Electroweak Axial Structure Functions and CKM Unitarity

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Axial Structure Functions

\[ F^{\nu p + \bar{\nu} p}_3 \sim \]
\[ F^{(0)}_3 = F^{\gamma Z}_{3,p} - F^{\gamma Z}_{3,n} \]
\[ F^{\nu p + \bar{\nu} p}_3 = F^{\gamma Z}_{3,p} + F^{\gamma Z}_{3,n} \]

- 4 SF’s, 2 equations express any 2 SF’s in terms of the other 2
- all are constrained by PDFs at high Q, but only \( F^{\nu p + \bar{\nu} p}_3 \) has experimental constraints at low Q
- \( F^{\gamma Z}_{3,p} \) has been previously modeled for the PV used for Qweak.
**CKM Unitarity**

- The Yukawa couplings between the quarks and Higgs fields is allowed to mix generations.
- One can then perform a basis change of the quark generations to diagonalize those terms.
- The cost is that we complicate the charged current interaction:

  \[
  -g \frac{1}{\sqrt{2}} (u_L, c_L, t_L) \gamma^\mu W^+_{\mu} V_{CKM} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} + \text{h.c.} \quad V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}
  \]

- The CKM matrix elements act like coupling constants between the W boson and two left-handed quarks of opposite isospin projection.
- Its unitarity means the sum of the squares of the top row elements is 1:

  \[
  |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1
  \]

- The largest term is extracted from superallowed Fermi beta decays:
  \[0^+ \rightarrow 0^+\]
- Extracted from \[K_{l3}\] decays:
  \[K \rightarrow \pi l \nu_l\]
- Extracted from \[B \rightarrow \pi l \nu\] decays
1 Loop Effects on Superallowed Fermi Transitions:

Master formula relating $V_{ud}$ to lifetime measurements:

$$|V_{ud}|^2 = \frac{2984.43s}{\mathcal{F}t(1+\Delta^V_R)}$$

Note: several universal RCs are common to both beta and muon decay and these cancel in

$$|V_{ud}|^2 \sim \frac{\text{beta decay}}{\text{muon decay}}$$

$\mathcal{F}t$ is a product of the statistical decay rate factor and decay lifetime and contains nuclear-dependent RCs

Nucleus-independent RCs:

$$\Delta^V_R = \frac{\alpha}{2\pi} \left[ 3\ln \frac{M_W}{M_p} - 4\ln c_W \right] + 2\Box^W_A$$

Sirlin 1978:

$$\Box^W_A = \frac{\alpha}{4\pi} \left[ \ln \frac{M_W}{M_A} + 2C_{Born} + A_g \right]$$

Marciano 2006:

$$\Box^W_A = \frac{\alpha}{8\pi} \int_0^\infty \frac{M_W^2}{Q^2 + M_W^2} F(Q^2) dQ^2$$

axial current interaction

form factor models hadron
If a function has an analytical structure in the complex plane, application of the Cauchy integral theorem using an appropriate contour can yield a Dispersion Relation.

**Dispersion Relations in QFT**

**Cutkosky Cutting Rule:**

\[
\text{Im}M(s) = -\frac{1}{2} \text{Disc}M(s)
\]

\[
M(s) = \begin{cases} 
\text{cut: intermediate state energy} \\
(m_e + m_p)^2 \approx m_p^2 
\end{cases}
\]

- cutting the diagram at the intermediate state, placing the intermediate state virtual particles on their mass shell
- sum over all possible phase space of these on shell particles

\[
\text{Im}M(s) = -\frac{1}{2} \text{Disc}M(s)
\]

\[
M(s) = \begin{cases} 
\text{cut: intermediate state energy} \\
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\end{cases}
\]

**Cauchy’s Integral theorem:**

\[
\Box(s_0) = \frac{1}{2\pi i} \oint \frac{\Box(s)}{s-s_0} \, ds
\]

