Pion and kaon parton distributions from their light-front wave functions

José Rodríguez-Quintero

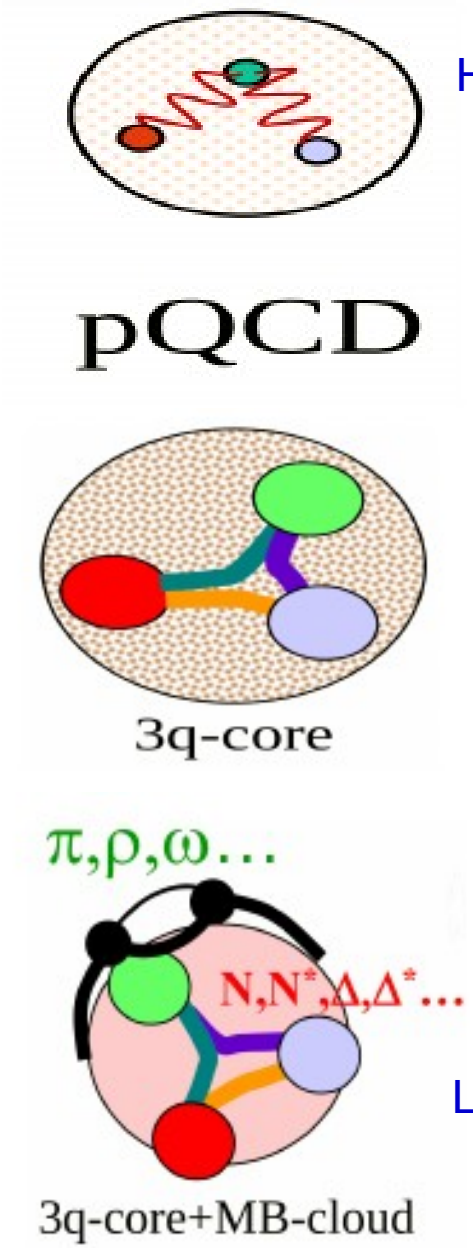
In collaboration with:

C. Lei, D. Binosi, C. Mezrag,

K. Raya, C. D. Roberts, M. Ding …

Pion and Kaon Structure Functions at the EIC; CFNS/Stony Brook U.: June 2-5, 2020.

General picture



High

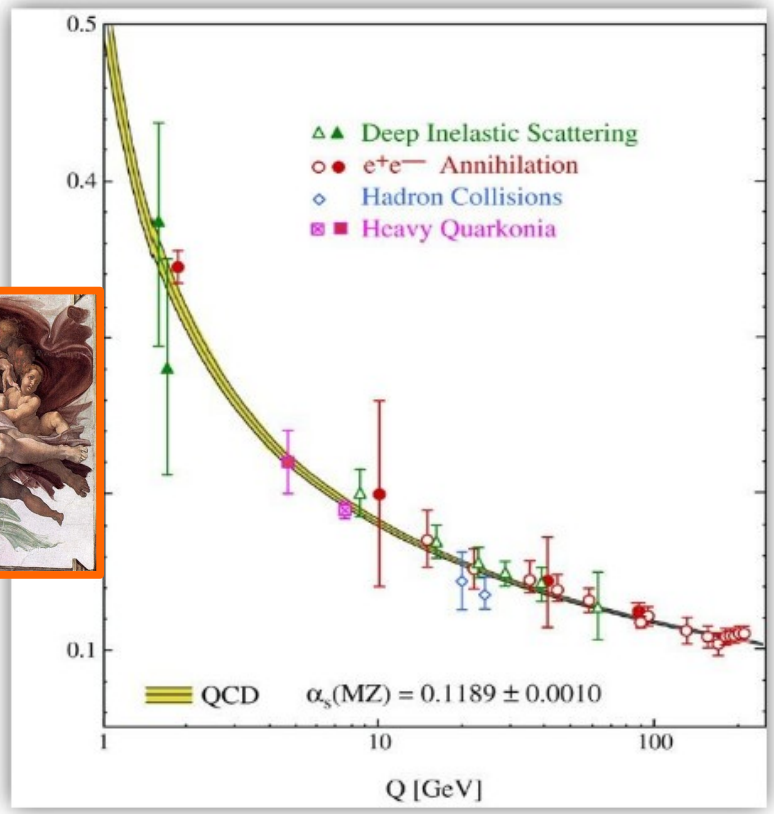
Q^2

Low

The QCD Holy Grail: the understanding of hadrons in terms of its elementary excitations; namely, quarks and gluons!

Confinement

Colored bound states have never been seen to exist as particles in nature



DCSB

Chiral symmetry appears dynamically violated in the Hadron spectrum

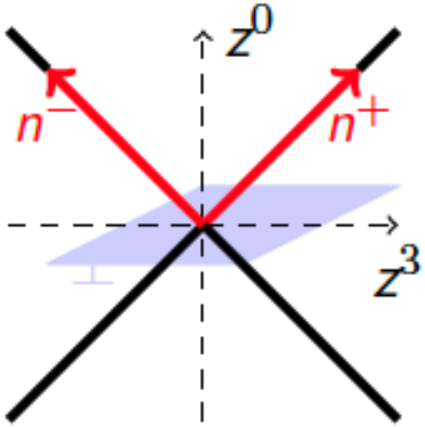
Emergent phenomena playing a dominant role in the real world dominated by the IR dynamics of QCD.

Antecedents:

GPD definition:

$$H^q_\pi(x,\xi,t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2}\middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right)\middle|\pi, P-\frac{\Delta}{2}\right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t=\Delta^2$ and $\xi=-\Delta^+/(2P^+)$.



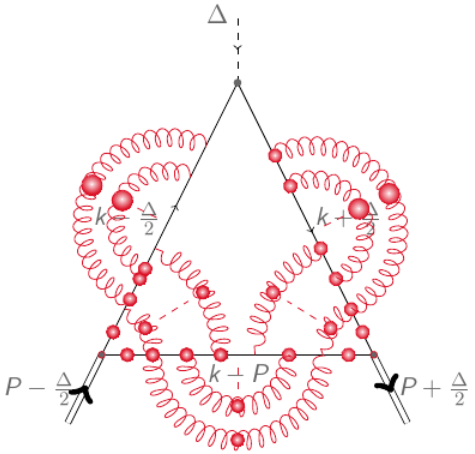
References

Muller et al., Fortchr. Phys. **42**, 101 (1994)
Radyushkin, Phys. Lett. **B380**, 417 (1996)
Ji, Phys. Rev. Lett. **78**, 610 (1997)

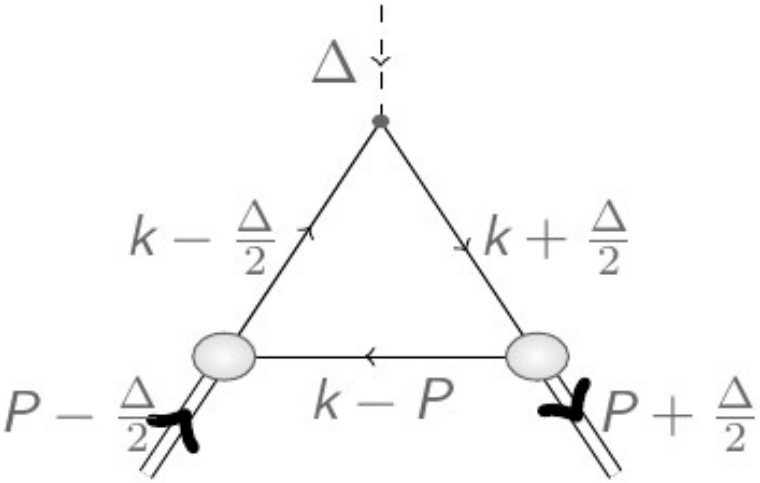
- From **isospin symmetry**, all the information about pion GPD is encoded in $H^u_{\pi^+}$ and $H^d_{\pi^+}$.
- Further constraint from **charge conjugation**: $H^u_{\pi^+}(x,\xi,t) = -H^d_{\pi^+}(-x,\xi,t)$.

Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach



$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD H.

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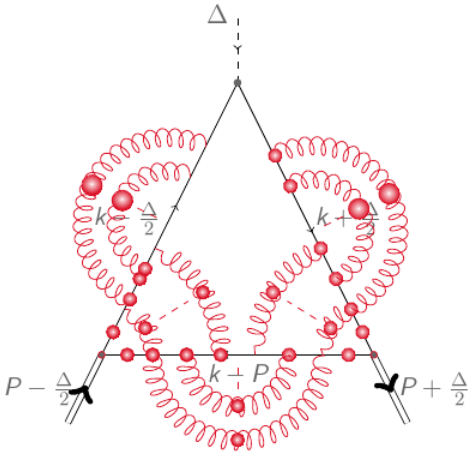

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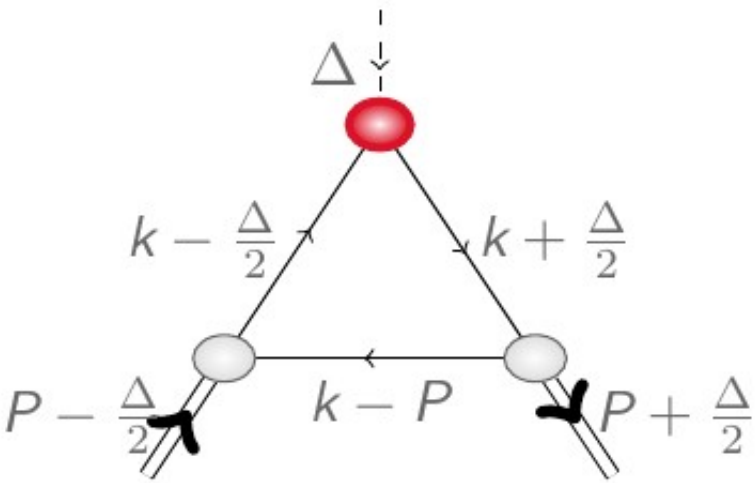

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- Triangle diagram approx.

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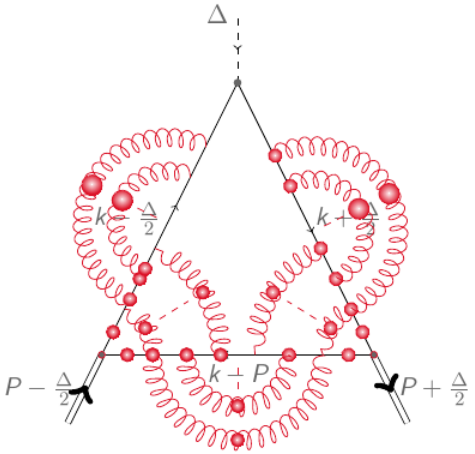

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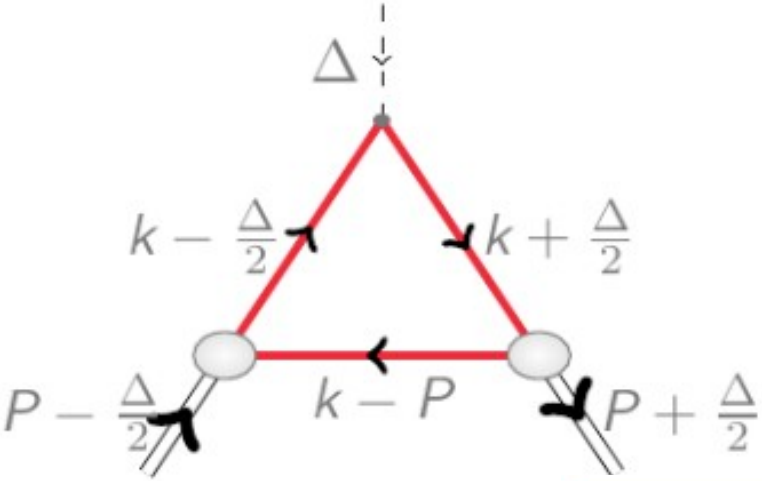
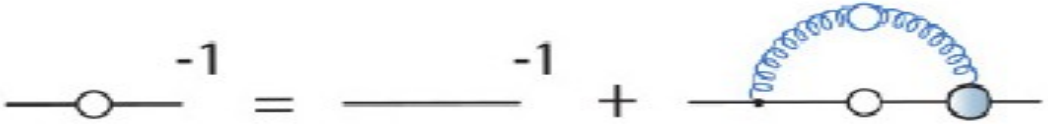


- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

Dyson - Schwinger equation



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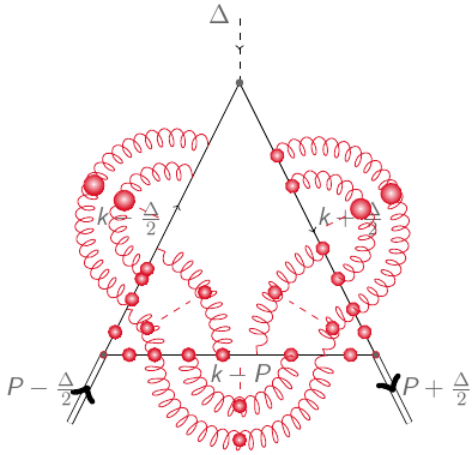
GPDs in the Schwinger-Dyson and Bethe-Salpeter approach



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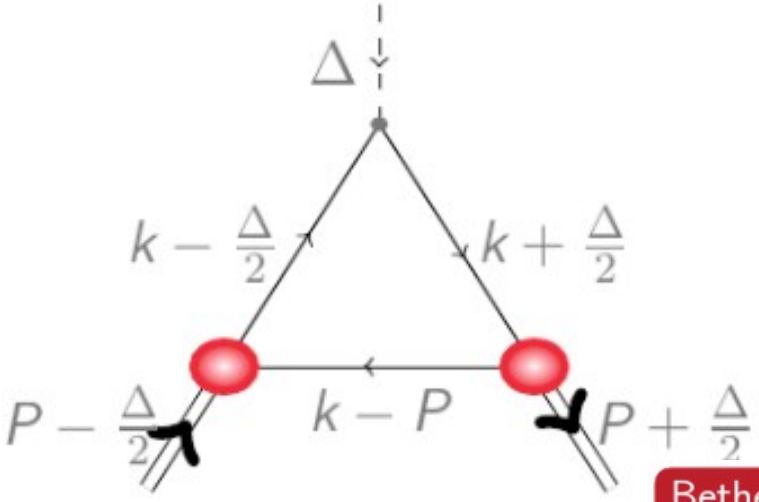

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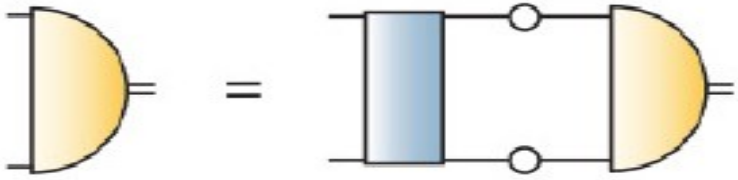
Bethe - Salpeter equation



Antecedents:

GPD asymptotic algebraic model:

- Expressions for vertices and propagators:

$$
\begin{aligned}
S(p) &= \left[-i\gamma\cdot p + M\right]\Delta_M(p^2)\\
\Delta_M(s) &= \frac{1}{s+M^2}\\
\Gamma_\pi(k,p) &= i\gamma_5\frac{M}{f_\pi}M^{2\nu}\int_{-1}^{+1}\mathrm{d}z\,\rho_\nu(z)\,\left[\Delta_M(k_{+z}^2)\right]^\nu\\
\rho_\nu(z) &= R_\nu(1-z^2)^\nu
\end{aligned}
$$

with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:
  - Dimension
  - Dimens
    asympt

Antecedents:

GPD asymptotic algebraic model:

- **Analytic expression** in the DGLAP region.

$$
\begin{aligned}
H^u_{x\geq\xi}(x,\xi,0) = \frac{48}{5}\Bigg\{ & \frac{3\left(-2(x-1)^4\left(2x^2-5\xi^2+3\right)\log(1-x)\right)}{20\left(\xi^2-1\right)^3}\\
& \frac{3\left(+4\xi\left(15x^2(x+3)+(19x+29)\xi^4+5(x(x(x+11)+21)+3)\xi^2\right)\tanh^{-1}\left(\frac{(x-1)}{x-\xi^2}\right.\right.}{20\left(\xi^2-1\right)^3}\\
& +\frac{3\left(x^3(x(2(x-4)x+15)-30)-15(2x(x+5)+5)\xi^4\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3}\\
& +\frac{3\left(-5x(x(x(x+2)+36)+18)\xi^2-15\xi^6\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3}\\
& +\frac{3\left(2(x-1)\left((23x+58)\xi^4+(x(x(x+67)+112)+6)\xi^2+x(x((5-2x)x+15)+\right.\right.}{20\left(\xi^2-1\right)^3}\\
& \qquad \xi^4+10x(3x(x+5)+11)\xi^2\Big)\log\left(1-\xi^2\right)\Big)\\
& \qquad \xi^2-1\Big)^3\\
& \qquad 3\Big)\log\left(1-\xi^2\right)\Bigg\}
\end{aligned}
$$

-ph]

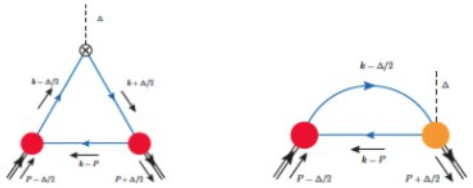
Can be done!

…and closed-form expressions can be found in simple cases.

