
Wilson fermions in the ϵ regime

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The benefits of rooting

They are obvious to any connoisseur . . .

The benefits of rooting

... as illustrated by



With some skill, this leads to ...

The benefits of rooting



But back to my original topic: **Wilson fermions in the ϵ regime**

Outline

- Wilson chiral perturbation theory
- Wilson random matrix theory
- Index of the Wilson-Dirac operator
- Gauge ensembles
- Index and topological charge
- Wilson eigenvalues and WRMT
- Clover eigenvalues and WRMT
- Summary

with Poul Damgaard and Kim Splittorff,
Phys. Rev. D85 (2012) 014505 [arXiv:1110.2851];
Phys. Rev. D86 (2012) 034503 [arXiv:1206.4786].

Wilson chiral perturbation theory

At low energy, the leading fermion discretization effects can be included in χ PT by considering a modified chiral Lagrangian

$$\mathcal{L} = \frac{f^2}{8} \text{Tr} \left(\partial_\mu U \partial_\mu U^\dagger \right) - \frac{1}{2} m \Sigma \text{Tr} \left(U + U^\dagger \right) + a^2 \mathcal{V} .$$

\mathcal{V} describes the chiral symmetry breaking terms applicable to Wilson fermions. In the ϵ -regime, and with the power counting $m \sim a^2$, we have (Sharpe & Singleton)

$$\mathcal{V} = W_8 \text{Tr} \left(U^2 + U^{\dagger 2} \right) + W_6 \left[\text{Tr} \left(U + U^\dagger \right) \right]^2 + W_7 \left[\text{Tr} \left(U - U^\dagger \right) \right]^2 .$$

The two-trace terms with low-energy constants W_6 and W_7 are suppressed at large N_c .

In the ϵ -regime of χ PT the zero momentum modes dominate and the first term in \mathcal{L} can be neglected.

Wilson random matrix theory

The ϵ -regime can equivalently be described by a chiral random matrix theory, with the Dirac operator, including the one-trace term with low-energy constants W_8 , represented as

$$\mathcal{D}_W = \begin{pmatrix} \tilde{a}A & iW \\ iW^\dagger & \tilde{a}B \end{pmatrix}$$

with W a random $(n + \nu) \times n$ complex matrix, and A and B random Hermitian matrices of size $(n + \nu) \times (n + \nu)$ and $n \times n$, respectively.

This is written in a chiral basis with

$$\gamma_5 = \text{diag}(1, \dots, 1, -1, \dots, -1) ,$$

and A and B represent the chiral symmetry breaking term corresponding to the Wilson term in the Wilson-Dirac operator.

Wilson RMT

With W χ PT Akemann, Damgaard, Splittorff and Verbaarschot have worked out the eigenvalue distribution of the Hermitian Wilson Dirac operator H_W or the RMT equivalent \mathcal{H}_W

$$H_W = \gamma_5 (D_W + m_0) , \quad \mathcal{H}_W = \gamma_5 (\mathcal{D}_W + \tilde{m})$$

with

$$\hat{m} = m \Sigma V = 2 \tilde{m} n \quad \text{and} \quad \hat{a}_8^2 = a^2 W_8 V = \frac{1}{2} \tilde{a}^2 n$$

held fixed. The eigenvalues are rescaled with ΣV or $2n$. m is related to m_0 by a suitable subtraction.

For more details see [G. Akemann *et al.*, Phys. Rev. D83 \(2011\) 085014 \[arXiv:1012.0752\]](#).

[G. Akemann and T. Nagao, JHEP 10 \(2011\) 060 \[arXiv:1108.3035\]](#), reproduced the results directly from WRMT.

Inclusion of the two-trace terms

It is useful to introduce a $\bar{\psi}\gamma_5\psi$ term into the chiral Lagrangian

$$\Delta\mathcal{L} = -\frac{1}{2}z\Sigma\text{Tr}\left(U - U^\dagger\right) .$$

With $\hat{z} = z\Sigma V$, the two-trace terms can be incorporated via two Gaussian integrations ($\hat{a}_j = a^2 W_j V$ for $j = 6, 7, 8$)

$$Z^\nu(\hat{m}, \hat{z}; \hat{a}_6, \hat{a}_7, \hat{a}_8) = \frac{1}{16\pi\hat{a}_6\hat{a}_7} \int_{-\infty}^{\infty} dy_6 dy_7 e^{-\frac{y_6^2}{16\hat{a}_6^2} - \frac{y_7^2}{16\hat{a}_7^2}} Z^\nu(\hat{m} - y_6, \hat{z} - y_7; 0, 0, \hat{a}_8) .$$

Here,

$$Z^\nu(\hat{m}, \hat{z}; 0, 0, \hat{a}_8) = \int dU \det^\nu U e^{-V(\mathcal{L} + \Delta\mathcal{L})}$$

is the fixed-index partition function with the one-trace $\mathcal{O}(a^2)$ term included.