If a function has an analytical structure in the complex plane, application of the Cauchy integral theorem using an appropriate contour can yield a Dispersion Relation.
Dispersion Relation for the $\gamma W$ Box

\[ M_{\Box}^{\gamma W} = W^+ \gamma q \approx 0 \]

\[ M_{\Box}^{\gamma W} |_{\text{fwd}} = \frac{-ig^2 e^2}{2M_W^2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2(1-k^2/M_W^2)} \bar{u}_e(l) \gamma \lambda \frac{k-l+m_e}{(l-k)^2-m_e^2} \gamma_\rho P_L u_\nu(l) T^{\lambda\rho}(k) \]

Numerator can be written as:
\[ L^{\gamma W}_{\mu\nu} H_{\gamma W}^{\mu\nu} \]

For on-shell states, hadronic tensor involves structure functions:
\[ H_{\gamma W}^{\mu\nu} = 4\pi \left[ \left( -g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2} \right) F_1^{\gamma W} + \frac{p^\mu p^\nu}{p \cdot k} F_2^{\gamma W} + \frac{i\epsilon^{\mu\nu\alpha\beta} p_\alpha k_\beta}{2p \cdot k} F_3^{\gamma W} \right] \]

only need axial piece

The axial part of the gW box correction is odd with respect to the neutrino’s incident energy $E$:
\[ \Rightarrow \text{Re} \Box_{\gamma W}^{(A)}(E) = \frac{2}{\pi} \int_{\nu,\pi}^{\infty} dE' \frac{E'}{E'^2-E^2} \text{Im} \Box_{\gamma W}^{(A)}(E') \]

\[ \Box_{\gamma W}^{(A)} = \frac{\alpha}{2\pi} \int_{W^2}^{\infty} dw^2 \int_0^{\infty} dQ^2 \frac{F_3^{\gamma W}(W^2, Q^2)}{1+Q^2/M_W^2} \frac{1}{M E_{\min}} \left( \frac{2}{\chi} - \frac{1}{4M E_{\min}} \right) \]

Depends on knowledge of the $F_3$ structure function at all $W^2$ and $Q^2$. 

Numerator can be written as:
\[ L^{\gamma W}_{\mu\nu} H_{\gamma W}^{\mu\nu} \]
• One needs different models for $F_3$ for different regions in the plane.
• The dispersion weight favours small $W^2$ and $Q^2$.
• The structure function should be continuous at the (moveable) boundaries, and the final box correction insensitive to their choice.

$$\square_A^{\gamma W} \sim \int \int dW^2 dQ^2 \omega(Q^2, W^2) F_3^{\gamma W}$$
Elastic Contribution:

\[ \Box_{\gamma W,el} = \frac{\alpha}{2\pi} \int_0^\infty dQ^2 \frac{[G^p_M(Q^2) + G^n_M(Q^2)]}{Q^2(1+Q^2/M_W^2)} \frac{g_A}{(1+Q^2/M_A^2)^2} \frac{1+2\sqrt{1+4M^2/Q^2}}{(1+\sqrt{1+4M^2/Q^2})^2} \]

\[ g_A = 1.2723 \quad M_A = 1.05(10) \text{ GeV} \]

\[ G^p_M, G^n_M \text{ taken from Ye, Arrington & Hill 2018 data} \]

\[ \Rightarrow \Box_{\gamma W,el}^{(A)} = (0.8967 \pm 0.0684) \frac{\alpha}{2\pi} = 1.04(6) \times 10^{-3} \]

Important note: the dispersion treatment doesn’t add anything new to the loop calculation of the elastic contribution

Box correction is sensitive to the axial mass parameter!
Resonance Contributions:

**Origin:** the first exchanged vector boson in the box diagrams can excite the neutron into an excited resonance state

the photon has an isovector (V) and an isoscalar (S) component
W boson only has isovector (V)

Box + XBOX \(\Rightarrow\) only (S) part of photon survives

\[ R = P_{11}(1440), D_{13}(1520), S_{11}(1535), \ldots \]