Antecedents:

GPD asymptotic algebraic model (completion):

The full model:

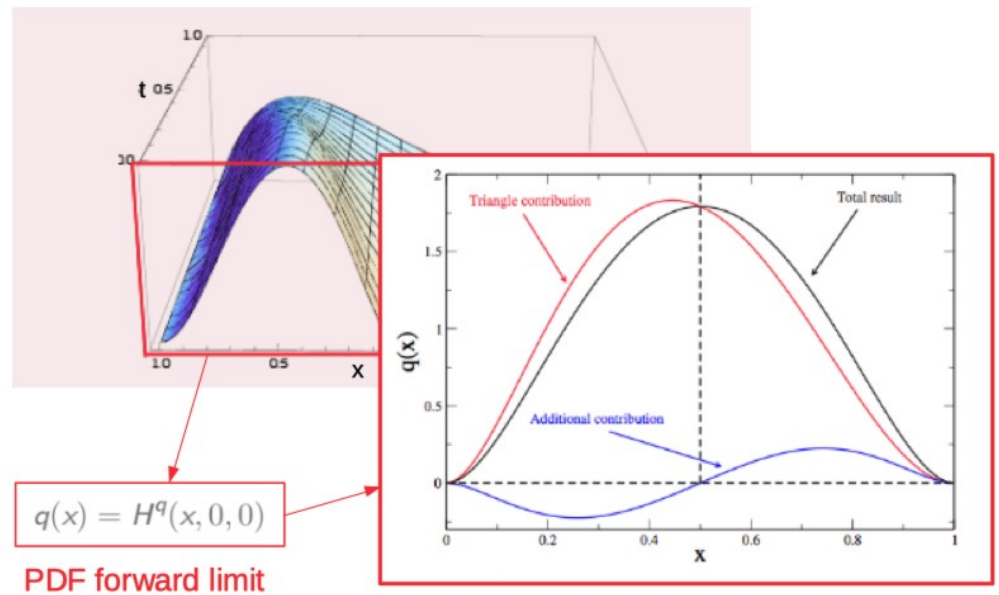


$$
\begin{aligned}
2(P\cdot n)^{m+1}\langle x^m\rangle^u = {} & \mathrm{tr}_{CFD}\int\frac{\mathrm{d}^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)\\
& S(k-\frac{\Delta}{2})\;i\gamma\cdot n\;S(k+\frac{\Delta}{2})\\
& \tau_- i\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P),
\end{aligned}
$$

$$
\begin{aligned}
2(P\cdot n)^{m+1}\langle x^m\rangle^u = {} & \mathrm{tr}_{CFD}\int\frac{\mathrm{d}^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)\\
& S(k-\frac{\Delta}{2})\tau_-\frac{\partial}{\partial k}\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P)
\end{aligned}
$$

C. Mezrag et al., PLB741(2015)190; ArXiv:1406.7425[hep-ph]

GPD asymptotic algebraic model (completion):

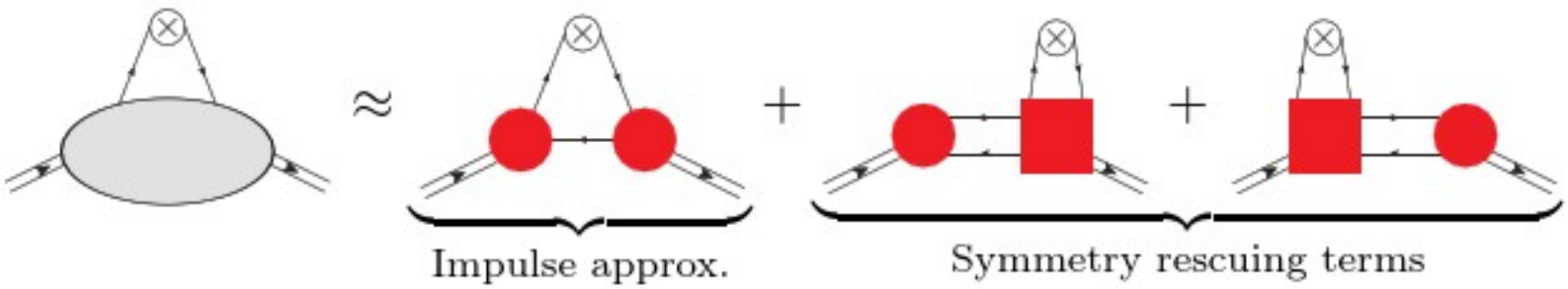


$q(x) = H^q(x,0,0)$

PDF forward limit

Antecedents:

GPD asymptotic algebraic model:



L.Chang *et al.*, Phys.Lett. B737 (2014) 23-29

4

Antecedents:

tic algebraic model:

region.

cf. Ding's & Mezrag's talks!

Symmetry-preserving contributions play a crucial role

$$q(x) = H^q(x,0,0)$$

C. Mezrag et al., Phys.Lett. B 741(2015)190
M. Ding et al., Chin.Phys. C (Lett)44(2020) 031002
M. Ding et al., Phys. Rev. D 101 (2020) 054014

$$q(x) \equiv q(1-x)$$

$$2(P\cdot n)^{m+1}\langle x^m\rangle$$

$$2(P\cdot n)^{m+1}\langle x^m\rangle^u = \mathrm{tr}_{CFD}\int\frac{d^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\ldots\right.$$

$$S(k-\tfrac{\Delta}{2})\tau_-\frac{\partial}{\partial k}\bar{\Gamma}_\pi\left((1-\eta)\left(k+\tfrac{\Delta}{2}\right)+\eta(k-P),P+\tfrac{\Delta}{2}\right)S(k-P)$$

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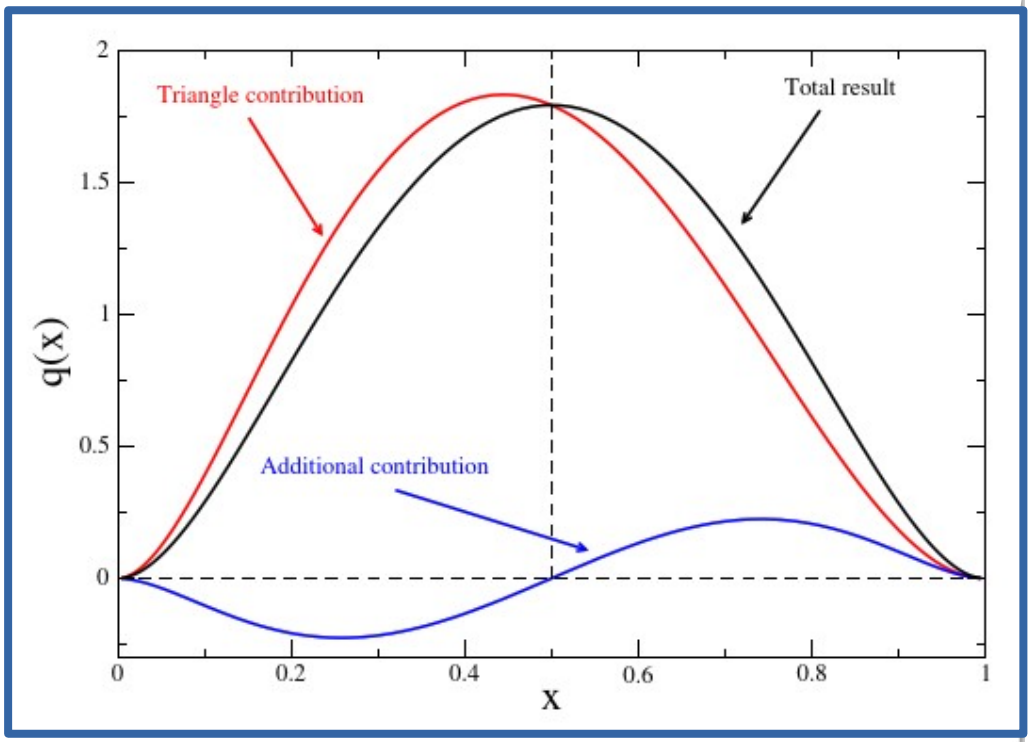

GPD overlap approach:

The overlap quark GPD for a meson in the DGLAP kinematic region reads

$$H^q(x,\xi,t) = \sum_{N,\beta} \sqrt{1-\xi}^{2-N} \sqrt{1+\xi}^{2-N} \sum_a \delta_{a,q} \int [\mathrm{d}\bar{x}]_N \left[\mathrm{d}^2\bar{\mathbf{k}}_\perp\right]_N \delta(x-\bar{x}_a)$$
$$\times\ \Psi^*_{N,\beta}\left(\hat{x}'_1, \hat{\mathbf{k}}'_{\perp 1}, \ldots, \hat{x}'_a, \hat{\mathbf{k}}'_{\perp a}, \ldots\right) \Psi_{N,\beta}\left(\tilde{x}_1, \tilde{\mathbf{k}}_{\perp 1}, \ldots, \tilde{x}_a, \tilde{\mathbf{k}}_{\perp a}, \ldots\right) ,$$

$$[\mathrm{d}x]_N = \prod_{i=1}^{N} \mathrm{d}x_i\ \delta\left(1 - \sum_{i=1}^{N} x_i\right),$$

$$[\mathrm{d}^2\mathbf{k}_\perp]_N = \frac{1}{(16\pi^3)^{N-1}} \prod_{i=1}^{N} \mathrm{d}^2\mathbf{k}_{\perp i}\ \delta^2\left(\sum_{i=1}^{N} \mathbf{k}_{\perp i} - \mathbf{P}_\perp\right)$$

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$$\times \boxed{\Psi^*_{N,\beta}\left(\hat{x}'_1, \hat{\mathbf{k}}'_{\perp 1}, ..., \hat{x}'_a, \hat{\mathbf{k}}'_{\perp a}, ...\right)} \boxed{\Psi_{N,\beta}\left(\tilde{x}_1, \tilde{\mathbf{k}}_{\perp 1}, ..., \tilde{x}_a, \tilde{\mathbf{k}}_{\perp a}, ...\right)} ,$$

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in terms of the meson LFWF

$$|H;P,\lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x]_N [\mathrm{d}^2\mathbf{k}_\perp]_N \boxed{\Psi^\lambda_{N,\beta}(x_1, \mathbf{k}_{\perp 1}, ..., x_N, \mathbf{k}_{\perp N})} |N,\beta;k_1,\ldots,k_N\rangle$$

which are the components in an expansion of the meson on a Fock basis, after light-front quantization.

GPD overlap approach:

The overlap valence-quark GPD for a meson in the DGLAP kinematic region reads

$$H^q(x,\xi,t) = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

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$$2P^+\Psi_{\uparrow\downarrow}(k^+, \mathbf{k}_\perp) = \int \frac{dk^-}{2\pi} \mathrm{Tr}\left[\gamma^+\gamma_5\chi(k,P)\right]$$

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$$\Gamma_\pi(q,P) = iN\gamma_5 \int_0^\infty d\omega \int_{-1}^{1} dz \frac{\rho(\omega,z)M^2}{\left(q - \frac{1-z}{2}P\right)^2 + M^2 + \omega}$$

$$\chi(q,P) = S(q)\Gamma_\pi(q,P)S(q-P)$$

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Bethe-Salpeter amplitudes and quark propagators can be obtained from applying continuum functional methods (DSE,BSE)

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$$S(q) = [-i\gamma\cdot q + M]/[q^2 + M^2]$$

Nakanishi weight

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Asymptotic case: $\rho(w,z) = \delta(w)(1-z^2)$

Bethe-Salpeter amplitudes and quark propagators can be obtained from applying continuum functional methods (DSE,BSE) or can be modeled as previously indicated.

GPD overlap approach:

The overlap valence-quark GPD for a meson in the DGLAP kinematic region reads

$$H^q(x,\xi,t) = 30\frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)$$

$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}$$

Encoding the correlation of kinematical variables

$$\boxed{S(q)} = [-i\gamma\cdot q + M]/[q^2+M^2]$$

Nakanishi weight

$$\boxed{\Gamma_\pi(q,P)} = iN\gamma_5\int_0^\infty d\omega\int_{-1}^{1}dz\frac{\rho(\omega,z)M^2}{\left(q-\frac{1-z}{2}P\right)^2+M^2+\omega}$$

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N. Chouika et al., PLB780(2018)287

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Encoding the correlation of kinematical variables

$$S(q) = [-i\gamma\cdot q + M]/[q^2+M^2]$$

Nakanishi weight

Forward limit: $\xi = 0, t = 0$

$$\Gamma_\pi(q,P) = iN\gamma_5\int_0^\infty d\omega\int_{-1}^{1}dz\frac{\rho(\omega,z)M^2}{\left(q-\frac{1-z}{2}P\right)^2+M^2+\omega}$$

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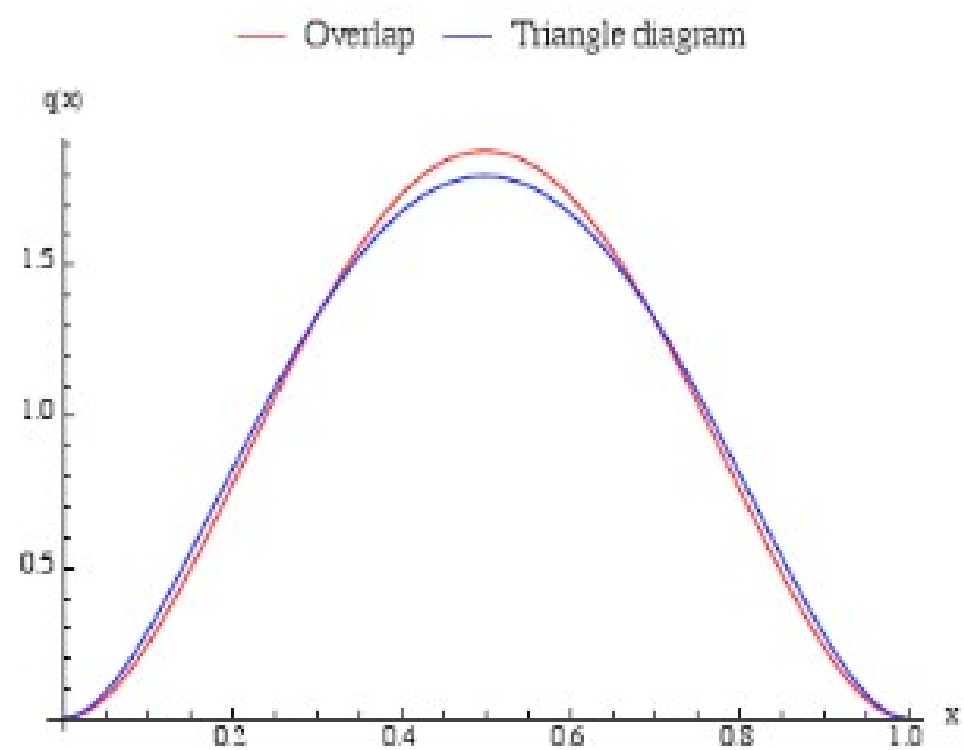

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Nakanishi weight

$$\Gamma_\pi(q,P) = iN\gamma_5\int_0^\infty d\omega\int_{-1}^1 dz\frac{\rho(\omega,z)M^2}{\left(q-\frac{1-z}{2}P\right)^2+M^2+\omega}$$

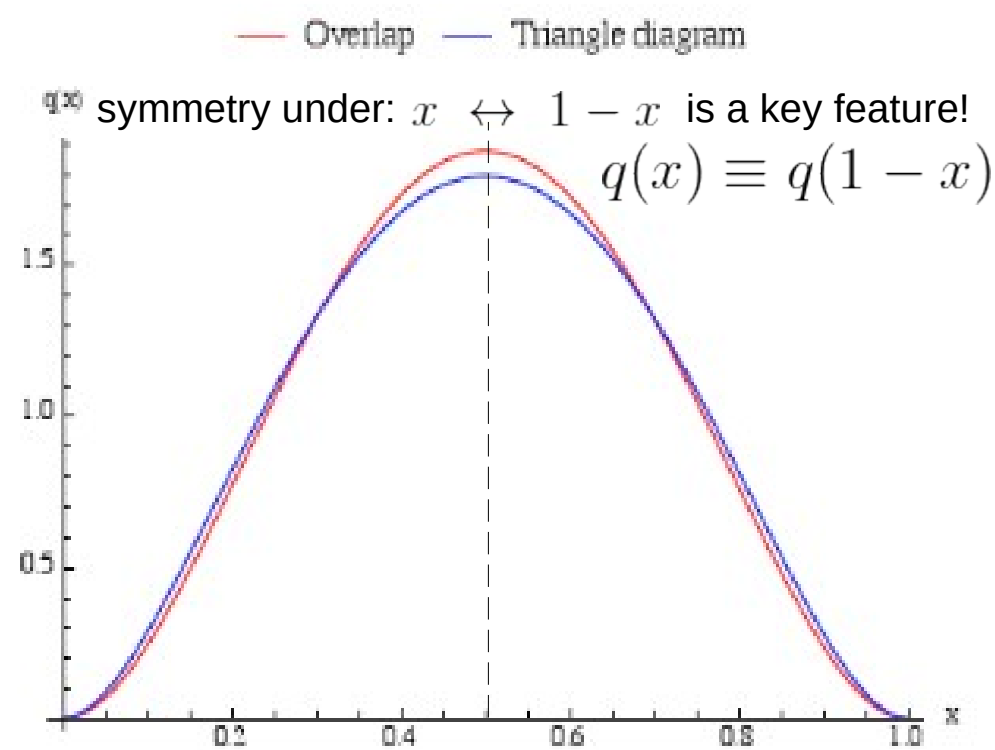
$$\chi(q,P) = S(q)\Gamma_\pi(q,P)S(q-P)$$

Forward limit: $\xi = 0, t = 0$

Asymptotic case: $\rho(w,z) = \delta(w)(1-z^2)$

Bethe-Salpeter amplitudes and quark propagators can be obtained from applying continuum functional methods (DSE,BSE) or can be modeled as previously indicated.

Results from the overlap and diagrammatic approaches compare very well (tested at the level of the PDF).



symmetry under: $x \leftrightarrow 1-x$ is a key feature!

$$q(x) \equiv q(1-x)$$

N. Chouika et al., PLB780(2018)287

Integral representation of LFWFs:

cf. Raya's talk!