Index of the Wilson-Dirac operator

RMT predictions are usually given for a gauge field sector with a particular index (or, in the continuum, topological charge). For the Wilson-Dirac operator, the index can be defined by

$$\nu \equiv \sum'_k \text{sign}(\langle k | \gamma_5 | k \rangle)$$

with $|k\rangle$ the k 'th eigenstate of the Wilson-Dirac operator. Only eigenvectors with real eigenvalues contribute, and the $'$ indicates that only the real eigenvalues in the branch near zero, with eigenvalues $< r_{cut}$, are kept. Using

$$H_W(m_0)|\psi\rangle = 0 \quad \Rightarrow \quad D_W|\psi\rangle = -m_0|\psi\rangle$$

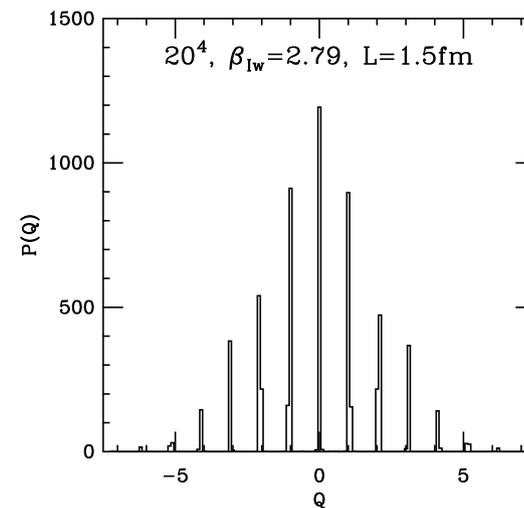
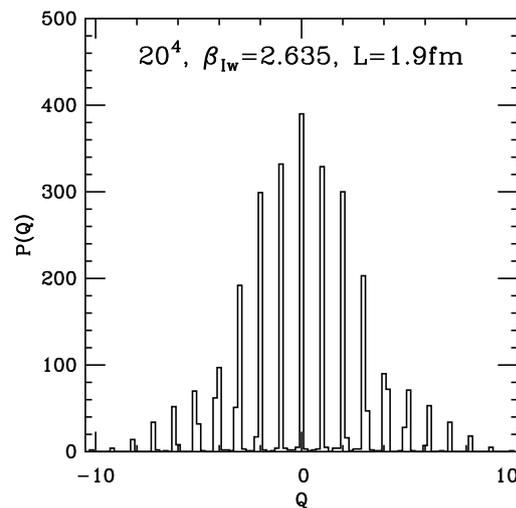
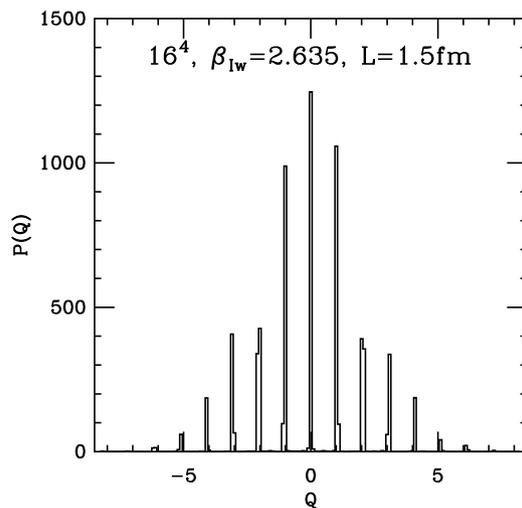
the index can equivalently be obtained from the zero crossings of the spectral flow of $H_W(m_0)$ to $m_{cut} = -r_{cut}$. It corresponds to the index of an overlap operator with kernel $H_W(m_{cut})$. The index is not unique, but depends on the choice of r_{cut} .

Gauge ensembles

For our test in the quenched case, we generated three ensembles using the Iwasaki gauge action, which suppresses dislocations, and gives a fairly unique index (topol. charge Q).

