## The pion PDF: from models to dynamical predictions

 in Mnkowski spaceGiovanni Salmè
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> some dynamical results....

- Pion:
de Paula et al PRD 94, 071901(R) (2016): Two-fermion bound systems de Paula et al EPJ C 77, 764 (2017) : Light-cone singularities and structure
- Nogueira et al PRD 100, 016021 (2019): Solving the Bethe-Salpeter Equation in Minkowski Space for a Fermion-Scalar system
- FVS PRD 85, 036009 (2012): General formalism for bound and scattering states
- FVS PRD 89, 016010 (2014): Bound states and LF momentum distributions for two scalars
- FVS EPJC 75, 398 (2015): Scattering lengths for two scalars
- Gutierrez et al PLB 759, 131 (2016): Spectra of excited states and LF momentum distributions


## Outline

(1) Motivations, tools and generalities
(2) Bird's-eye view of phenomenological results
(3) The exact projection of the BSE and the BS amplitude onto the hyper-plane $x^{+}=0$

4 BSE for fermions in Minkowski space
(5) Conclusions \& Perspectives

## Motivations and Tools

M: Presently, many and collaborative efforts to address relevant dynamical quantities, like PDFs, TMDs and GPDs are carried out within LQCD, widely considered the elective tool for non perturbative studies. Of particular interest, the ongoing investigation of the X. Ji proposal (PRL 110, 262002 (2013)) and the A. Radyushkin one (PRD 96, 034025 (2017)) on Quasi-PDFs, that aim at the evaluation of PDFs from LQCD, (see, e.g., David Richards talk).
M: To have workable alternatives is highly desirable, even with a lower degree of complexity than LQCD can achieve. A reference approach, quite popular and very effective in predicting the dynamical behavior inside hadrons, is the so-called continuum-QCD. Based on both the Dyson-Schwinger equation (for self-energies) and the Bethe-Salpeter one, cQCD is able to get the ingredients for calculating dynamical observables (see, e.g., Craig Roberts talk).
M: Our perspective evolves within a continuum approach, but directly played in Minkowski momentum-space. The aim is to achieve a fully covariant and non perturbative description for bound systems, with spin dof, incorporating, step by step and in a controlled way, dynamical effects, at the level of the interaction kernel, self-energy and vertex corrections. The first milestone has been the actual solutions of the Bethe-Salpeter equation (BSE), in ladder approximation. The formal extension to the gap-equations for particle and quanta is ready for $Q D E_{3+1}$. One can also pursue an hybrid approach: self-energies from LQCD.

M : Once the BSE is solved in Minkowski momentum-space, one can determine from the BS amplitude, the relevant momentum distributions. A straightforward outcome: the light-cone valence distributions can be formally obtained by projecting the BS amplitude of the bound system onto the null-plane.

T: Pivotal role of the Nakanishi Integral Representation (NIR) of the BS amplitude for implementing the numerical approach and solving the BSE
T: Light-front (LF) variables, $x^{ \pm}=x^{0} \pm x^{3}$ and $\mathrm{x}_{\perp} \equiv\left\{x^{1}, x^{2}\right\}$, very suitable for managing analytic integration and spin dof in a very effective way, in Minkowski space. More generally, LF framework is extremely useful in hadron physics (see the valence distribution).
T: Standard LAPACK routines for the numerical evaluations, to emphasize that the calculations are not too much demanding, at this initial stage.
To better understand the quality of the efforts in progress for elaborating a genuinely dynamical approach, it is interesting to compare a wide set of physical observables of the Pion and the corresponding outcomes from phenomenological models that share the common playground:

> the Bethe-Salpeter framework in Minkowski space.

## Bird's-eye view of phenomenological results

The key ingredient of the models is the Pion Bethe-Salpeter amplitude (BSA), with suitable Dirac structure and momentum dependence (containing a minimal number of adjustable parameters, usually only one). The applications range from elastic form factor, in space- and time-like regions, to the leading-twist GPD's and the parton distribution function.

