(1-\eta_6)}
$$

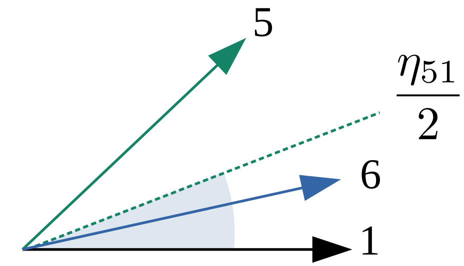
and

$$
\eta_{56} = \frac{|\eta_5-\eta_6|^2}{D} \qquad\qquad \sin^2\varphi_{56} = 4\lambda(1-\lambda)\frac{|\eta_5-\eta_6|^2}{D^2}
$$

- In the different sectors we perform the substitutions

$$
\begin{array}{lll}
\text{(a)} & \eta_5 = x_3 & \eta_6 = \frac{x_3 x_4}{2} \\
\text{(b)} & \eta_5 = x_3 & \eta_6 = x_3\left(1-\frac{x_4}{2}\right) \\
\text{(c)} & \eta_5 = \frac{x_3 x_4}{2} & \eta_6 = x_3 \\
\text{(d)} & \eta_5 = x_3\left(1-\frac{x_4}{2}\right) & \eta_6 = x_3
\end{array}
$$

Phase space parametrization [Czakon]



- For instance in sector (a) $\eta_5 = x_3 \quad \eta_6 = \frac{x_3 x_4}{2}$ we then obtain

$$\mathrm{d}\Omega_{56}^{(a)} = \left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\frac{\mathrm{d}\Omega_b^{(d-2)}}{\Omega^{d-2}}\frac{\mathrm{d}\Omega_a^{(d-3)}}{\Omega^{d-3}}\boxed{\frac{\mathrm{d}x_3}{x_3^{1+2\epsilon}}\frac{\mathrm{d}x_4}{x_4^{1+2\epsilon}}}\frac{\mathrm{d}\lambda}{\pi[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}}(256F_\epsilon)^{-\epsilon}4F_0x_3^2x_4$$

where

$$F_\epsilon = \frac{(1-x_3)(1-\frac{x_3x_4}{2})(1-\frac{x_4}{2})^2}{2N(x_3,x_4,\lambda)^2} \qquad F_0 = \frac{1-\frac{x_4}{2}}{2N(x_3,\frac{x_4}{2},\lambda)}$$

and

$$N(x_3,x_4,\lambda) = 1 + x_4(1-2x_3) - 2(1-2\lambda)\sqrt{x_4(1-x_3)(1-x_3x_4)}$$

- This parametrization accounts for the angular ordering of sector $\theta^{(a)} = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right)$ by construction.
- The double $(6||1)$ and triple $(5||6||1)$ collinear singularities in this sector are $x_4 = 0$ and $x_3 = 0$; they are factored out explicitly.
- The same happened for sectors $\theta^{(b)}$ to $\theta^{(d)}$.
- For a simpler analytic integration we define the single collinear limits to also act on the phase space.

Numerical validation of building blocks (e.g. DIS)

- Check analytic subtraction terms and regulated matrix elements against existing (inclusive) results [Kazakov et al. ‘90; Zijlstra, van Neerven ‘92; Moch, Vermaseren ‘00; …]
- Simplest possible set-up: Only photon exchange and one quark flavour
- Permille agreement only on **NNLO correction** $\sigma_{\text{NNLO}} = \sigma_{\text{LO}} + \Delta\sigma_{\text{NLO}} + \boxed{\Delta\sigma_{\text{NNLO}}}$

partonic channel	numerical result	analytic result
$\Delta\sigma_{\text{q,ns}}^{\text{NNLO}}$	$[33.1(2) - 2.18(1)\cdot n_f]\,\text{pb}$	$[33.1 - 2.17\cdot n_f]\,\text{pb}$
$\Delta\sigma_{\text{q,s}}^{\text{NNLO}}$	$9.19(2)\,\text{pb}$	$9.18\,\text{pb}$
$\Delta\sigma_{\text{g}}^{\text{NNLO}}$	$-142.4(4)\,\text{pb}$	$-142.7\,\text{pb}$

$$\sqrt{s} = 100\,\text{GeV},\;\; 10\,\text{GeV} < Q < 100\,\text{GeV},\;\; \mu_R = \mu_F = 100\,\text{GeV}$$

A glimps on efficiency

- Permill precision on full σ_{NNLO} cross section in ~ 50 CPU hours
- To compare with $\mathcal{O}(500)$ CPU hours with current methods even for simpler Drell-Yan process [Grazzini, Kallweit, Wiesemann, ‘18]

Different subtraction schemes and slicing methods

qt	slicing	[Catani, Grazzini]
Jettiness	slicing	[Boughezal et al., Gaunt et al.]
Antenna	subtraction	[Gehrmann-de Ridder, Gehrmann, Glover et al.]
Projection-to-Born	subtraction	[Cacciari et al.]
Colorful NNLO	subtraction	[Del Duca, Troscanyi et al.]
Stripper	subtraction	[Czakon]
Nested soft-collinear	subtraction	[Caola, Melnikov, Röntsch]
Local Analytic Sector	subtraction	[Magnea, Maina et al.]
Geometric	subtraction	[Herzog]

	Analytic	FS Colour	IS Colour	Local
Antenna	✓	✓	✓	✗
qT	✓	✗	✓	✗ **(slicing)**
Colourful	✓	✓	✗	✓
Stripper	✗	✓	✓	✓
N-jettiness	✓	✓	✓	✗ **(slicing)**

Updated and adapted from [Nigel Glover, Amplitudes '15]