- The pseudoscalar LFWF can be written:

$$f_K \psi_K^{\uparrow\downarrow}(x, k_\perp^2) = \mathrm{tr}_{CD} \int_{dk_{||}} \delta(n \cdot k - x n \cdot P_K) \gamma_5 \gamma \cdot n \chi_K^{(2)}(k_-^K; P_K) \ .$$

- The moments of the distribution are given by:

$$< x^m >_{\psi_K^{\uparrow\downarrow}} = \int_0^1 dx x^m \psi_K^{\uparrow\downarrow}(x, k_\perp^2) = \frac{1}{f_K n \cdot P} \int_{dk_{||}} \left[\frac{n \cdot k}{n \cdot P}\right]^m \gamma_5 \gamma \cdot n \chi_K^{(2)}(k_-^K; P_K)$$

$$\int_0^1 d\alpha \alpha^m \left[\frac{12}{f_K} \mathcal{Y}_K(\alpha; \sigma^2)\right] \ , \quad \mathcal{Y}_K(\alpha; \sigma^2) = [M_u(1-\alpha) + M_s \alpha] \mathcal{X}(\alpha; \sigma_\perp^2) \ .$$

Uniqueness of Mellin moments →

$$\boxed{\psi_K^{\uparrow\downarrow}(x, k_\perp^2) = \frac{12}{f_K} \mathcal{Y}_K(x; \sigma_\perp^2)}$$

$$\boxed{\chi_K(\alpha; \sigma^3) = \left[\int_{-1}^{1-2\alpha} d\omega \int_{1+\frac{2\alpha}{\omega-1}}^{1} dv + \int_{1-2\alpha}^{1} d\omega \int_{\frac{\omega-1+2\alpha}{\omega+1}}^{1} dv\right] \frac{\rho_K(\omega)}{n_K} \frac{\Lambda_K^2}{\sigma^3} \ .}$$

$$\boxed{\Rightarrow \psi_K^{\uparrow\downarrow}(x, k_\perp^2) \sim \int d\omega \ \cdots \rho_K(\omega) \cdots}$$

The Nakanishi weight $\rho_K(z)$ can be modeled...
...Or taken with BSE solutions as an input!

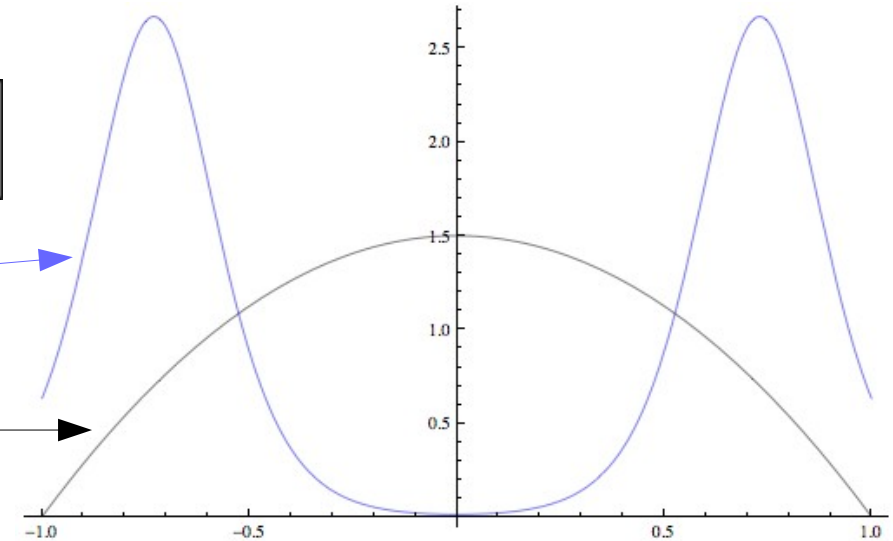
Integral representation of LFWFs: pion case

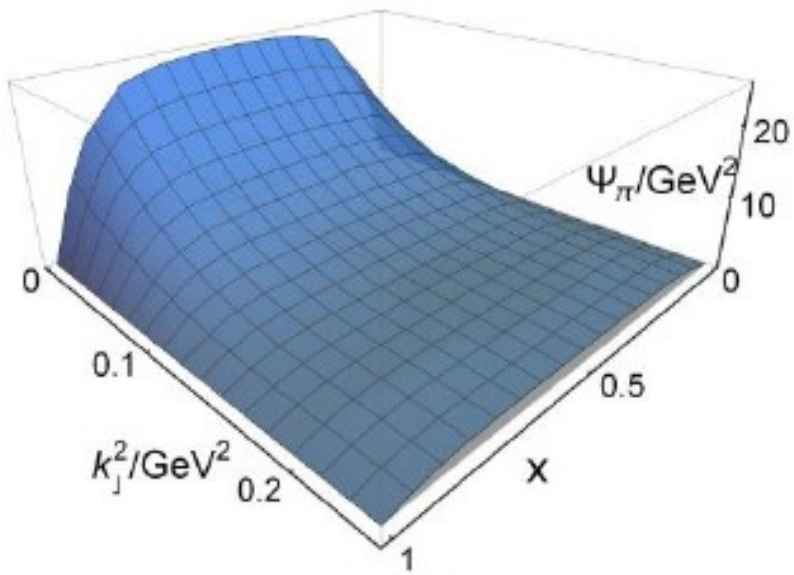
Nakanishi weight parametrization:

$$\rho(z) = \frac{1}{2b_0}\left[\mathrm{sech}^2\left(\frac{z-z_0}{2b_0}\right) + \mathrm{sech}^2\left(\frac{z+z_0}{2b_0}\right)\right]$$

Phenomelogical model: $b_0^\pi = 0.1, z_0^\pi = 0.73;$

Asymptotic case: $\rho(z) = (1 - z^2)$





S-S Xu et al., PRD97(2018)094014

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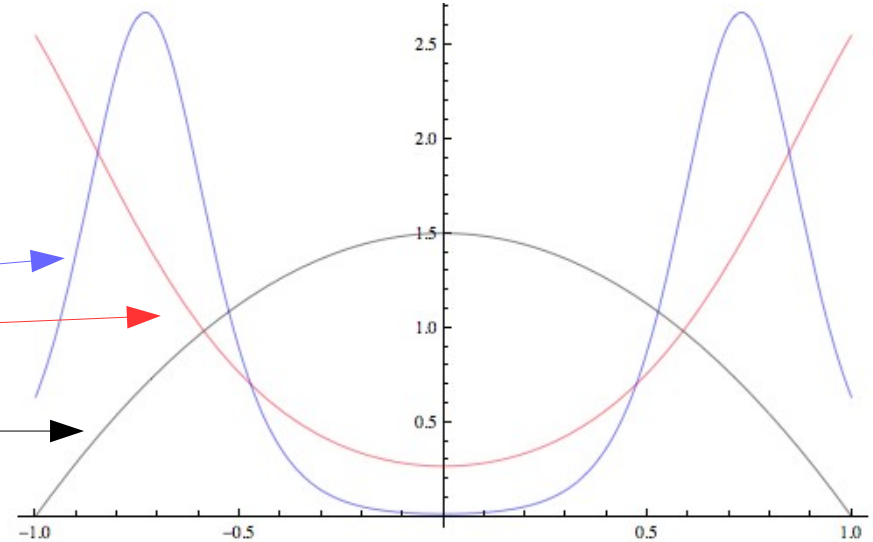
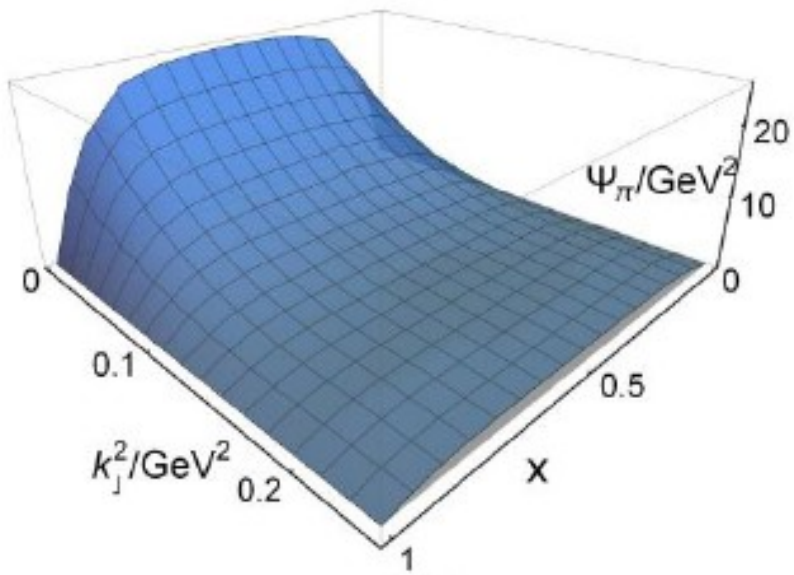
Phenomelogical model: $b_0^\pi = 0.1, z_0^\pi = 0.73;$

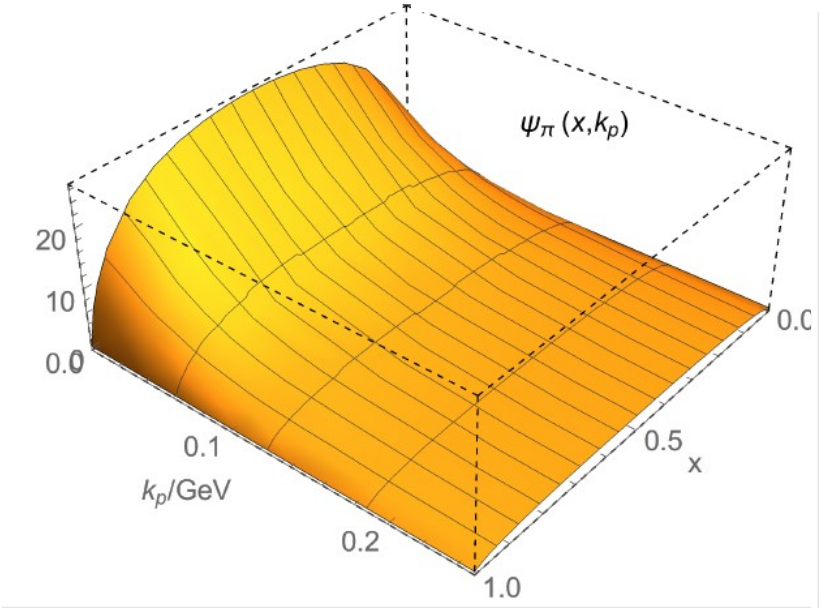
Realistic case: $b_0^\pi = 0.275, z_0^\pi = 1.23;$

Asymptotic case: $\rho(z) = (1 - z^2)$



See Khépani's talk: PDF as a benchmark!



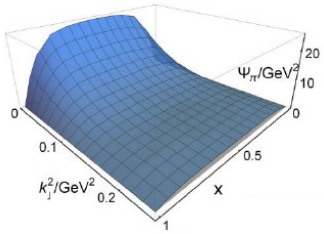


S-S Xu et al., PRD97(2018)094014

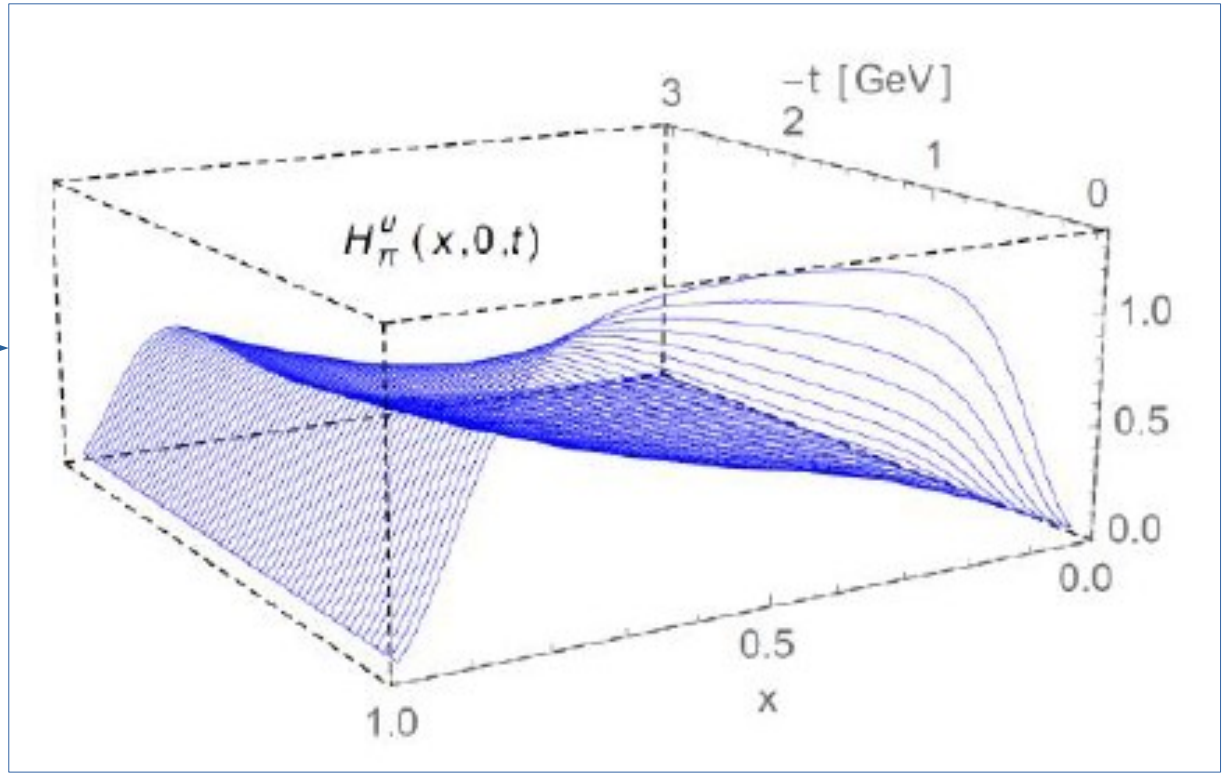
GPD overlap approach: pion case

The overlap valence-quark GPD for a meson in the DGLAP kinematic region reads

$$H^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$



Phenomenological model



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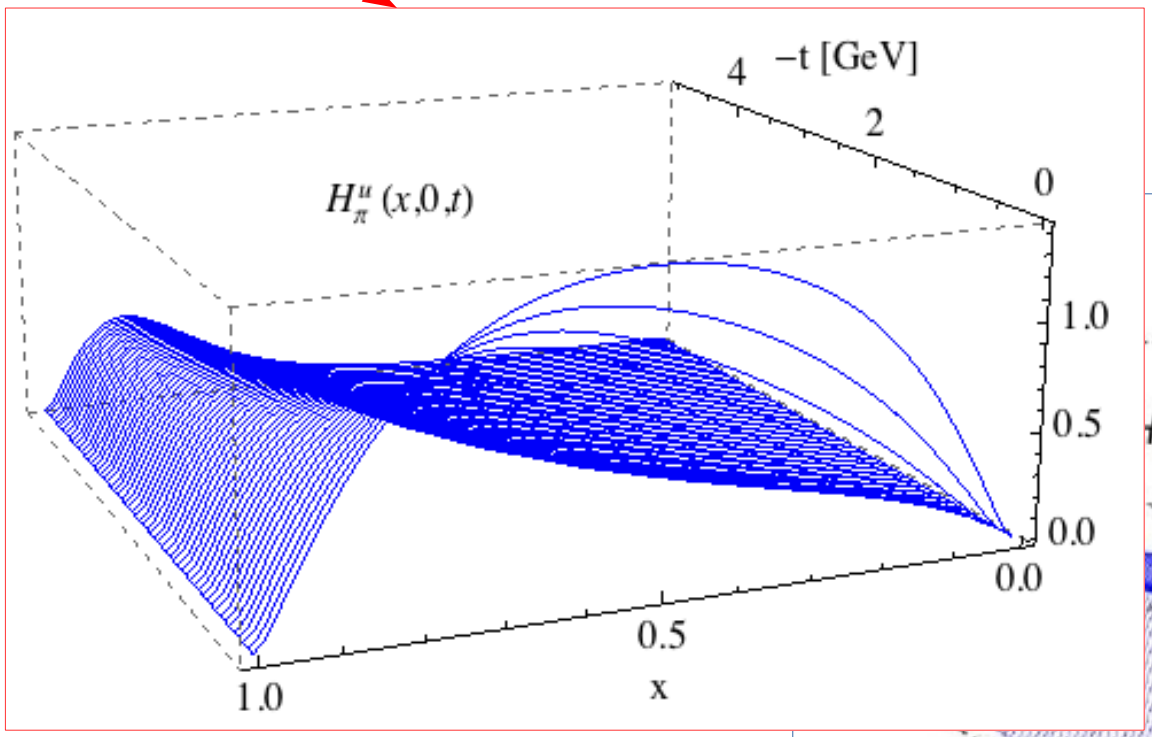
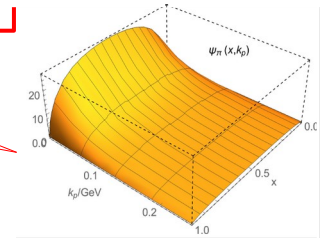