The first investigation, retaining only the valence component of the BSA, had as a target the Pion em form factor in the whole kinematical region, i.e both space-and time-like regions: $\rightarrow$ LF-CQM (N.B. $q^{+} \neq 0$ frame).


Formally exact : $\quad \int \frac{d k^{-}}{(2 \pi)} \Phi_{B S}\left(k, P_{\pi}\right) \propto \psi_{\text {val }}\left(k^{+},\left|\mathbf{k}_{\perp}\right|\right)$
Recall that within the Light-front QFT the Fock expansion becomes meaningful, once only massive dofs are present.

The Pion EM Form Factor in SL and TL regions within the LFCQM.
Only 2 adjusted parameters. N.B. $m_{q}=$ 200 MeV and (LF) instantaneous contributions.


Solid line: calculation with BSA and the Frederico-Pauli-Zhou vector-meson spectrum (PRD 66, 116011 (2002)).
[From de Melo et al PLB. 581 (2004) 75; PRD 73, 074013 (2006)]
Dashed line: the same as the solid line, but with the asymptotic pion w.f.

- : Compilation from R. Baldini et al. EPJ. C11 (1999) 709.
$\square$ : TJLAB SL data [J. Volmer et al., PRL. 86, 1713 (2001)].

To gain a covariant approach and study the Pion GPD's, the full $\Phi_{B S}$ was introduced

$$
\begin{aligned}
& \Phi_{B S}\left(k, P_{\pi}\right)=-\frac{m}{f_{\pi}} S\left(k+P_{\pi} / 2\right) \gamma_{5} \Lambda\left(k, P_{\pi}\right) S\left(k-P_{\pi} /\right) \\
& \Lambda\left(k, P_{\pi}\right)=C \frac{1}{\left[\left(k+P_{\pi} / 2\right)^{2}-m_{R}^{2}+\imath \epsilon\right]} \frac{1}{\left[\left(k-P_{\pi} / 2\right)^{2}-m_{R}^{2}+\imath \epsilon\right]}
\end{aligned}
$$

with $m_{R}$ fixed by $f_{\pi}$ and $C$ from the charge. N.B. It is assumed that all the expected, four, Dirac structures have the same momentum dependence.
The Pion has two leading-twist quark GPDs: i) the vector, or no spin-flip, GPD, $H_{\pi}^{\prime}(x, \xi, t)$, and ii) the tensor, or spin-flip, GPD, $E_{\pi, T}^{\prime}(x, \xi, t)(I=$ isoscalar and isovector GPDs).
One can expand the corresponding Mellin moments in terms of Generalized Form Factors (GFFs), $A_{n+1,2 i}(t)$ and $B_{n+1,2 i}(t)$. For a $u$ quark

$$
\begin{aligned}
& \int_{-1}^{1} d x x^{n} H_{\pi}^{u}(x, \xi, t)=\sum_{i=0}^{[(n+1) / 2]}(2 \xi)^{2 i} A_{n+1,2 i}^{u}(t) \\
& \int_{-1}^{1} d x x^{n} E_{\pi, T}^{u}(x, \xi, t)=\sum_{i=0}^{[(n+1) / 2]}(2 \xi)^{2 i} B_{n+1,2 i}^{u}(t)
\end{aligned}
$$

A striking feature is the so-called polinomiality, i..e. the dependence, in the rhs, upon finite powers of the variable $\xi$. Polinomiality follows from general properties like covariance, parity and time-reversal invariance. It is a needed test for any covariant model.