We can use the Lalakulich or MAID helicity amplitudes to find \(F_3\) from these resonances.
example: Lalakulich \(D_{13}\)

\[
F_3^{\gamma W}(D_{13}) = -\frac{4\nu}{3M} \left[ -C_4^S(Q^2 - \nu M) + C_5^S \nu M + C_3^S M_R (2M_R^2 - 2M M_R + Q^2 - \nu M) \right] C_5^A \Gamma_R(W, M_R)
\]

Using MAID:

<table>
<thead>
<tr>
<th>Resonance</th>
<th>(\Box_{A, \text{res}}^{\gamma W} \times 10^{-3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_{13})</td>
<td>0.054</td>
</tr>
<tr>
<td>(P_{11})</td>
<td>-0.009</td>
</tr>
<tr>
<td>(S_{11})</td>
<td>-0.002</td>
</tr>
<tr>
<td>total</td>
<td>0.04</td>
</tr>
</tbody>
</table>

\(C_i^{A,S}\) are form factors found from scattering data
DIS Contribution:

High $Q^2$ means the hadron looks like individual free quarks, so we use pQCD and factorization:

$$
\Box_{\gamma W}^{A(DIS)} = \frac{1}{2\pi} \int_{Q_0^2}^{\infty} dQ^2 \frac{\alpha_{EM}(Q^2)}{Q^2(1 + Q^2/M_W^2)} \int_0^{x_{\text{max}}} dx \ F_{3,DIS}^{(0)}(x, Q^2) \left( \frac{2r - 1}{r^2} \right)
$$

$$
r \equiv 1 + \sqrt{1 + 4M^2x^2/Q^2}
$$

Perturbatively include effects of the strong interaction at NLO

$$
F_{3,DIS}^{(0)}(x, Q^2) = \int_x^1 \frac{dz}{z} C_{3}^{(1)}(z)
$$

$$
\times \frac{1}{3} \left( u_v(x/z, Q^2) - d_v(x/z, Q^2) \right)
$$

- The effect of the NLO pQCD correction suppresses the LO prediction by $1 - \frac{\alpha_S}{\pi}$
- The running of $\alpha_{EM}(Q^2)$ enhances the box correction by $4\%$ from atomic limit

$$
\Box_{A,DIS}^{\gamma W} = 2.29(3) \times 10^{-3} \quad \langle Q^2 \rangle = 12 \text{ GeV}^2
$$
Regge Contribution:

At low $Q^2$ and high $W^2$, the strong interaction becomes nonperturbative

**Model 1 for $F_3$:** Seng, Gorchtein, Ramsey-Musolf, Phys. Rev. Lett. 121, 241804 (2018)

\[
F_{3,\text{Reg}}^{(0)}(W^2, Q^2) = \frac{f(1 + gQ^2)}{(1 + Q^2/m_{\rho}^2)(1 + Q^2/m_{a_1}^2)} f_{\text{th}}(W) \left(\frac{\nu}{\nu_0}\right)^{\alpha_0}
\]

\[
f_{\text{th}}(W) = \Theta(W^2 - W_{\text{th}}^2)(1 - e^{(W_{\text{th}}^2 - W^2)/\Lambda_{\text{th}}^2})
\]

$W_{\text{th}} = M + m_\pi$

The true $Q^2$-dependence of this structure function is not well-determined by theory.

VMD Processes:


**Idea:** match this function to the well-known value in the DIS region around $Q^2 = 2$ GeV$^2$ AND constrain it from available data on $F_{3}^{\nu p + \bar{\nu} p}$

\[
\frac{F_{3}^{\nu p + \bar{\nu} p}}{F_{3}^{(0)}} \approx 9
\]