Realistic case



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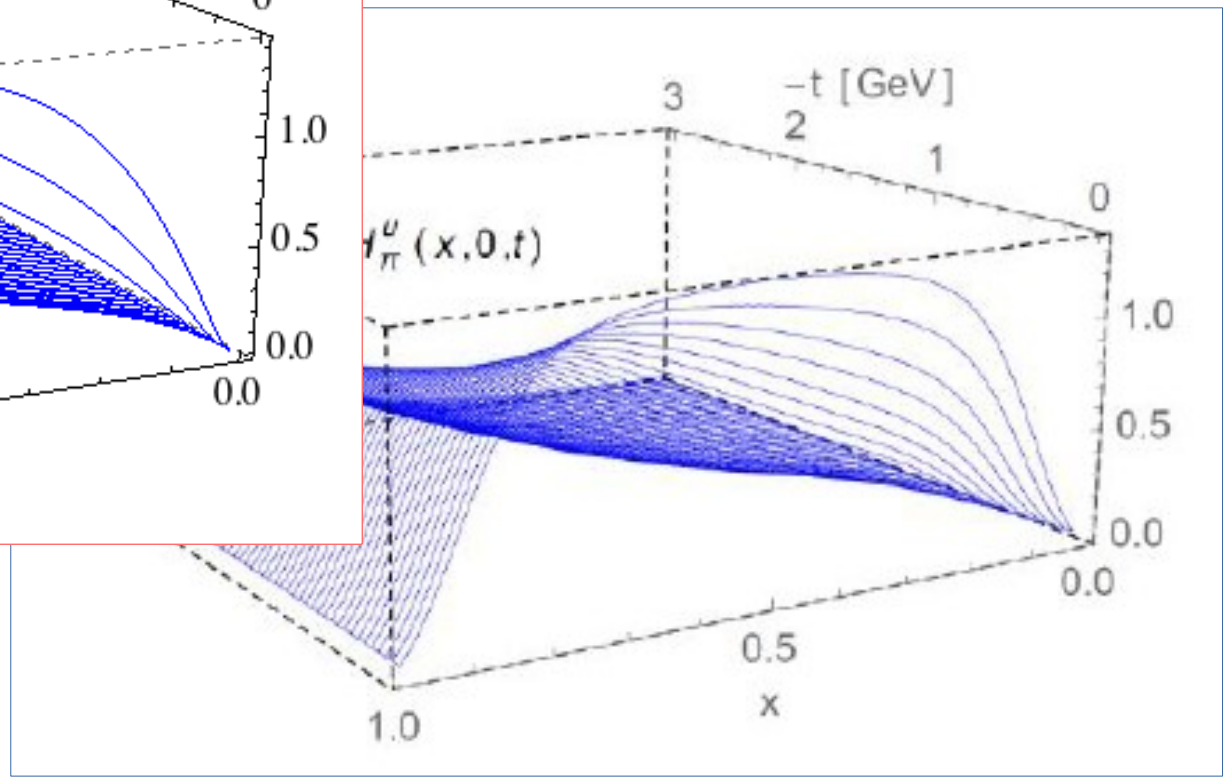

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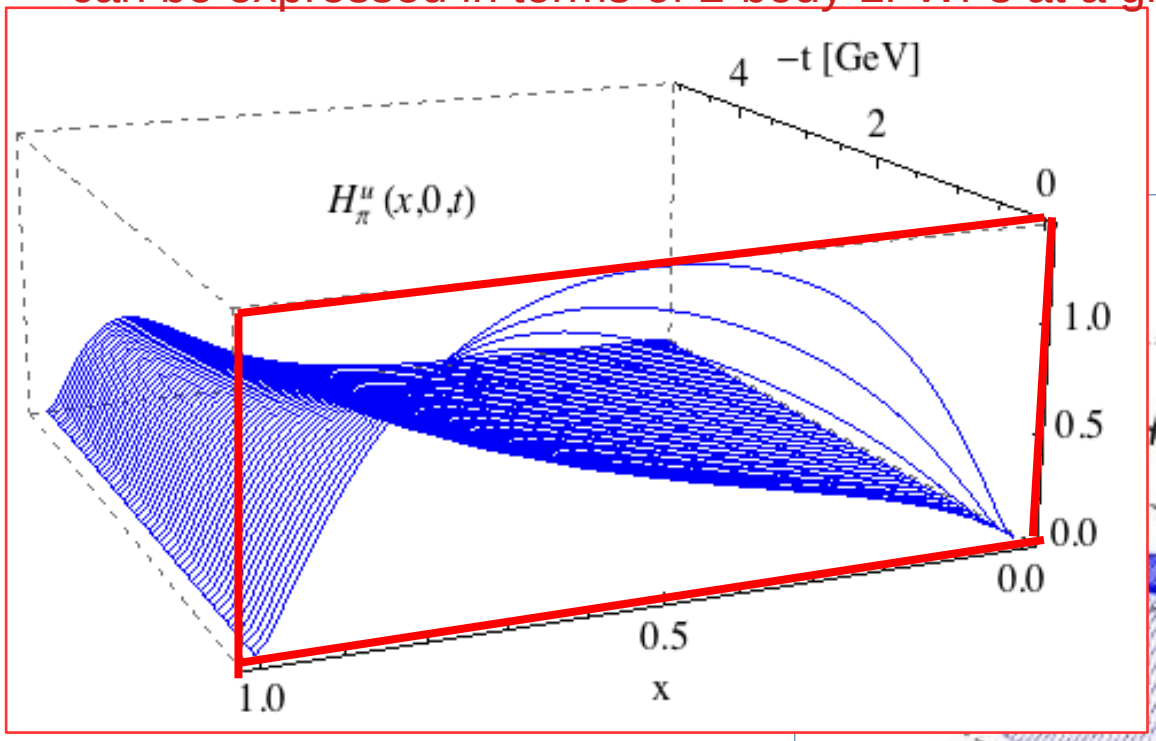
Focus on the forward limit: the PDF that, in the overlap representation at low Fock space, can be expressed in terms of 2-body LFWFs at a given hadronic scale



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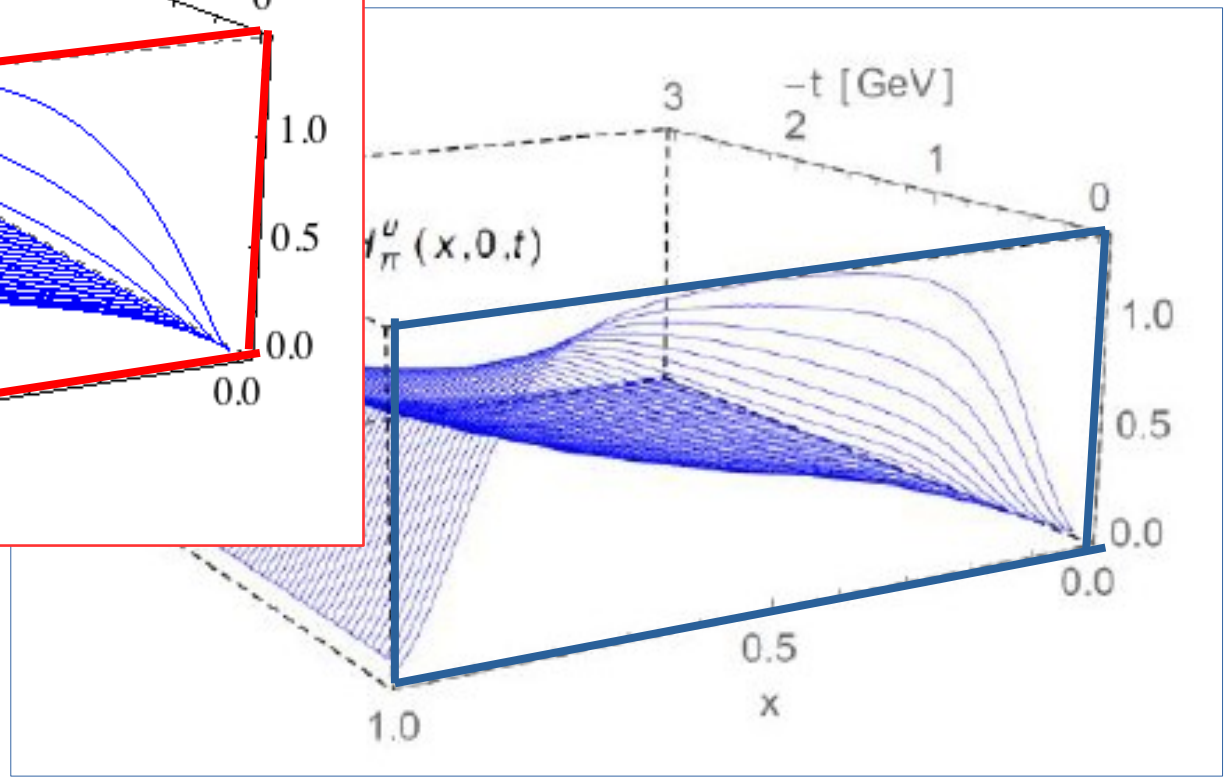

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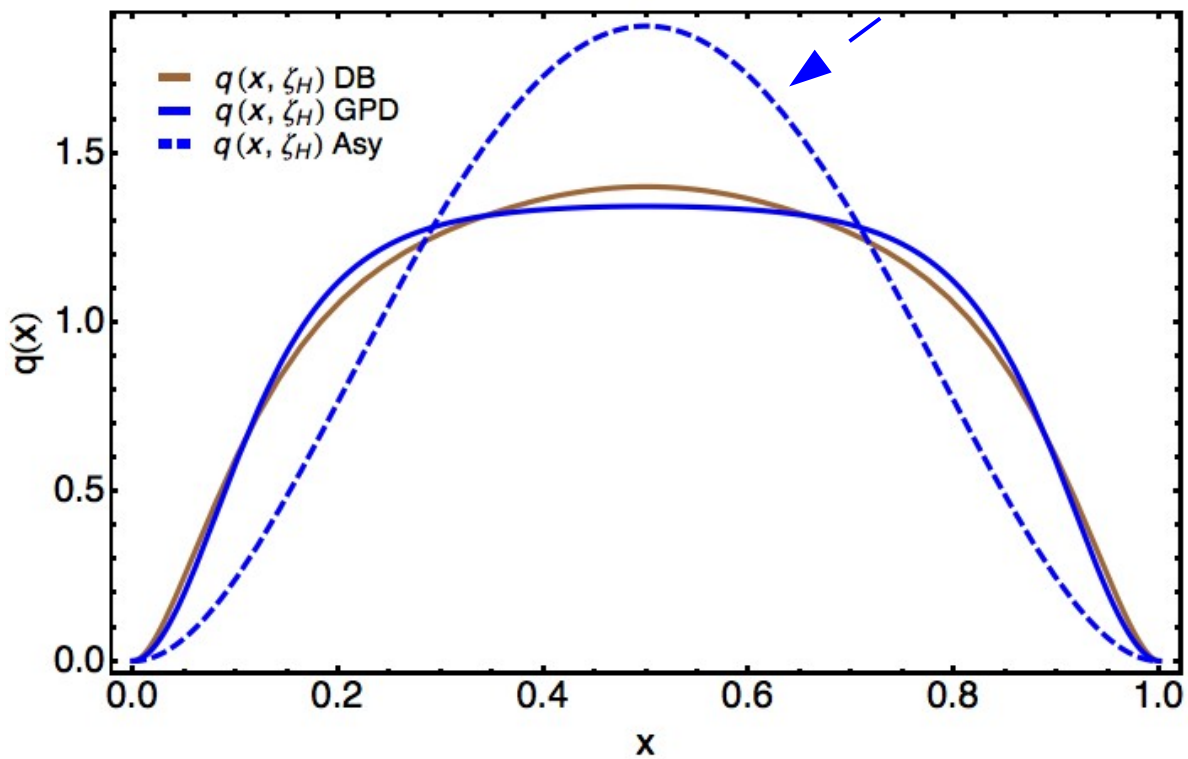
$$q^\pi(x;\zeta_H) = 30x^2(1-x)^2$$

LFWF leading to asymptotic PDA



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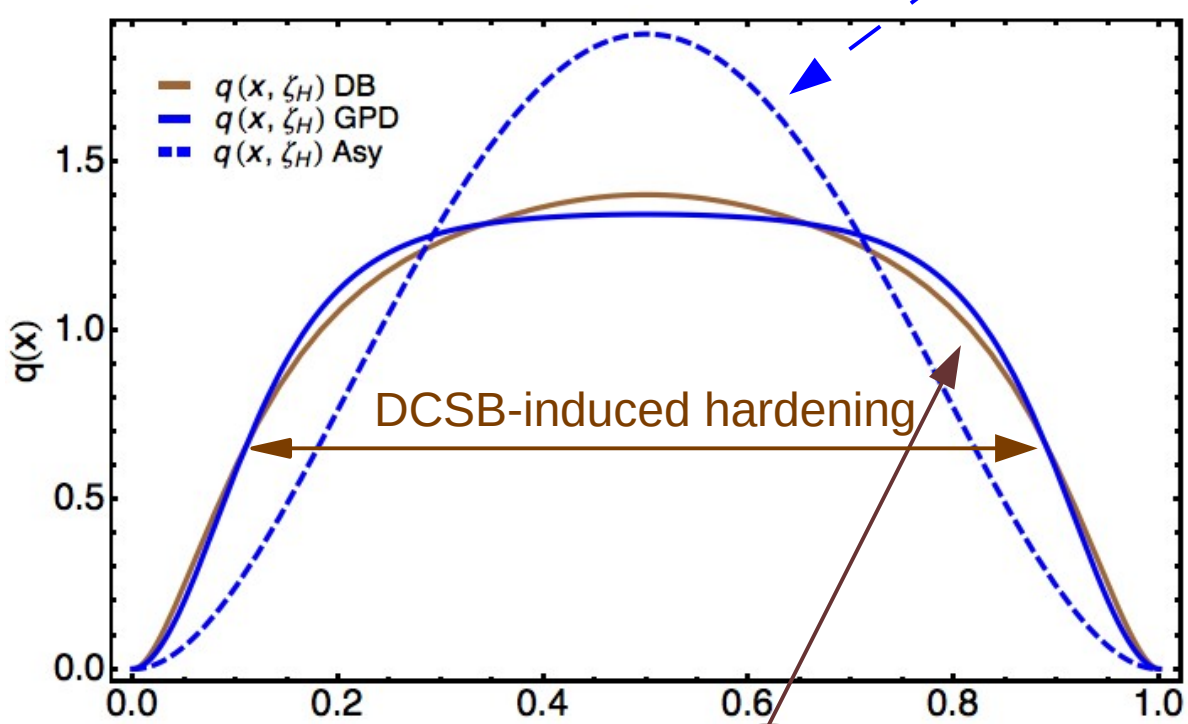
LFWF leading to asymptotic PDA

cf. Ding's & Chang's talk



Direct computation of Mellin moments:

$$\langle x^m\rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$

$$= \frac{N_c}{n\cdot P}\mathrm{tr}\int_{dk}\left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar\eta},P)\,S(k_{\bar\eta})\,n\cdot\partial_{k_\eta}\left[\Gamma_\pi(k_\eta,-P)S(k_\eta)\right]$$

M. Ding et al., PRD01(2020)054014

$$q^\pi(x;\zeta_H) = 213.32\,x^2(1-x)^2 \times\left[1-2.9342\sqrt{x(1-x)}+2.2911\,x(1-x)\right]$$

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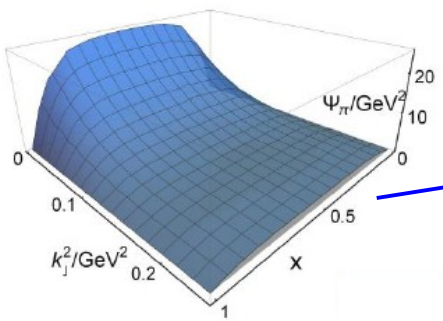

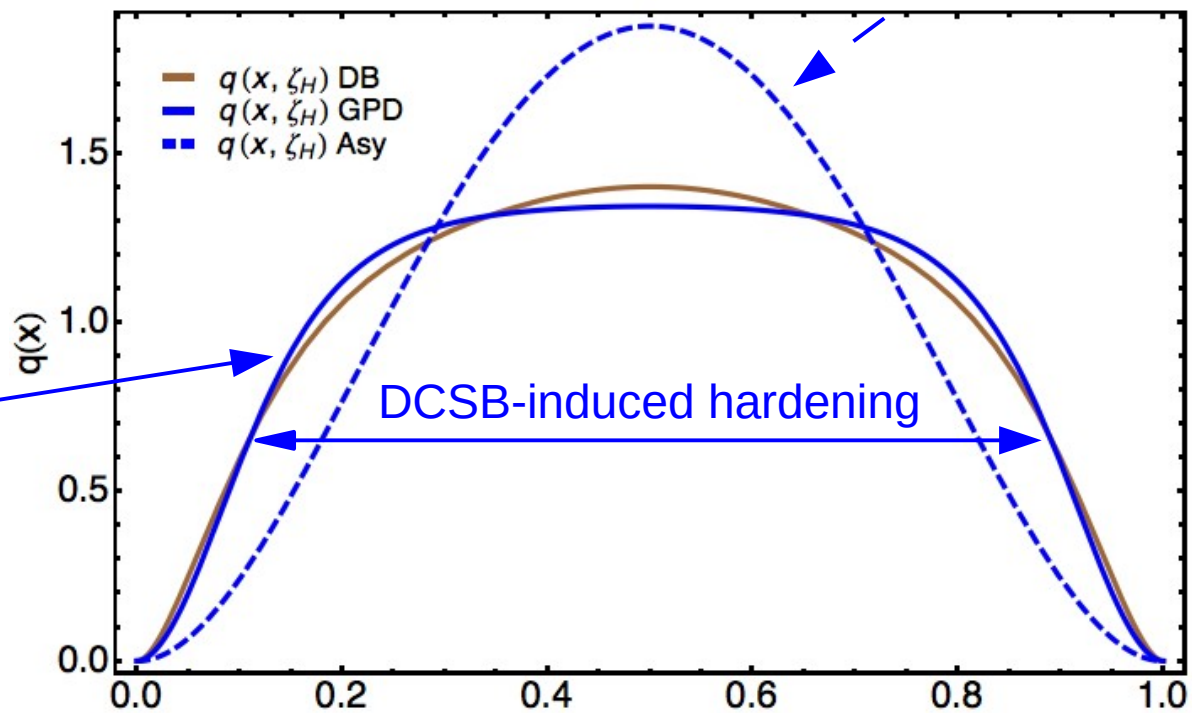

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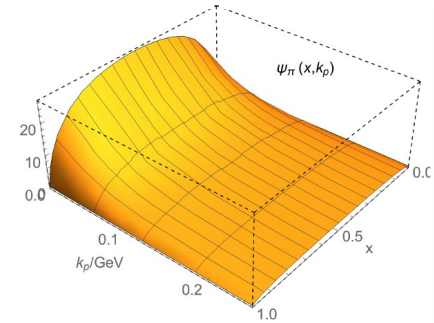