Space-like Pion form factor in C-CQM.
T. Frederico et al, PRD 80, 054021 (2009), Fanelli et al EPJ C 76, 253 (2016)


Dashed line: monopole fit to lattice data extrapolated to $m_{\pi}=0.140 \mathrm{GeV}$ [D. Brömmel et al, Eur. Phys. J. C 51, 335(2007]
Double-dot-dashed line: CCQM with $m_{q}=0.200 \mathrm{GeV}$ and $m_{R}=1.453 \mathrm{GeV}$
Dotted line: $\quad$ CCQM with $m_{q}=0.210 \mathrm{GeV}$ and $m_{R}=1.320 \mathrm{GeV}$.
Dot-dashed line: $\quad$ CCQM with $m_{q}=0.220 \mathrm{GeV}$ and $m_{R}=1.192 \mathrm{GeV}$.
Nice dependence upon the CQ mass in the high-momentum region. [Fanelli et al EPJ C 76, 253 (2016)]

Evolved Tensor GFFs $B_{1,0}^{q}(t)$ and $B_{2,0}^{q}(t)$ within C-CQM


Dashed line: C-CQM result LO-evolved to the lattice scale from $\mu_{0}=.500 \mathrm{GeV}$, CQM mass $=0.220 \mathrm{GeV}$ and $m_{R}=1.192 \mathrm{GeV}$, [Fanelli et al EPJ C 76, 253 (2016)] Shaded area: the lattice data with scale $\mu=2 \mathrm{GeV}$ [From D. Brömmel et al., Phys. Rev. Lett. 101, 122001 (2008)]

Evolved Vector GFF's $A_{2,0}^{q}(t)$ and $A_{2,2}^{q}(t)$ within C-CQM


Dashed line: C-CQM result LO-evolved to the lattice scale, from $\mu_{0}=.500 \mathrm{GeV}$, CQM mass $=0.220 \mathrm{GeV}$ and $m_{R}=1.192 \mathrm{GeV}$, [Fanelli et al, EPJ C 76, 253 (2016)] Shaded area: the lattice data with scale $\mu=2 \mathrm{GeV}$. [From D. Brömmel et al, Phys. Rev. Lett. 101, 122001 (2008)]

- Original E615 data (LO analyzed) from J.S.Conway et al., Phys. Rev. D 39, 92 (1989)

Diamonds: Rescaled E615 data, following by Aicher, Schafer, and Vogelsang, Phys. Rev. Lett. 105, 252003 (2010), with $\mu=5.2$ GeV .


Dashed line: PDF from the full BSA
Solid line : PDF from the valence component, $\int_{0}^{1} u_{V}^{\pi}(x)=1$
Dashed line: Evolved to 5.2 GeV PDF from the full BSA
Solid line: Evolved to 5.2 GeV PDF from the valence component
Evolution from the initial scale $\mu_{0}=0.496 \mathrm{GeV}\left(\langle x\rangle_{C C Q M}=0.47\right.$ at the initial scale $)$ to $\mu=5.2 \mathrm{GeV}$.

The wide phenomenology we explored encourages the attempt of including more dynamics.

This motivates the investigation of the Bethe-Salpeter equation for bound systems, directly in Minkowski space, with the aim of gaining more deep insights in the phenomenological description of Hadrons in non perturbative regime, and eventually establishing correlations between the emerging dofs and the underlying dynamics, without the intermediate step: Euclidean $\rightarrow$ Minkowskian

For instance, the procedure for reconstructing Pion PDF from the Mellin moments can be fully tested without leaving the Minkowski space.

$$
\Phi_{B S}=\mathcal{K} \otimes \Phi_{B S} \quad \Rightarrow \quad \text { Nakanishi Integral Representation for } \Phi_{\mathrm{BS}}
$$



## Projecting BSE onto the LF hyper-plane $x^{+}=0$

- NIR contains the needed freedom for exploring non perturbative problems, once the Nakanishi weight functions are taken as unknown REAL quantities.
- Even adopting NIR, BSE spin dof still remains a highly singular integral equation in the 4D Minkowski momentum space. BUT exploiting an expression á la Nakanishi for the $\Phi_{B S}$, then its analytic structure is displayed in full, allowing formal manipulations
- Noteworthy, in the LF framework one recovers a probabilistic interpretation by expanding the BS amplitude on a Fock basis, and then singling out the valence component. Hence the probability of finding two constituents in the fully interacting state can be evaluated, $|\pi\rangle=|q \bar{q}\rangle+|q \bar{q} q \bar{q}\rangle+|q \bar{q} g\rangle \ldots .$.