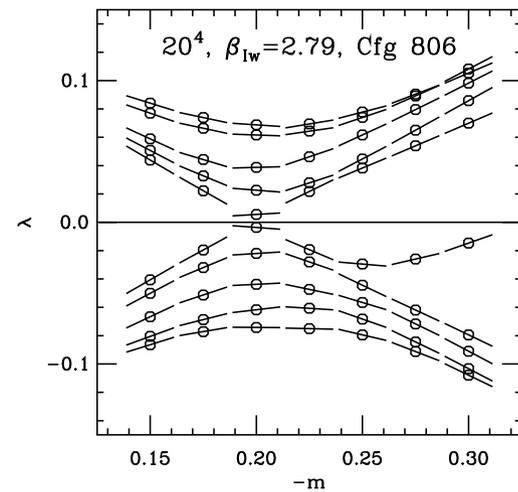
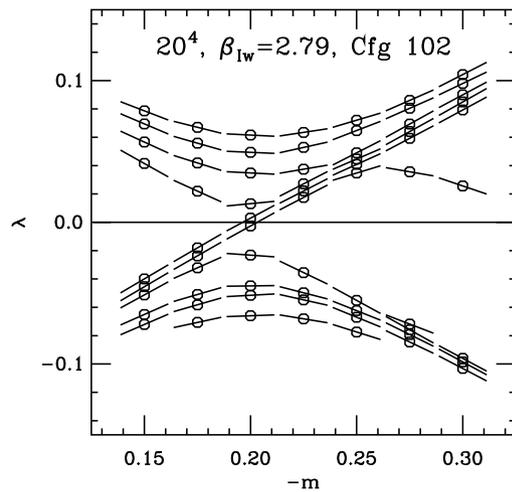
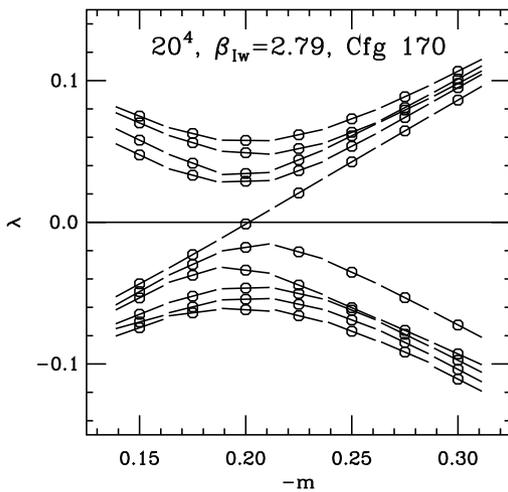
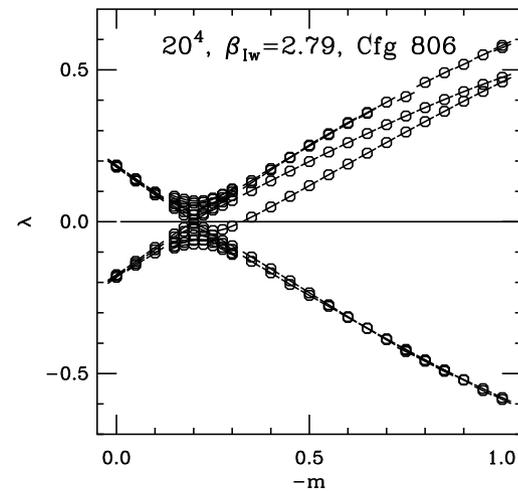
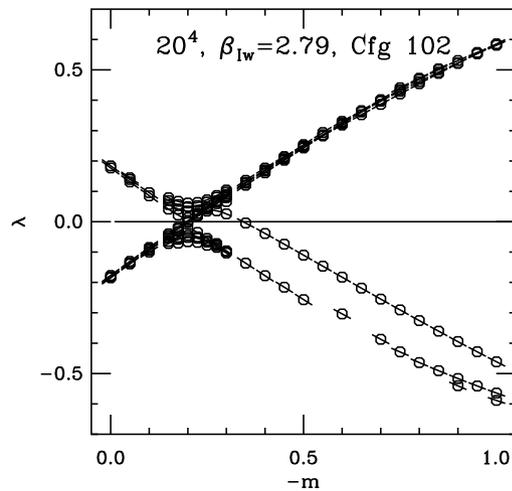
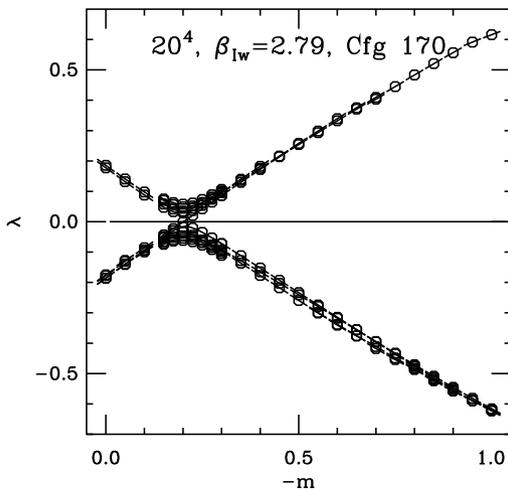
β_{Iw}	r_0/a	a [fm]	size	L [fm]	cfgs	$\nu = 0, 1, -1$
2.635	5.37	0.093	16^4	1.5	6500	1246, 1088, 1045
2.635	5.37	0.093	20^4	1.9	3000	379, 319, 322
2.79	6.70	0.075	20^4	1.5	6000	1172, 990, 988