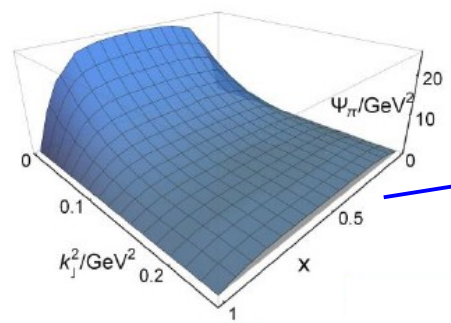

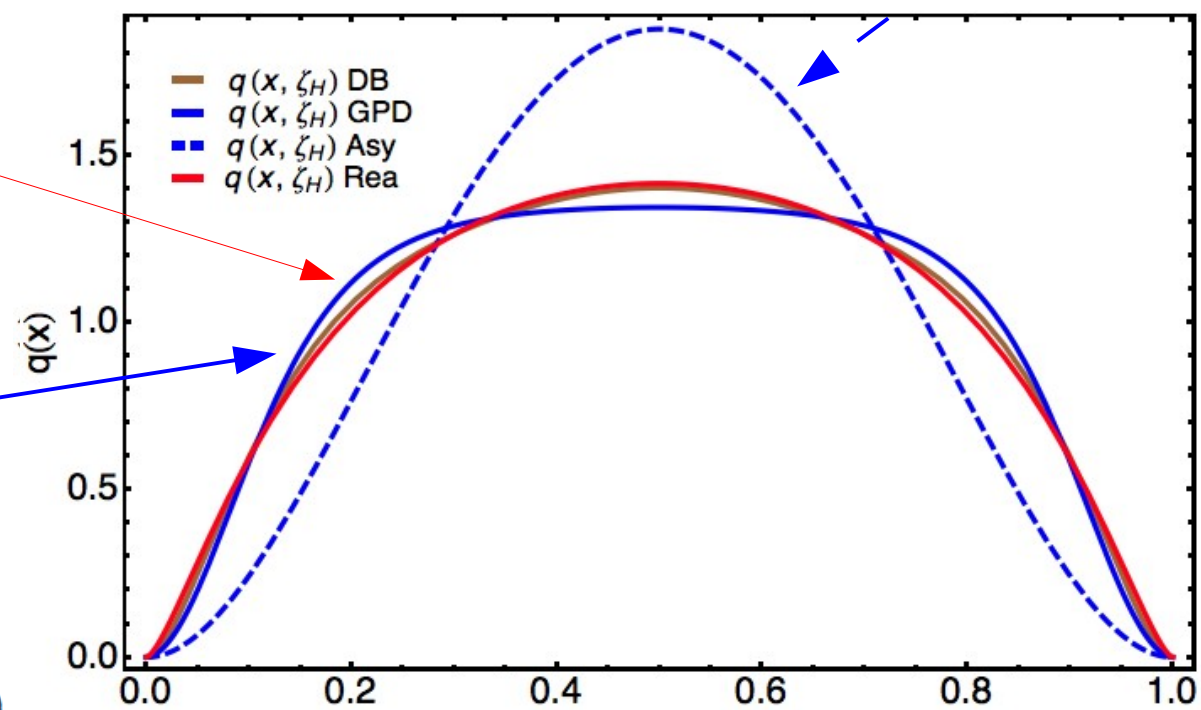

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Comparison with ~~experiment~~: DGLAP evolution pseudo-data

Integro-differential equation for the evolution of non-singlet and singlet combinations

$$\left\{ \zeta^2 \frac{d}{d\zeta^2} \int_0^1 dy \delta(y-x) - \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \begin{pmatrix} P_{qq}^{\rm NS}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\rm S}\left(\frac{\mathbf{x}}{\mathbf{y}}\right) \end{pmatrix} \right\} \begin{pmatrix} H_\pi^{\rm NS,+}(y,t;\zeta) \\ \mathbf{H}_\pi^{\rm S}(y,t;\zeta) \end{pmatrix} = 0$$

DGLAP non-skewed quark GPDs

$\xi = 0\,;\, x\,,\, y > 0$

$$\mathbf{H}_\pi^{\rm S}(y,t;\zeta) = \begin{pmatrix} H_\pi^{\rm S,+}(y,t;\zeta) \\ \frac{1}{x} H_\pi^{g}(y,t;\zeta) \end{pmatrix}$$

$$\mathbf{P}^{\rm S}\left(\frac{x}{y}\right) = \begin{pmatrix} P_{qq}^{\rm S}\left(\frac{x}{y}\right) & 2n_f P_{qg}^{\rm S}\left(\frac{x}{y}\right) \\ P_{gq}^{\rm S}\left(\frac{x}{y}\right) & P_{gg}^{\rm S}\left(\frac{x}{y}\right) \end{pmatrix}$$

mixing of gluon and and singlet quark combination

Comparison with ~~experiment~~: DGLAP evolution pseudo-data

Integro-differential equation for the evolution of non-singlet and singlet combinations that, in Mellin space, reads

$$\left\{ \zeta^2 \frac{d}{d\zeta^2} \mathbb{1} + \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma_{qq}^{(n)} & 0 & 0 \\ 0 & \gamma_{qq}^{(n)} & 2n_f \gamma_{qg}^{(n)} \\ 0 & \gamma_{gq}^{(n)} & \gamma_{gg}^{(n)} \end{pmatrix} \right\} \begin{pmatrix} \langle x^n \rangle_{\rm NS}(\zeta) \\ \langle x^n \rangle_{\rm S}(\zeta) \\ \langle x^n \rangle_g(\zeta) \end{pmatrix} = 0$$

DGLAP non-skewed quark GPDs

$\xi = 0, x, y > 0$

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$

Mellin's moments anomalous dimensions

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Integro-differential equation for the evolution of non-singlet and singlet combinations that, in Mellin space, reads

Let us focus on the valence-quark sector

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Forward limit: Parton DFs

$\xi = 0\,,\ t = 0\,;\ x\,,\ y > 0$

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$

Mellin's moments anomalous dimensions

Comparison with experiment: DGLAP evolution pseudo-data

$$\langle x^n(t)\rangle = \int_0^1 dx\, x^n q(x,z)$$

$$z = \ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

A master equation for the (1-loop) moments' evolution:

$$\frac{d}{dz} q(x,z) = \frac{\alpha(z)}{4\pi} \int_x^1 \frac{dy}{y} q(y,z) P\left(\frac{x}{y}\right) + \ldots$$

$$-\int_0^1 dx\, x^n P(x) = \gamma^{(n)}$$

Moments' evolution (1-loop):

$$\frac{d}{dz}\langle x^n(z)\rangle = -\frac{\alpha(z)}{4\pi}\gamma^{(n)}\langle x^n(z)\rangle + \ldots$$

$$\frac{d}{dz}\alpha(z) = -\frac{\alpha^2(z)}{4\pi}\beta_0 + \ldots$$

$$P(x) = \frac{8}{3}\left(\frac{1+z^2}{(1-x)_+} + \frac{3}{2}\delta(x-1)\right)$$

$$\gamma^{(n)} = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - 4\sum_{i=1}^{n+1}\frac{1}{i}\right)$$

$$\alpha(z) = \frac{4\pi}{\beta_0(z - z_\Lambda)} + \ldots$$

$$z_\Lambda = \ln\left(\frac{\Lambda^2}{\zeta_0^2}\right)$$

$$\langle x^n(z)\rangle = \langle x^n(z_0)\rangle \left(\frac{\alpha(z)}{\alpha(z_0)}\right)^{\gamma^{(n)}/\beta_0}$$

Λ defines the coupling renormalisation scheme (and evolution depends on it because of truncation)

Comparison with experiment: DGLAP evolution pseudo-data

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$$\gamma^{(n)} = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - 4\sum_{i=1}^{n+1}\frac{1}{i}\right)$$

$$\langle x^n(z)\rangle = \langle x^n(z_0)\rangle \exp\left(-\frac{\gamma_0^n}{4\pi}\int_{z_0}^{z} ds\, \alpha(s)\right)$$

The scheme can be defined in such a way that one-loop DGLAP is exact at all orders (Grunberg's effective charge). And we are thus left with a would-be evolution from the nonperturbative hadronic scale up to the experimental one.

Λ defines the coupling renormalisation scheme (and evolution depends on it because of truncation)

Comparison with ~~experiment~~: DGLAP evolution pseudo-data

Integro-differential equation for the evolution of non-singlet and singlet combinations that, in Mellin space, reads

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Forward limit: Parton DFs

$\xi = 0\,,\ t = 0\,;\ x\,,\ y > 0$

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$

Mellin's moments anomalous dimensions

Let us focus on the singlet sector:

$$\int_{z_0}^{z} ds\, \alpha(s)$$

Basis transformation

$$\begin{pmatrix} \langle x^n(\zeta_f) \rangle_{\sum_q q+\bar{q}} \\ \langle x^n(\zeta_f) \rangle_g \end{pmatrix} = W \begin{pmatrix} \exp\left(-\frac{\lambda_+^{(n)}}{4\pi} S(\zeta_0, \zeta_f)\right) & 0 \\ 0 & \exp\left(-\frac{\lambda_-^{(n)}}{4\pi} S(\zeta_0, \zeta_f)\right) \end{pmatrix} W^{-1} \begin{pmatrix} \langle x^n(\zeta_0) \rangle_{\sum_q q+\bar{q}} \\ \langle x^n(\zeta_0) \rangle_g \end{pmatrix}$$

Comparison with ~~experiment~~: DGLAP evolution pseudo-data

Integro-differential equation for the evolution of non-singlet and singlet combinations that, in Mellin space, reads

$$\left\{ \zeta^2 \frac{d}{d\zeta^2} \mathbb{1} + \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma_{qq}^{(n)} & 0 & 0 \\ 0 & \gamma_{qq}^{(n)} & 2n_f \gamma_{qg}^{(n)} \\ 0 & \gamma_{gq}^{(n)} & \gamma_{gg}^{(n)} \end{pmatrix} \right\} \begin{pmatrix} \langle x^n \rangle_{\rm NS}(\zeta) \\ \langle x^n \rangle_{\rm S}(\zeta) \\ \langle x^n \rangle_g(\zeta) \end{pmatrix} = 0$$

Forward limit: Parton DFs

$\xi = 0\,,\ t = 0\,;\ x\,,\ y > 0$

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$

Mellin's moments anomalous dimensions

$$\int_{z_0}^{z} ds\, \alpha(s)$$

Let us focus on the singlet sector:

$$\begin{pmatrix} \langle x^n(\zeta_f) \rangle_{\sum_q q+\bar{q}} \\ \langle x^n(\zeta_f) \rangle_g \end{pmatrix} = W \begin{pmatrix} \exp\left(-\frac{\lambda_+^{(n)}}{4\pi} S(\zeta_0, \zeta_f)\right) & 0 \\ 0 & \exp\left(-\frac{\lambda_-^{(n)}}{4\pi} S(\zeta_0, \zeta_f)\right) \end{pmatrix} W^{-1} \begin{pmatrix} \langle x^n(\zeta_0) \rangle_{\sum_q q+\bar{q}} \\ \langle x^n(\zeta_0) \rangle_g \end{pmatrix}$$

At the hadronic scale, ζ_H, all the momentum is carried out by valence quarks!

$$\begin{pmatrix} \langle x^n(\zeta_0) \rangle_{\sum_q q+\bar{q}} \\ \langle x^n(\zeta_0) \rangle_g \end{pmatrix} \to \begin{pmatrix} \langle x^n(\zeta_H) \rangle_u + \langle x^n(\zeta_H) \rangle_{\bar{d}} \\ 0 \end{pmatrix} \underset{n=1}{\to} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Comparison with ~~experiment~~: DGLAP evolution pseudo-data

Therefore:

- All the Mellin moments for gluon, sea- and valence-quarks can be obtained at any scale from the evolution of those computed at the hadronic scale for the valence-quark (in DSE-BSE or LFWF approach). One only needs to know the first valence-quark moment a this scale. E.g.,

$$\langle x^n(\zeta_f)\rangle_q = \langle x^n(\zeta_H)\rangle_q \left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_q^{(1)}} = \langle x^n(\zeta_H)\rangle_q \left(\langle 2x(\zeta_f)\rangle_q\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

- In the case of the momentum fraction averages, the expressions are very simple:

$$\begin{aligned}
\langle x(\zeta_f)\rangle_{\text{sea}} &= \langle x(\zeta_f)\rangle_{\sum_q q+\bar{q}} - \left(\langle x(\zeta_f)\rangle_u + \langle x(\zeta_f)\rangle_{\bar{d}}\right), \\
&= \boxed{\frac{3}{7} + \frac{4}{7}\langle 2x(\zeta_f)\rangle_u^{7/4} - \langle 2x(\zeta_f)\rangle_u} \\
\langle x(\zeta_f)\rangle_g &= \boxed{\frac{4}{7}\left(1 - \langle 2x(\zeta_f)\rangle_u^{7/4}\right)}
\end{aligned}$$

- Sum rule: $\boxed{\langle 2x(\zeta_f)\rangle_q + \langle x(\zeta_f)\rangle_{\text{sea}} + \langle x(\zeta_f)\rangle_g = 1}$

Comparison with ~~experiment~~: DGLAP evolution pseudo-data

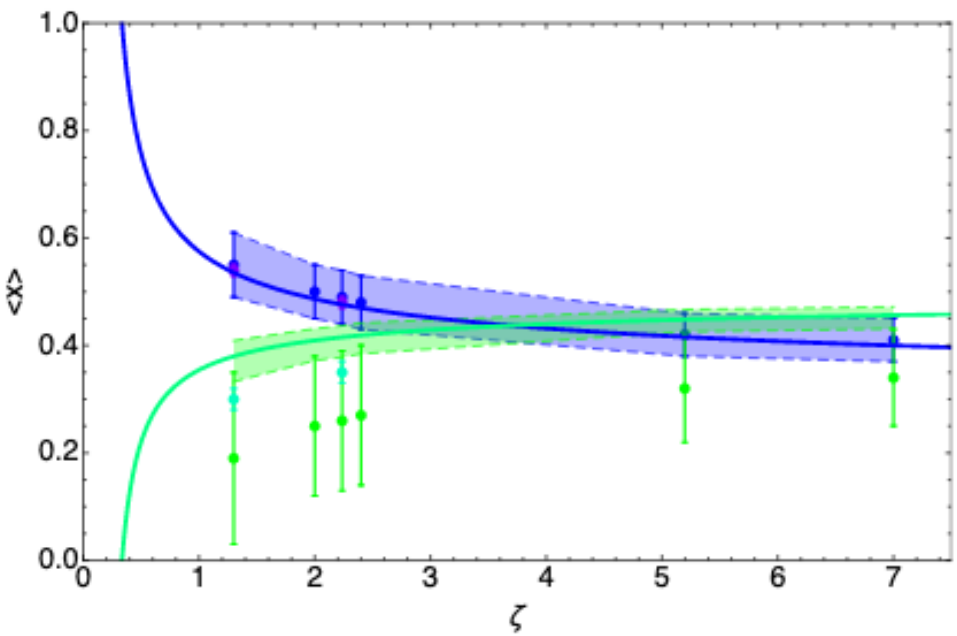
$$\langle x(\zeta_f)\rangle_{\rm sea} = \langle x(\zeta_f)\rangle_{\sum_q q+\bar{q}} - (\langle x(\zeta_f)\rangle_u + \langle x(\zeta_f)\rangle_{\bar{d}}) \,,$$

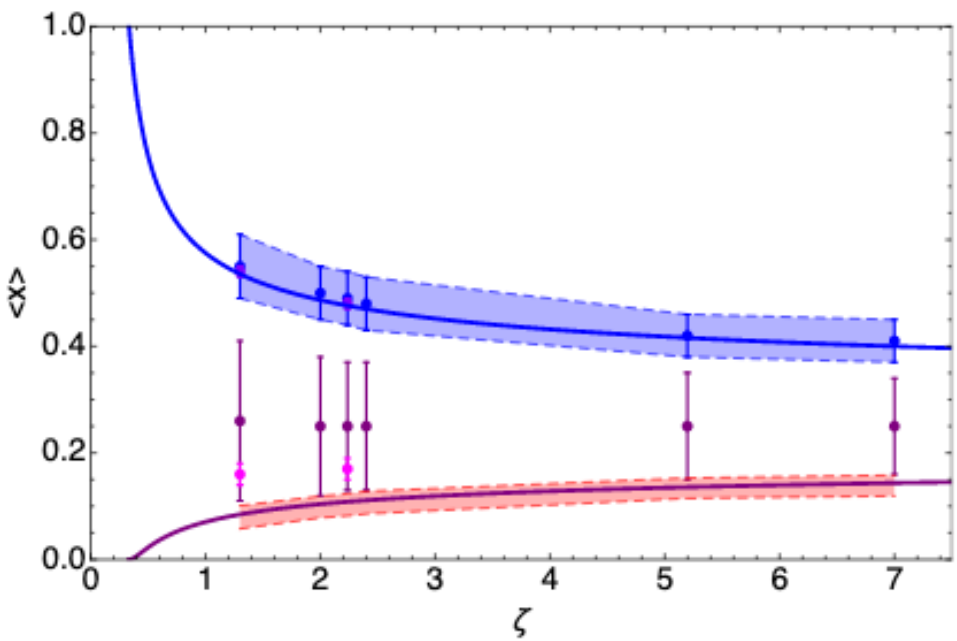
$$= \boxed{\frac{3}{7} + \frac{4}{7}\langle 2x(\zeta_f)\rangle_u^{7/4} - \langle 2x(\zeta_f)\rangle_u}$$

$$\langle x(\zeta_f)\rangle_g = \boxed{\frac{4}{7}\left(1 - \langle 2x(\zeta_f)\rangle_u^{7/4}\right)}$$