## BSE for fermions

$$
\begin{gathered}
\Phi_{B S}(k, p)=S(p / 2+k) \int d^{4} k^{\prime} F^{2}\left(k-k^{\prime}\right) i \mathcal{K}\left(k, k^{\prime}\right) \Gamma_{1} \Phi_{B S}\left(k^{\prime}, p\right) \bar{\Gamma}_{2} S(k-p / 2) \\
S(q)=i \frac{\not q+m}{q^{2}-m^{2}+i \epsilon}, \quad F\left(k-k^{\prime}\right)=\frac{\left(m_{g}^{2}-\Lambda^{2}\right)}{\left[\left(k-k^{\prime}\right)^{2}-\Lambda^{2}+i \epsilon\right]} \\
\Gamma_{1}=\Gamma_{2}=1(\text { scalar }), \gamma_{5}(\text { pseudo }), \gamma^{\mu}(\text { vector })
\end{gathered}
$$

For a $0^{+}$state, one can decompose $\Phi_{B S} \quad \Rightarrow$

$$
\Phi_{B S}(k, p)=S_{1} \phi_{1}(k, p)+S_{2} \phi_{2}(k, p)+S_{3} \phi_{3}(k, p)+S_{4} \phi_{4}(k, p)
$$

with

$$
S_{1}=\gamma_{5}, \quad S_{2}=\frac{p}{M} \gamma_{5}, \quad S_{3}=\frac{k \cdot p}{M^{3}} \dot{p} \gamma_{5}-\frac{1}{M} k \gamma_{5} \quad, S_{4}=\frac{i}{M^{2}} \sigma^{\mu \nu} p_{\mu} k_{\nu} \gamma_{5}
$$

and $\phi_{i} \equiv$ unknown scalar functions, with well-defined symmetry under the exchange $1 \rightarrow 2$, from the symmetry of both $\Phi_{B S}(k, p)$ and $S_{i}$.

$$
\text { NIR applied to } \phi_{i} \text { !! }
$$

$\star$ For each $\phi_{i}$, use NIR and apply the LF projection to obtain the valence amplitudes

$$
\psi_{i}(\gamma, z)=\int \frac{d k^{-}}{2 \pi} \phi_{i}(k, p)=-\frac{i}{M} \int_{0}^{\infty} d \gamma^{\prime} \frac{g_{i}\left(\gamma^{\prime}, z ; \kappa^{2}\right)}{\left[\gamma+\gamma^{\prime}+m^{2} z^{2}+\left(1-z^{2}\right) \kappa^{2}-i \epsilon\right]^{2}}
$$

- $\gamma \equiv\left|\mathbf{k}_{\perp}\right|^{2} \in[0, \infty]$
- $z \equiv \in[-1,1]$
- $\kappa^{2}=4 m^{2}-M^{2}$ with $M=2 m-B .(B \equiv$ binding energy $)$.
$\star \star$ Hence, BSE formally reduces to a system of 4 coupled integral equations for a $0^{+}$state, where the 4 Nakanishi weights are the unknowns

$$
\psi_{i}(\gamma, z)=g^{2} \sum_{j} \int_{-1}^{1} d z^{\prime} \int_{0}^{\infty} d \gamma^{\prime} g_{j}\left(\gamma^{\prime}, z^{\prime} ; \kappa^{2}\right) \mathcal{L}_{i j}\left(\gamma, z, \gamma^{\prime}, z^{\prime} ; p\right)
$$

Again, if the coupled system admits solutions, then we know how to reconstruct the BS amplitude!.
In Ladder approximation the adopted interaction kernel is (Feynman gauge)

$$
\mathcal{K}_{V}^{\mu \nu}=\frac{g^{2} g^{\mu \nu}}{\left[\left(k-k^{\prime}\right)^{2}-m_{g}^{2}+i \epsilon\right]}
$$

## Spin dof and BSE (short detour, but not trivial)

Adding spin dof is a challenge, both on formal and numerical sides, since while projecting onto the null plane, one faces with integrals that could become singular for some values of an external variable.