(more on this later)
Some data exists on the 1st Nachtmann moment of $F_{3}^{\nu p+\bar{\nu}p}$

\[ M_{3}^{\nu p+\bar{\nu}p}(1, Q^2) \bigg|_{\text{low } Q^2} = \frac{2}{3} \int_{0}^{1} dx \frac{\xi}{x^2} (2x - \frac{\xi}{2}) \left[ F_{3, \text{el}}^{\nu p+\bar{\nu}p} + F_{3, \text{res}}^{\nu p+\bar{\nu}p} + 9F_{3, \text{Reg}}^{(0)} \right] \]

\[ \xi = \frac{2x}{1 + \sqrt{1+4M^2x^2/Q^2}} \]


**Nonlinear fit:**

\[
\begin{align*}
    f & = 0.80(3) \\
    g & = 0.63(10) \text{ GeV}^{-2}
\end{align*}
\]

**Box: **

\[
\gamma_{A, \text{Reg}}^W = 0.37(10) \times 10^{-3}
\]

(Model 1 result)

$$F_3^{(0)} = A_p-nx^{-\alpha_R}(1-x)^c \left( \frac{Q^2}{Q^2 + \Lambda_R^2} \right)^{\alpha_R}$$

Similarly, we can model the purely axial $F_3^{\nu p+\bar{\nu}p}$ in the same way. P.C. Bosetti et al., Nucl. Phys. B 203, 362 (1982):

Fit parameters:

$$A_{p+n} = 2.16(3)$$
$$c = 0.61(1)$$
$$\Lambda_R = 0.49(7)$$

also include high-weight data points from DIS region at $Q^2 = 2$ GeV$^2$
Is the ratio really 9?

No! But the observable correction is proportional to \( \int_0^1 dx F_3(x, Q^2) \) so taking a ratio of 9 is still meaningful.

Two choices:

\[
\begin{align*}
A_{p-n} &= \frac{A_{p+n}}{I(x)} \quad \Rightarrow \quad \Box \gamma_W^{A,\text{Reg}} = 0.34 \times 10^{-3} \quad \text{(PDF-dependent)} \\
A_{p-n} &= \frac{A_{p+n}}{9} \quad \Rightarrow \quad \Box \gamma_W^{A,\text{Reg}} = 0.38(4)_{\text{sys}}(3)_{\text{stat}} \times 10^{-3}
\end{align*}
\]

The Regge contribution is poorly constrained, and multiple models lead to a similar central value, e.g. from gZ axial contribution to Qweak:

\[
F_{3,\text{Reg}}^{(0)} = \frac{1+\Lambda^2/Q_0^2}{1+\Lambda^2/Q^2} F_{3,\text{DIS}}^{(0)}(x, Q_0^2), \quad \Lambda \approx 0.8 \text{ GeV}
\]

Background Contribution:

The background is a smoothly decreasing curve which goes to 0 at the pion threshold and matches the DIS and Regge regions at $W^2 = 4$ GeV$^2$.

By using PDF info and valence quark arguments one can show the proportionality statement:

$$F_{3,\text{bgd}}(0) \sim F_{1,\text{bgd}}^{\gamma\gamma}$$

at fixed $Q^2$.

Rescaled Bosted-Christy parametrization:

$$F_{3,\text{bgd}}(0) = \eta_S(Q^2) \frac{W^2 - M^2}{8\pi^2\alpha} \left[ 1 + \frac{W^2 - (M + m_{\pi})^2}{Q^2 + Q_0^2} \right]^{-1} \sum_{i=1}^{2} \sigma_{N,i}^{R}(0) \frac{[W - (M + m_{\pi})]^{(i+1)/2}}{(Q^2 + a_i^T)^2(b_i^T + c_i^T Q^2 + d_i^T Q^4)}$$

$$\square_{A,\text{bgd}}^{\gamma W} = 0.15(1) \times 10^{-3}$$

this is a similar box contribution as compared to extending the DIS & Regge models up to $x=1$.
Boundary Matching:

- The SF should be continuous over all region boundaries
- All models agree within uncertainties
- The boundaries shown \( Q_0^2 = 2 \text{ GeV}^2, W_{\text{min}}^2 = 4 \text{ GeV}^2 \) are not unique, and we find the total Box correction is insensitive to their choice

\[ W^2 = 4 \text{ GeV}^2 \]
\[ W^2 = 10 \text{ GeV}^2 \]
\[ W^2 = 50 \text{ GeV}^2 \]

\[ Q^2 = 0.05 \text{ GeV}^2 \]
\[ Q^2 = 2 \text{ GeV}^2 \]
\[ Q^2 = 3 \text{ GeV}^2 \]
\[ Q^2 = 4 \text{ GeV}^2 \]
Revising $V_{ud}$: Simple Approach

Total Box correction:

<table>
<thead>
<tr>
<th>$\Box_A^W \times 10^{-3}$</th>
<th>SBM</th>
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<th>CMS</th>
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<tbody>
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<td>elastic</td>
<td>1.04(6)</td>
<td>1.06(6)</td>
<td>0.99(10)</td>
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<td>resonance</td>
<td>0.04(2)</td>
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<td>DIS + high-$Q^2$ bgd</td>
<td>2.29(2)</td>
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<td>total</td>
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*computed at $\alpha(0) = 1/137.036$

Extract $V_{ud}$ from super allowed beta decays:

$$|V_{ud}|^2 = \frac{2984.43s}{\mathcal{F}t(1+\Delta_V^R)}$$

$$\Delta_V^R = 0.017007 + 2\Box_A^W$$

$$\Delta_V^R = 0.02479(20)$$

includes re-summed log

$$\sum_{CKM}^{3 \times 3} = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = .9983(4) \neq 1$$

The effect of computing $\Box_A^W$ via a dispersion relation has led to the top row of the CKM matrix to fall short of unitarity by 4$\sigma$
Revising \( V_{ud} \): CMS Approach

**Total Box correction:**

\[
\Box_A^{\gamma W} (\times 10^{-3}) \quad |SBM| \quad |SGRM| \quad |CMS|
\]

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*computed at \( \alpha(0) = 1/137.036 \)

**Extract \( V_{ud} \) from super allowed beta decays:**

\[
|V_{ud}|^2 = \frac{2984.43 s}{\mathcal{F} t (1 + \Delta V_R)} \quad \Delta V_R = 0.01671 + 1.022 \left[ 2 \Box_A^{\gamma W} (Q^2 \geq Q_0^2) + 0.0014 \right] + 1.065 \left[ 2 \Box_A^{\gamma W} (Q^2 < Q_0^2) + 2 \Box_A^{\gamma W, el} \right]
\]

\[
\Delta V_R = 0.02486(26)
\]

\[
\sum_{CKM}^{3 \times 3} |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9982(5) \neq 1
\]

The effect of computing \( \Box_A^{\gamma W} \) via a dispersion relation has led to the top row of the CKM matrix to fall short of unitarity by \( 3.8 \sigma \)
EIC Contribution?

- One could improve the $V_{ud}$ extraction by better constraining the neutral current axial structure functions:

$$F_{3}^{\nu p + \bar{\nu} p} = F_{3,p}^{\gamma Z} + F_{3,n}^{\gamma Z}$$

$$F_{3}^{(0)} = F_{3,p}^{\gamma Z} - F_{3,n}^{\gamma Z}$$

Need more data at low $Q$, low $x$

$$eN \rightarrow eX$$

$e^\pm$ DIS cross section:

$$\frac{d^2\sigma^{NC}}{dxdy} = 4\pi\alpha^2 \frac{\eta^{NC}}{xyQ^2} \left\{ \left( 1 - y - \frac{x^2y^2M^2}{Q^2} \right) F_{2}^{NC} + y^2xF_{1}^{NC} + \left( y - \frac{y^2}{2} \right) xF_{3}^{NC} \right\}$$

$$F_{3}^{NC} \sim -(g_{A}^{e} \pm \lambda g_{V}^{e})\eta_{\gamma Z}xF_{3}^{\gamma Z} + \ldots$$