Data from Novikov et al., arXiv:2002.02902:

	$\langle xv\rangle$	$\langle xS\rangle$	$\langle xg\rangle$	Q^2 (GeV^2)
JAM [26]	0.54 ± 0.01	0.16 ± 0.02	0.30 ± 0.02	1.69
JAM (DY)	0.60 ± 0.01	0.30 ± 0.05	0.10 ± 0.05	1.69
this work	0.55 ± 0.06	0.26 ± 0.15	0.19 ± 0.16	1.69
Lattice-3 [16]	0.428 ± 0.030			4
SMRS [20]	0.40 ± 0.02			4
Han et al. [42]	0.428 ± 0.03			4
DSE [7]	0.52			4
this work	0.50 ± 0.05	0.25 ± 0.13	0.25 ± 0.13	4
JAM	0.48 ± 0.01	0.17 ± 0.01	0.35 ± 0.02	5
this work	0.49 ± 0.05	0.25 ± 0.12	0.26 ± 0.13	5
Lattice-1 [14]	0.558 ± 0.166			5.76
Lattice-2 [15]	0.48 ± 0.04			5.76
this work	0.48 ± 0.05	0.25 ± 0.12	0.27 ± 0.13	5.76
WRH [21]	0.434 ± 0.022			27
ChQM-1 [11]	0.428			27
ChQM-2 [13]	0.46			27
this work	0.42 ± 0.04	0.25 ± 0.10	0.32 ± 0.10	27
SMRS [20]	0.49 ± 0.02			49
this work	0.41 ± 0.04	0.25 ± 0.09	0.34 ± 0.09	49



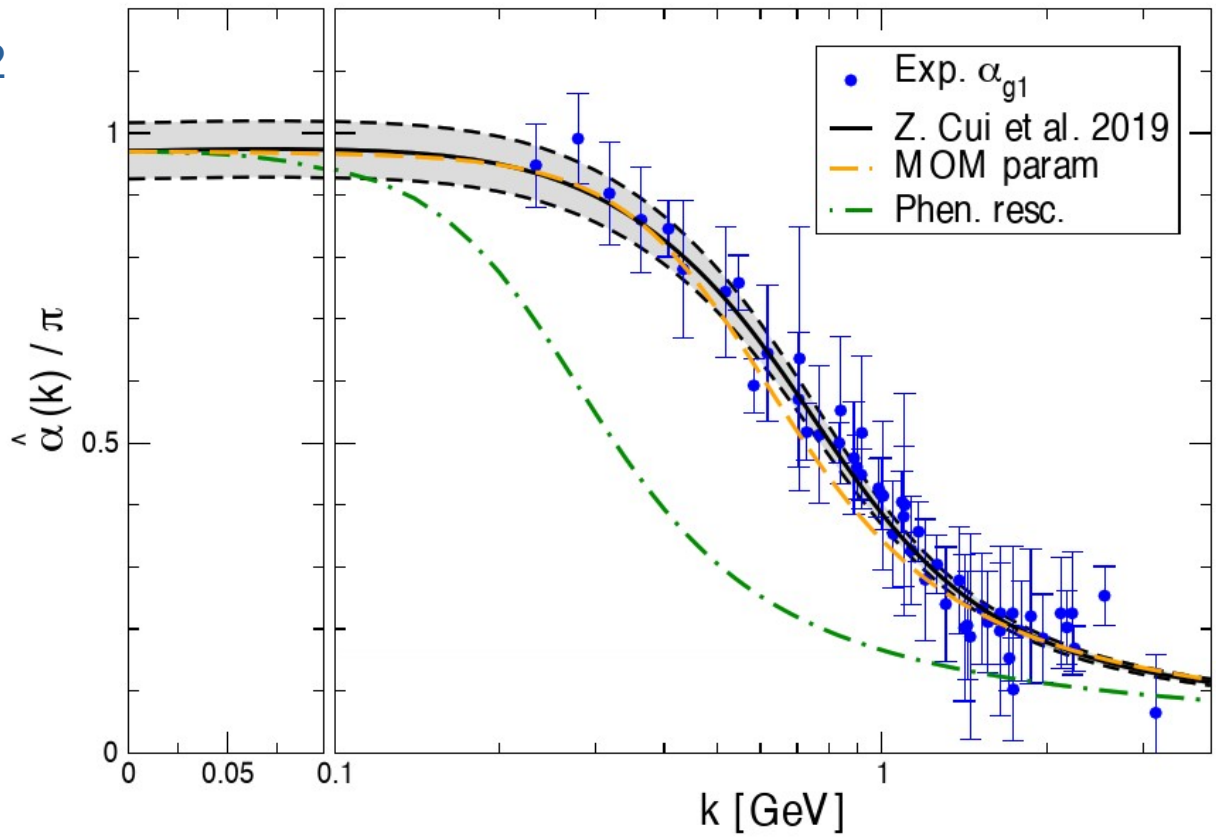


Comparison with ~~experiment~~: PI effective charge pseudo-data

D.B et al., PRD96(2017)054026
J.R-Q et al., FBS59(2018)121
Z-F Cui et al., arXiv:1912.08232

Process-independent charge, defined as an analogue of the QED Gell-Man-low, on the basis of the PT-BFM truncation of DSEs in the gluon sector

Gauge-independent, no Landau pole, fully determined by the gluon sector, known to unify a wide range of observables, it compares very well with the Bjorken sum rule charge…



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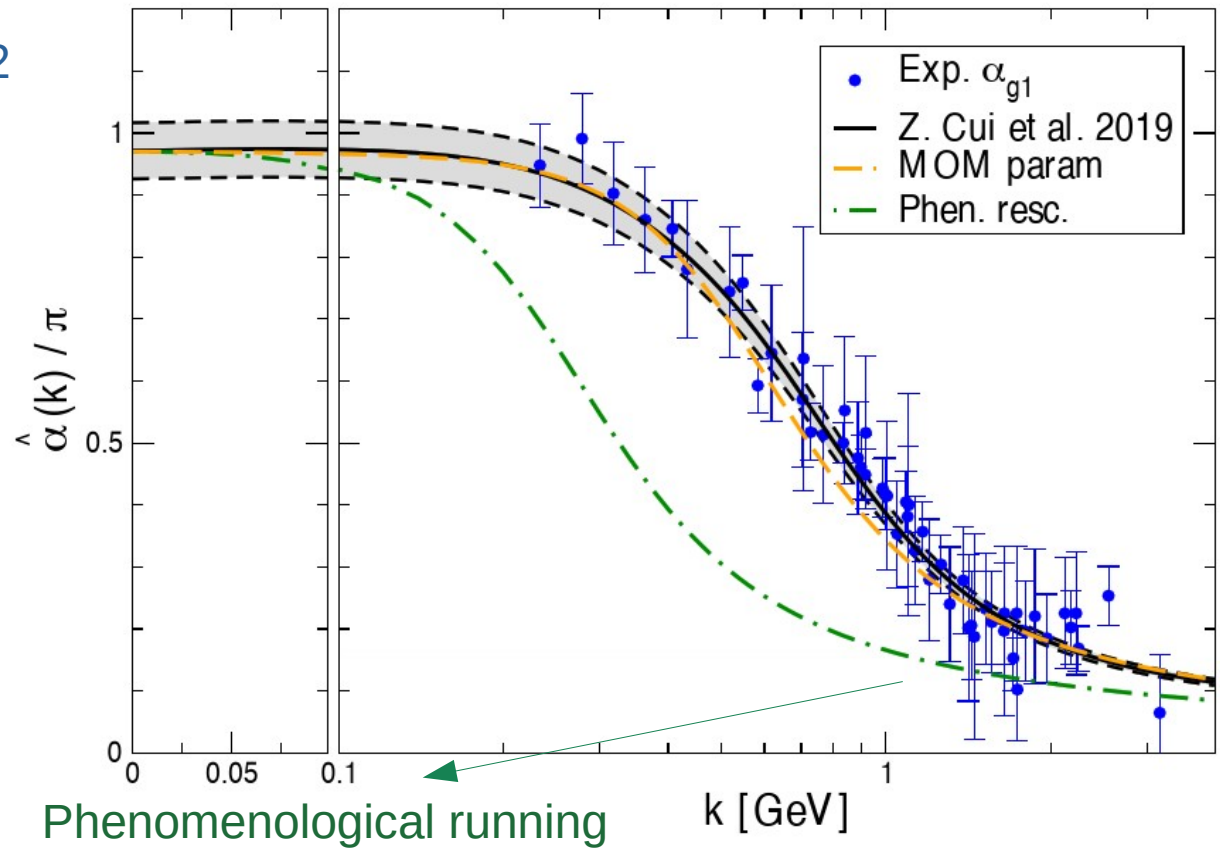


Assumption: The PI charge and the one for the all-order DF evolution correspond with each other within the IR, while at large momenta the latter is defined by the phenomenological value of $\Lambda_{QCD} = 234$ MeV.

$$\boxed{\zeta_H = m_G = 0.331(2)\ \text{GeV}}$$

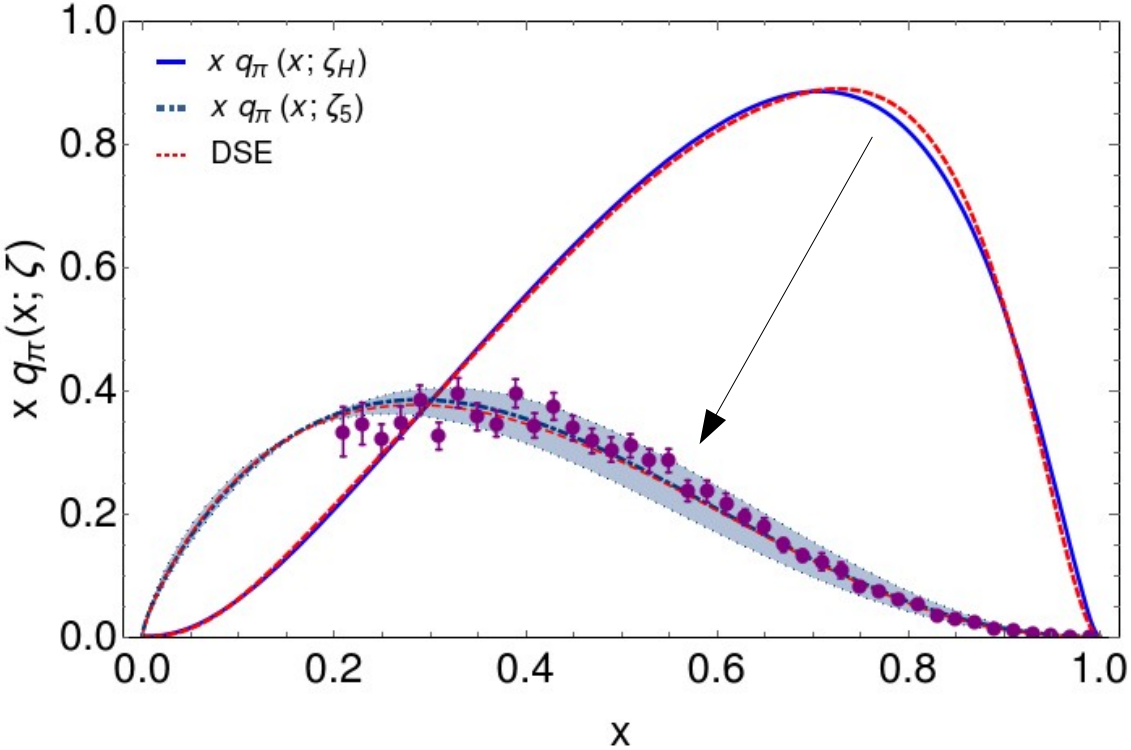
Comparison with ~~experiment~~: pseudo-data

$$\langle x(z)\rangle = \langle x(z_H)\rangle \exp\left(-\frac{\gamma^{(1)}}{4\pi}\int_{z_H}^{z} ds\,\alpha(s)\right)$$

Then, one can evolve the pion PDF, by using the effective charge, from the hadronic scale up to the relevant one for the E615 experiment:

[Aicher et al., PRL105(2010)252003]

$$\zeta_H \equiv m_G \rightarrow \zeta_2 = 5.2 \text{ GeV}$$



After identifying $m_0 \equiv \zeta_H$, all the scales (and the evolution between them) appear thus fixed. And the agreement with E615 ~~data~~ pseudo-data is perfect!!!

Comparison with ~~experiment~~: pseudo-data

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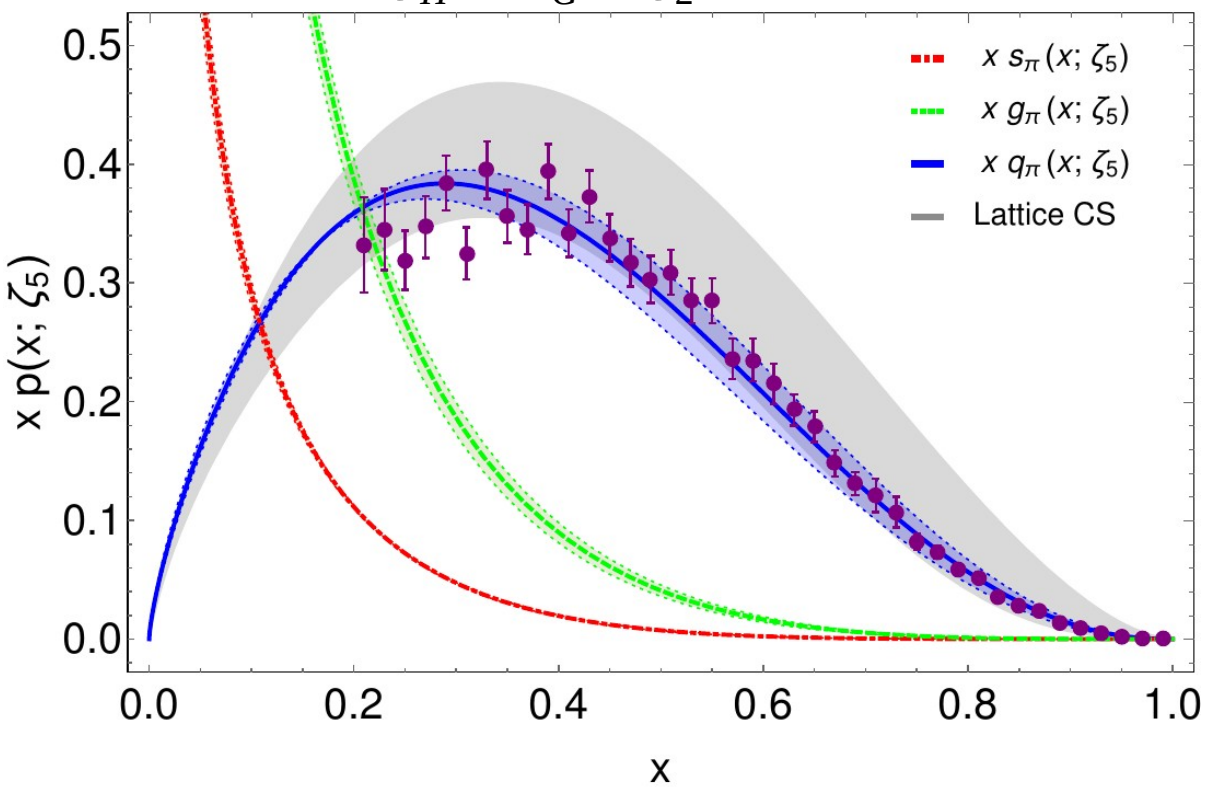


Lattice: [Sufian et al., arXiv:2001.04960]

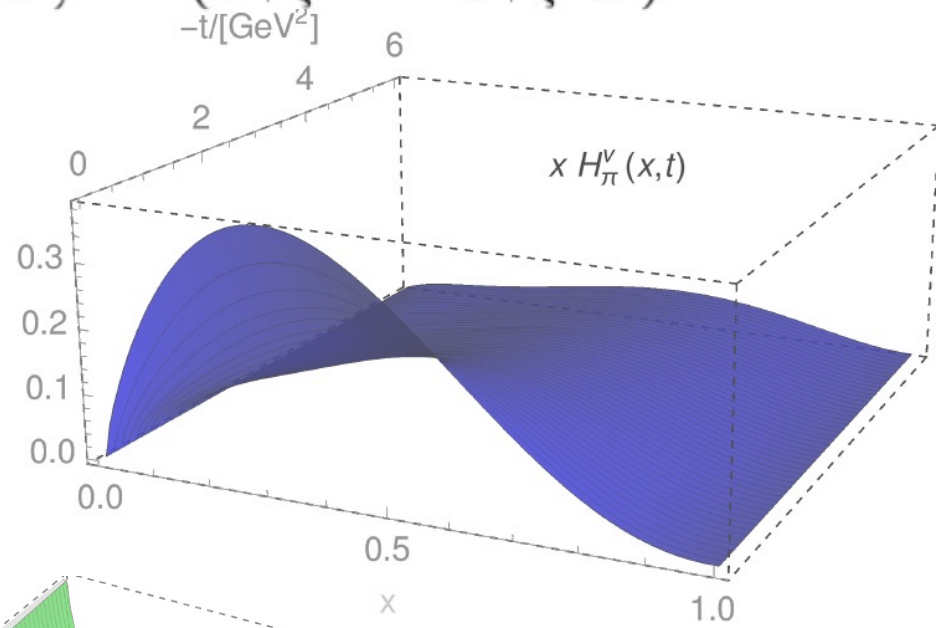
ζ	$\langle x\rangle_V$	$\langle x\rangle_G$	$\langle x\rangle_S$
2 GeV	0.483(42)	0.411(24)	0.106(18)
5 GeV	0.412(36)	0.449(19)	0.138(17)