Fortunately, the prototype of such singular integrals was studied by Yan (PRD 7 (1973) 1780) in the context of the field theory in the Infinite Momentum frame viz

$$
\mathcal{I}(\beta, y)=\int_{-\infty}^{\infty} \frac{d x}{[\beta x-y \mp i \epsilon]^{2}}= \pm \frac{2 \pi i \delta(\beta)}{[-y \mp i \epsilon]}
$$

$\star$ In the fermionic BSE case, one can rigorously evaluate the singular integrals by applying the Yan result and some simple extensions, leading to derivative of the delta-functions. This is not an issue since we use an orthonormal basis (infinitely derivable... ) for expanding the Nakanishi weight functions.

Pion PDF from the valence component of the Pion state (I) Calculated $\Phi_{B S} \Rightarrow$ valence component through LF projection

- Original E615 data (LO analyzed) from J.S.Conway et al., Phys. Rev. D 39, 92 (1989)

Diamonds: Rescaled E615 data at $\mu=5.2 \mathrm{GeV}$ [cf Aicher, Schafer, and Vogelsang, Phys. Rev. Lett. 105, 252003 (2010)].


Thick line: valence component from BS equation with $m_{q}=215 \mathrm{MeV}, m_{g}=310 \mathrm{MeV}$. Thick line: valence component from BS equation with $m_{q}=255 \mathrm{MeV}, m_{g}=640 \mathrm{MeV}$. Thick line: valence component in the Covariant Model

Thin line: evolved results from 0.496 GeV to 5.2 GeV Thin line: evolved results from 0.496 GeV to 5.2 GeV . Thin line: evolved results from 0.496 GeV to 5.2 GeV .

Pion PDF from the valence component of the Pion state, (II)

- Original E615 data (LO analyzed) from J.S.Conway et al., Phys. Rev. D 39, 92 (1989)

Diamonds: Rescaled E615 data at $\mu=5.2 \mathrm{GeV}$ [cf Aicher, Schafer, and Vogelsang, Phys. Rev. Lett. 105, 252003 (2010)].


Thin line: evolved results from 0.496 GeV to 5.2 GeV Thin line: evolved results from 0.496 GeV to 5.2 GeV . Thin line: evolved results from 0.496 GeV to 5.2 GeV .
.... One-gluon exchange $\Rightarrow$ high-x tail

## Conclusions \& Perspectives

- A systematization of the technique for solving BSE with and without spin dof has been reached, and the cross-check among results obtained by different groups, for different interacting systems (with kernels in ladder and cross-ladder contributions) has produced a clear numerical evidence of the validity of NIR for obtaining actual solutions.
- A first message from the the PDF is very encouraging, particularly if it is coupled to the one from the comparison with experimental elastic form factor [see Tobias Frederico talk]
- This can open a new dynamical era, in Minkowski space, but it is fundamental to include needed improvements, i.e. self-energies and vertex corrections evaluated possibly within the same framework, or even, as an intermediate step, from LQCD.


## Thank you for your attention!

## Back-up slides

## Vector coupling and high-momentum tails: $\gamma \equiv\left|\mathbf{k}_{\perp}\right|^{2}$

The LF amplitudes $\psi_{i}$, components of the valence momentum distributions have the correct tail (!), for the massless-vector coupling.
Power one is expected for the pion valence amplitude from dimensional arguments by X . Ji et al, PRL 90 (2003) 241601 (cf also Brodsky \& Farrar (PRL 31 (1973) 1153) for the counting rules of exclusive amplitudes)

$\psi_{i} \times \gamma / m^{2}$ at fixed $z=0$
$(\xi=1 / 2)$, for the masslessvector coupling.
$B / m=0.1$ (thin lines) and 1.0 (thick lines).
-_ : $\left(\gamma / m^{2}\right) \psi_{1}$.
$--:\left(\gamma / m^{2}\right) \psi_{2}$.
$-\quad:\left(\gamma / m^{2}\right) \psi_{4}$.
$\psi_{3}=0$ for $z=0$ (odd function)