Q is measured with 6 HYP steps and the “Boulder” $F\tilde{F}$:



Index and topological charge

On the configurations with “Boulder charge” $Q = -1, 0$ and 1 , we did the expensive measurement of the index with $r_{cut} = 1$.



Index and topological charge

For all our tests of $W\chi$ PT (WRMT) discussed here, we did one HYP smearing of the gauge fields, before constructing the Wilson-Dirac operator. The HYP smearing smooths the gauge fields and further suppresses dislocations.

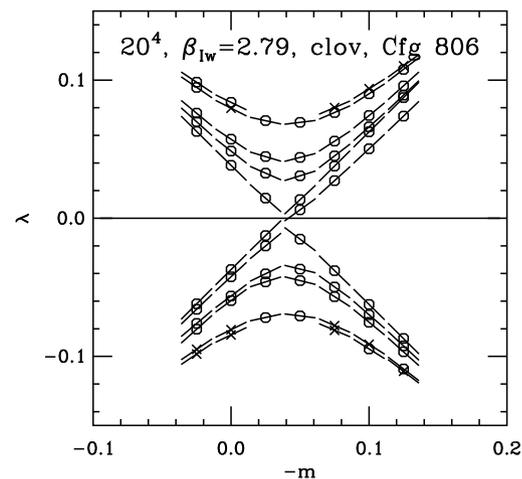
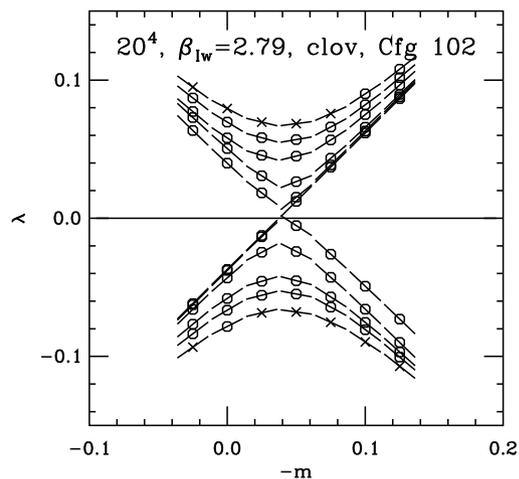
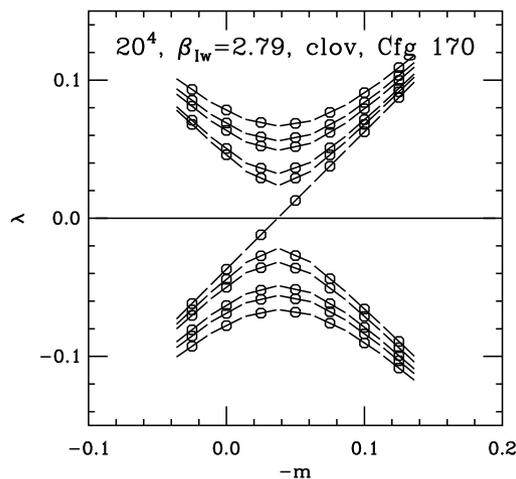
For **only** 1.3%, 2.9% and 0.1% configs did the index ν with $r_{cut} = 1$ differ from the “Boulder charge” Q .

The good agreement between “Boulder charge Q ” and Wilson index ν supports our choices in gauge action and implementation of the Wilson-Dirac operator.

Changing r_{cut} to as low as 0.5 would not affect the agreement much.