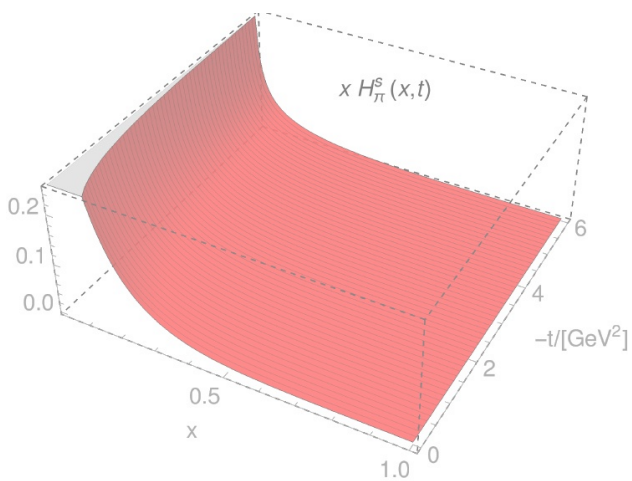
Evolved GPDs: pion case

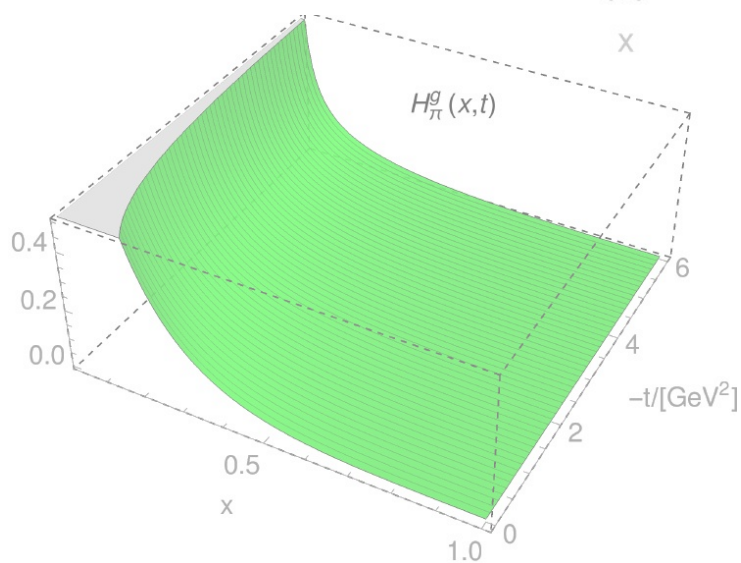
The overlap valence-quark GPD for a meson in the DGLAP kinematic region reads

$$H^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

$\zeta_0 = \zeta_H = 0.33$ GeV $\rightarrow \zeta_5 = 5.2$ GeV



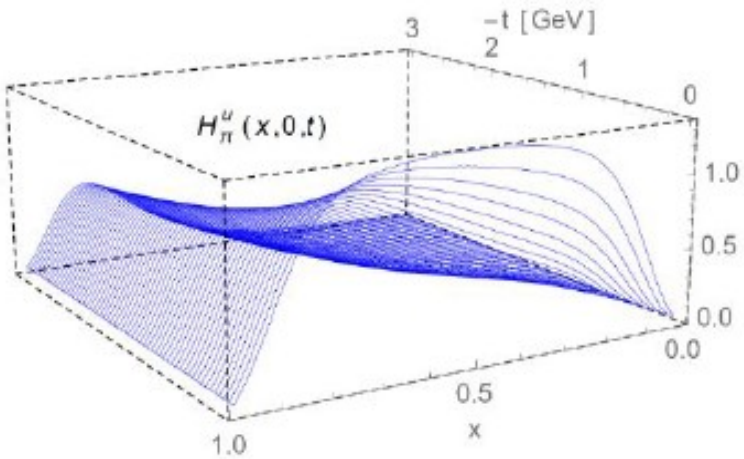


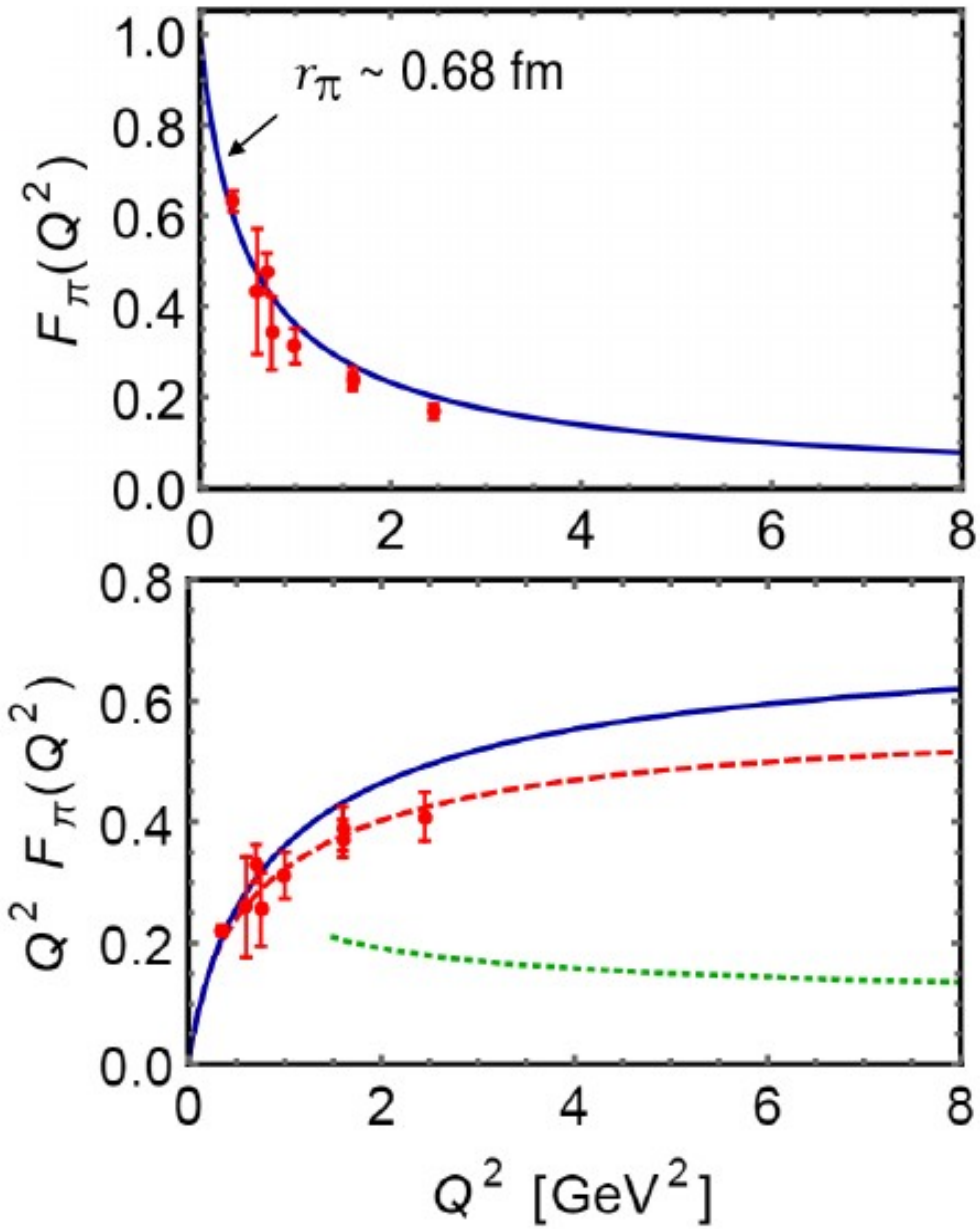


Pion realistic picture: Electromagnetic Form Factor

$$F_M(\Delta^2) = e_u F_M^u(\Delta^2) + e_f F_M^f(\Delta^2)\ ,\ F_M^q(-t=\Delta^2) = \int_{-1}^{1} dx\ H_M^q(x,\xi,t)$$

Electric charges





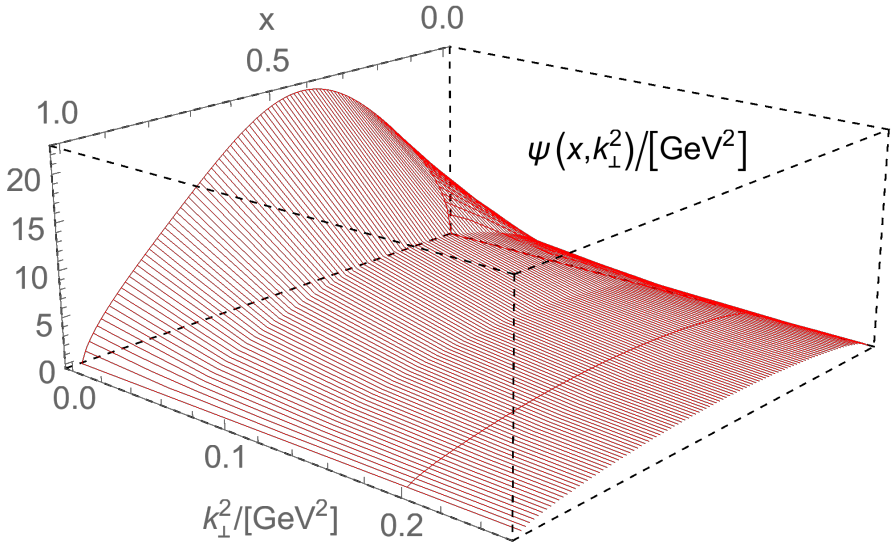
Blue: Computed from GPD

Green: Computed from HS formula

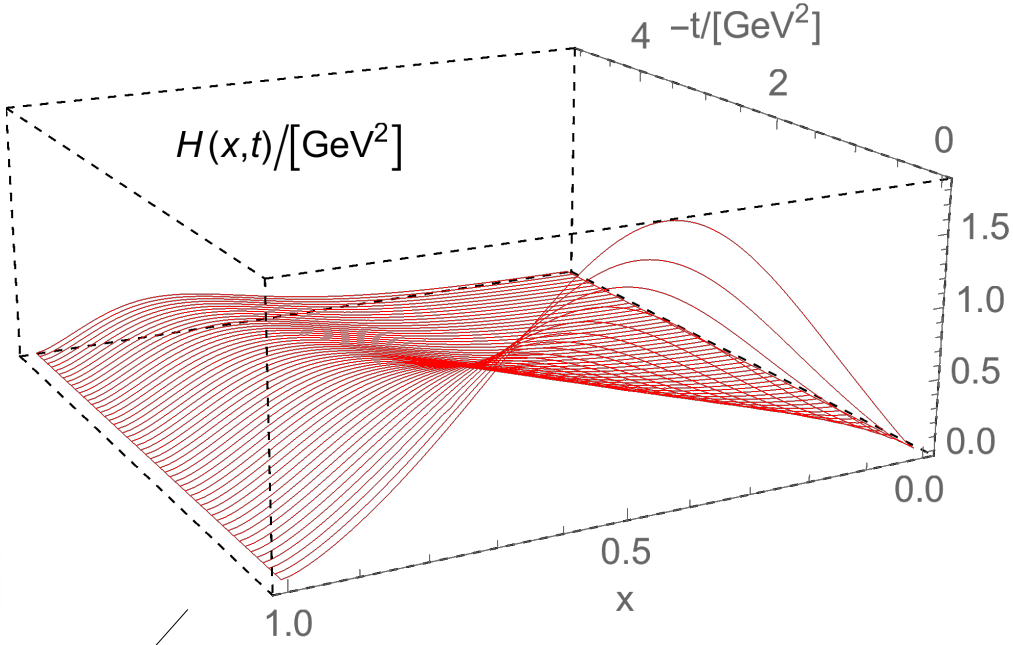
Red: '*Evolved*' form factor

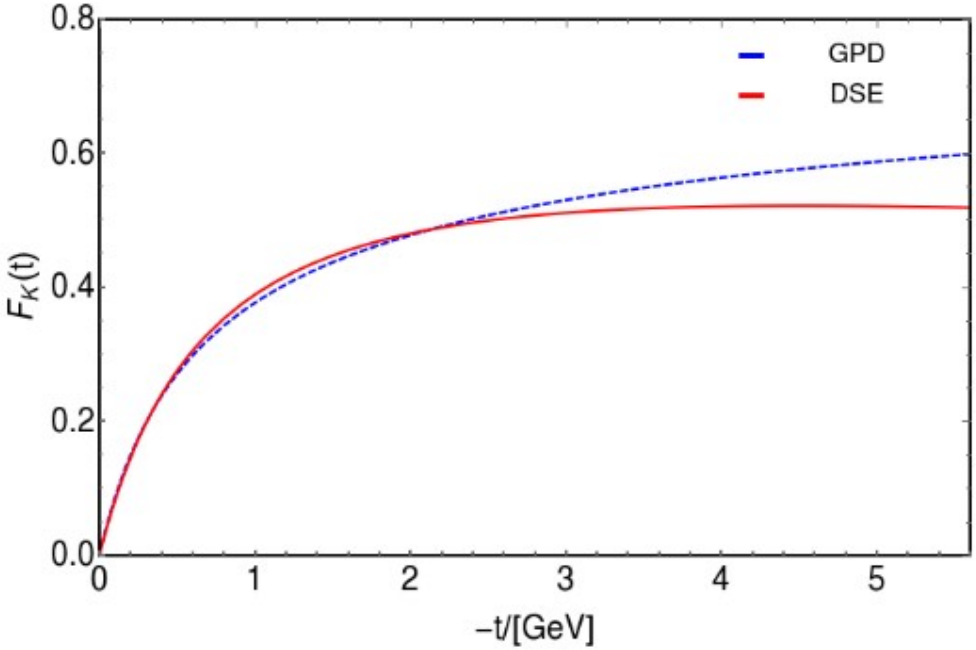
Kaon preliminary results:

Identifying first the LFWF (cf. Khépani's talk)



one obtains the DGLAP GPD:





and the electromagnetic form factor.

A word about gravitational form factors:

First, polynomiality:

$$\int_{-1}^{1} dx\, x^m\; H(x,\xi,t) \;=\; \sum_{k=0}^{m+1} C_k^{(m)}(t)\; \xi^k$$

(Time reversal symmetry implies k even)

If one defines a function D such that:

$$\int_{-1}^{1} dx\; x^m D(x,t) = C_{m+1}^{m}(t)$$

where $D(x,t) = 0 \quad \forall x \in [-\infty,-1)U(1,\infty]$

$$\int_{-1}^{1} dx\; x^m \left(H(x,\xi,t) - \mathrm{sign}(\xi) D\left(\frac{x}{\xi}\right)\right) = \sum_{k=0}^{m} C_k^{(m)}(t))\; \xi^k$$

$$\boxed{\int_{\Omega} d\beta d\alpha\; h_{PW}(\beta,\alpha;t)\delta(x-\beta-\alpha\xi) \;=\; \mathcal{R}\left[h_{\mathrm{PW}}\right]}$$

$$\frac{1}{|\xi|} D\left(\frac{x}{\xi},t\right) = \int_{\Omega} d\beta d\alpha\; \delta(\beta) D(\alpha,t)\delta(x-\beta-\alpha\xi) \;=\; \mathcal{R}\left[\delta D)\right]$$

PW D-term
(Pure ERBL contribution)

Specializing for the case m=1

$$\int_{-1}^{1} dx\; x\; H(x,\xi,t) \;=\; c_0^{(1)}(t) \;+\; \xi^2 \int_{-1}^{1} dz\; z\; D(z,t)$$

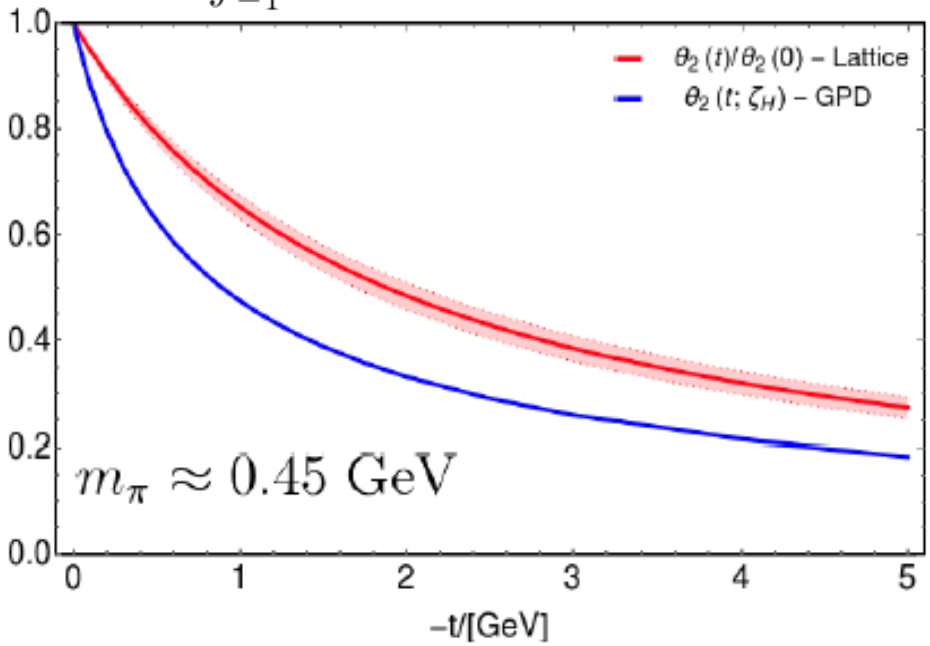
A word about gravitational form factors:

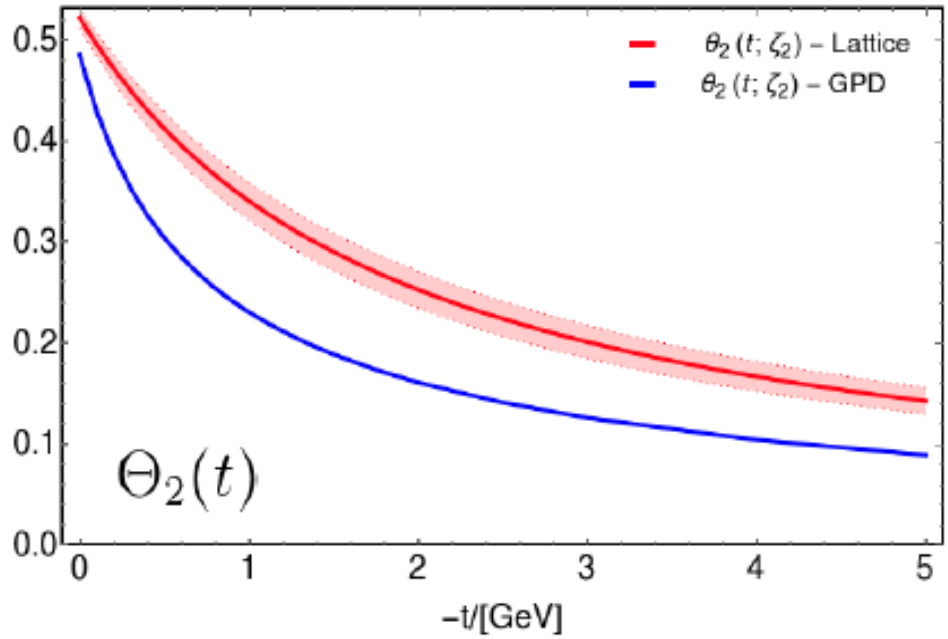
$$\int_{-1}^{1} dx\ x\ H(x,\xi,t) = \boxed{c_0^{(1)}(t)} + \xi^2 \int_{-1}^{1} dz\ z\ D(z,t)$$

$$\boxed{\frac{1}{2}\theta_2(t)}$$

Isospin-symmetric limit

$$\theta_2(t) = \int_{-1}^{1} dx\ x\left(H^u_{\pi^+}(x,0,t) + H^d_{\pi^+}(-x,0,t)\right)$$





Lattice: (2007) Brömmel's dissertation. $\Theta_2(0)/2 =< x >= 0.261(5)$

GPD + Ding et al. $\Theta_2(0)/2 =< x >= 0.242(20)$

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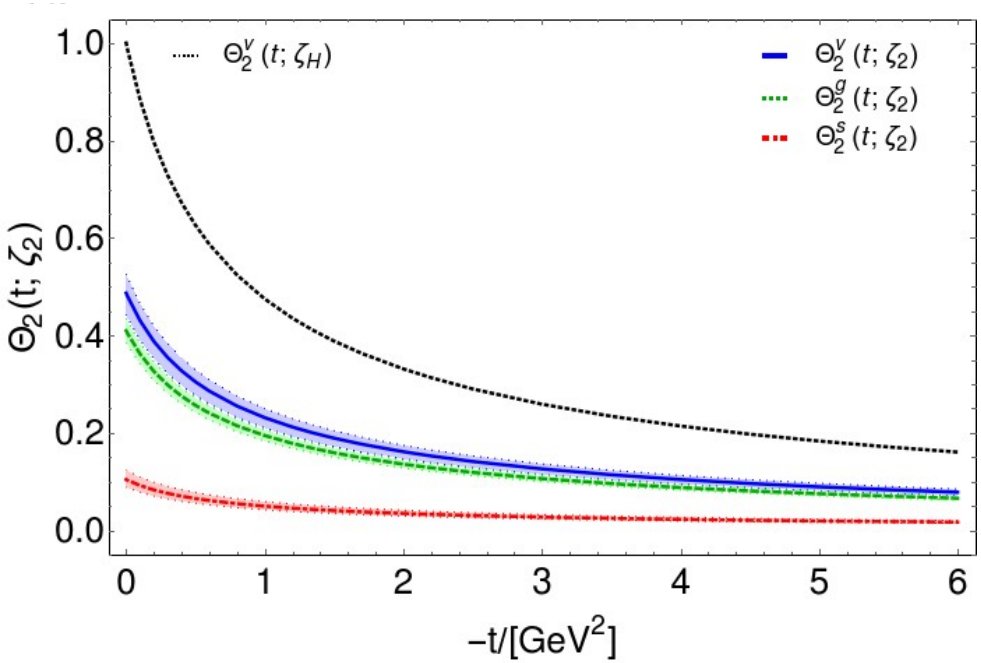

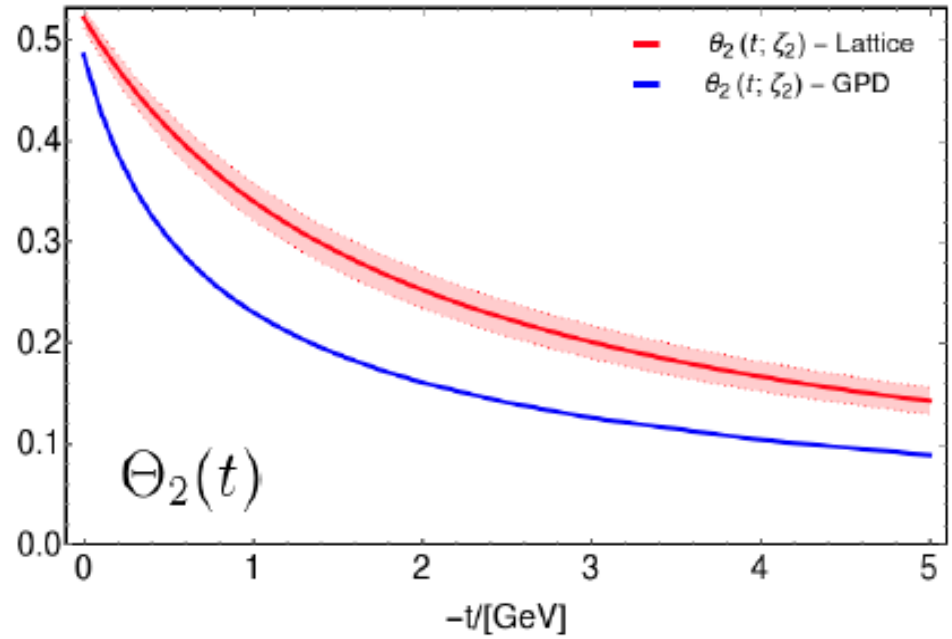

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A word about gravitational form factors:

$$\int_{-1}^{1} dx\; x\; H(x,\xi,t) \;=\; c_0^{(1)}(t) \;+\; \xi^2 \boxed{\int_{-1}^{1} dz\; z\; D(z,t)}$$

3/2-Gegenbauer expansion

$$D(z,t) \;=\; (1-z^2) \sum_{k=1,\mathrm{odd}}^{\infty} d_k(t)\; C_k^{(3/2)}(x)$$

$$\theta_2(t) \;=\; \int_{-1}^{1} dx\; x\left(H^u_{\pi^+}(x,0,t) + H^d_{\pi^+}(-x,0,t)\right)$$

$$\boxed{\frac{4}{5}\, d_1(t)}$$

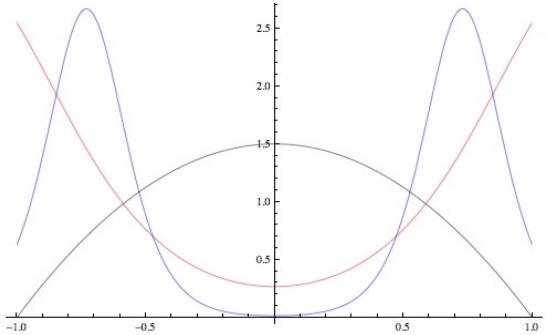
Only the first coefficient is needed!

LFWF + overlap approach cannot give access to the second gravitational moment. The Radon transform inversion of the DGLAP GPD cannot either (as it is nothing but a D-term contribution).

A possible way-out is considering unsubtracted t-channel dispersion relations to provide with a representation of the D-term form factor
[See Pasquini et al., PLB739(2014)133, precisely determining $d_1(t)$ for a nucleon case]

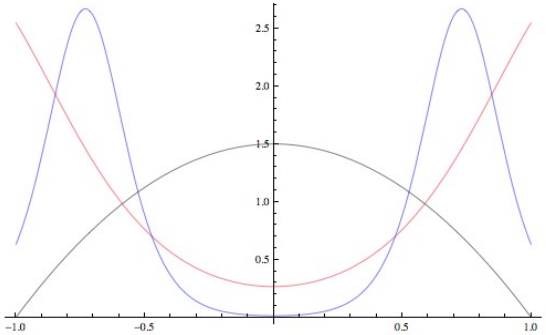
On the other hand, the soft-pion theorem opens a window to fully constraint the D-term and thus this second gravitational factor can be estimated at t=0. E.g., in the algebraic asymptotic model, one is left at the hadronic scale with: $\int_{-1}^{1} dz\, z\, D(z,0) = -1$

Conclusions



Owing to a sensible parametrisation of the BSA grounded on the so-called Nakanishi representation, one is left with a flexible algebraic model for the LFWF in terms of a spectral density.

Conclusions



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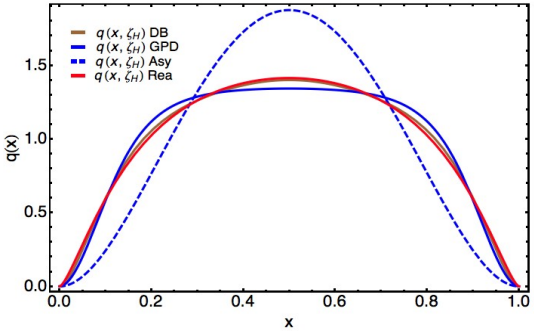
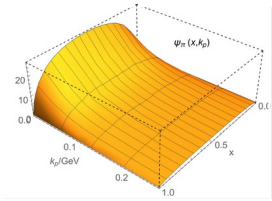


A direct calculation of the PDF from realistic quark gap and Bethe-Salpeter equations' solutions (in the forward kinematical limit) delivers a benchmark result to identify the spectral density which corresponds to the realistic LFWF.



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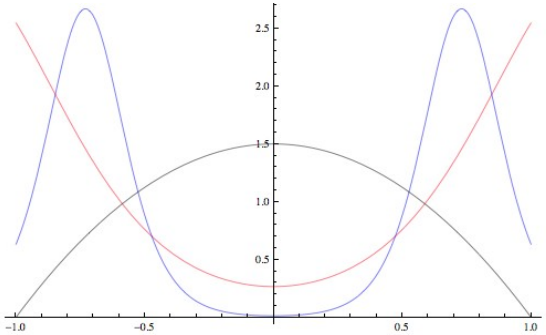

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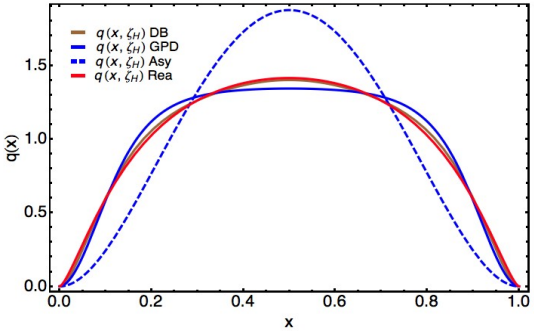

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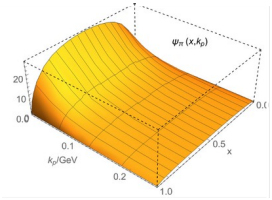
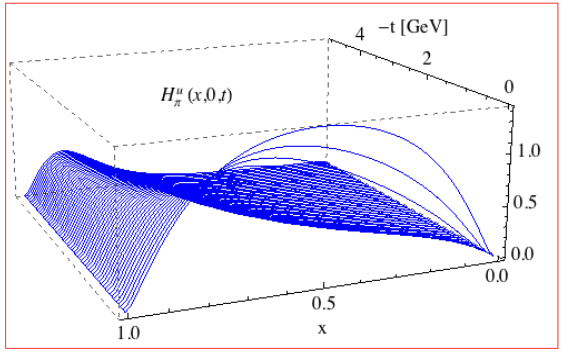
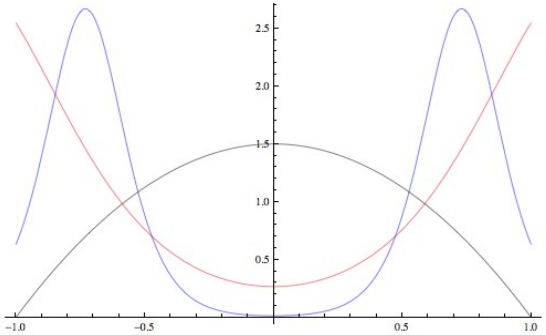




The overlap representation provides with a simple way to calculate beyond the forward kinematic limit, and thus obtain the GPD, although only in the DGLAP region.

Conclusions



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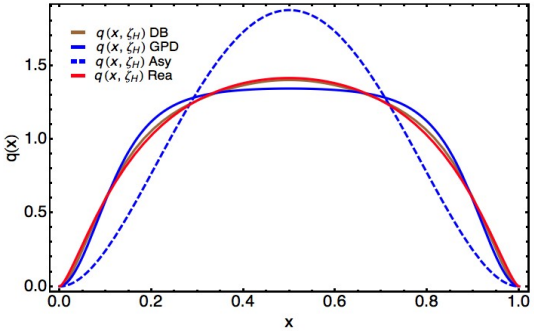
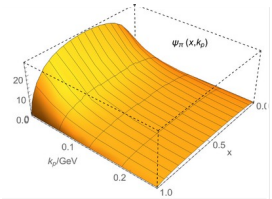
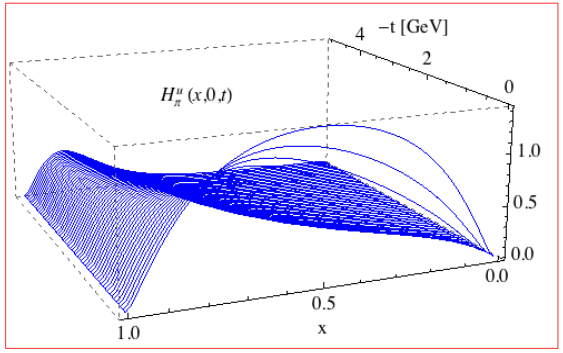


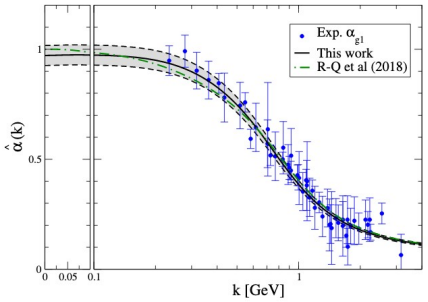
A direct calculation of the PDF from realistic quark gap and Bethe-Salpeter equations' solutions (in the forward kinematical limit) delivers a benchmark result to identify the spectral density which corresponds to the realistic LFWF.



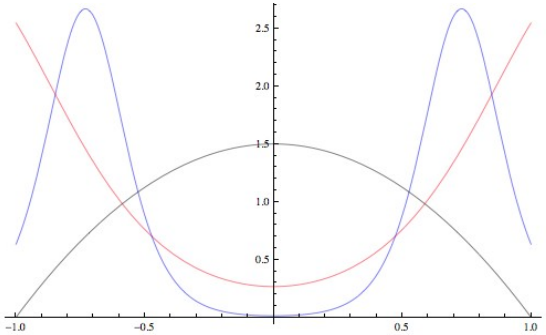


The overlap representation provides with a simple way to calculate beyond the forward kinematic limit, and thus obtain the GPD, although only in the DGLAP region.

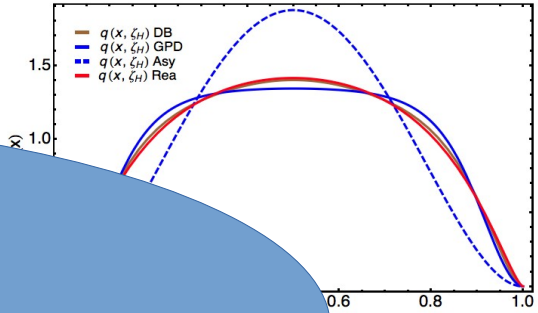
A recently proposed PI effective charge can be used to make the DGLAP GPD evolve from the hadronic scale (where quasi-particle DSE's solutions are the correct degrees-of-freedom) up to any other relevant scale.



Conclusions

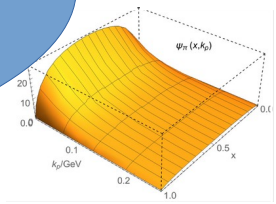


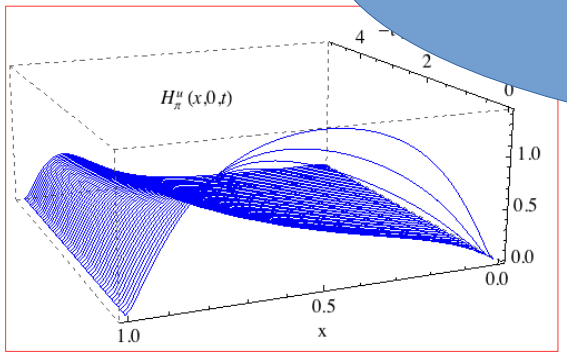
Owing to a sensible parametrisation of the BSA grounded on the so-called Nakanishi representation, one is left with a flexible algebraic model for the LFWF in terms of a spectral density.



A direct calculation of the PDF f
Bethe-Salpeter equ
kinematical limit)
spectral density

Thank you!!





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