Index and topological charge

On the same configs, with a clover-improved ($c_{SW} = 1.0$) Dirac operator – again with one HYP smearing, so that the tree-level c_{SW} is close to the nonperturbative value –

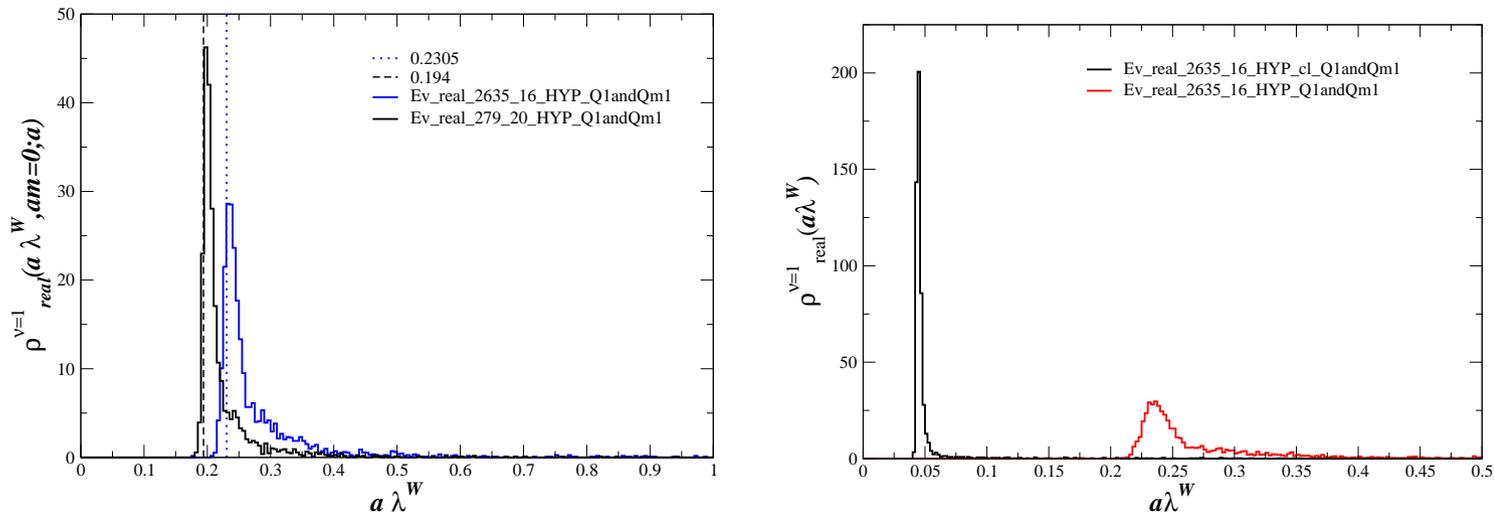


Multiple zero crossings become rarer, and the crossing modes (topological modes) more chiral: the slope at the crossing, which is equal to the chirality, is close to ± 1 .

With clover improvement the “Boulder charge” Q and the index differed on 3.1% of the $\beta_{Iw} = 2.635 \cdot 16^4$ cfs.

Index and topological charge

We show the distribution of the real eigenvalues for the $|Q| = 1$ configs of the two ensembles with $L = 1.5$ fm (left, with estimates of $-m_{crit}$) and the impact of clover improvement (right):



Some real eigenvalues are smaller than $-m_{crit}$ (on “exceptional configs”).

Clover improvement decreases the additive mass renormalization and makes the distribution of the real eigenvalues narrower and more symmetric.

Wilson eigenvalues and WRMT

We computed the lowest (in magnitude) 20 eigenvalues of $H_W(m_0)$ for:

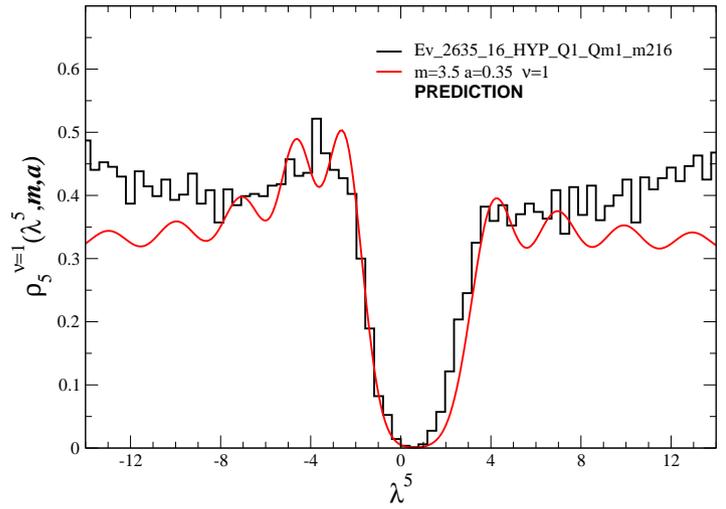
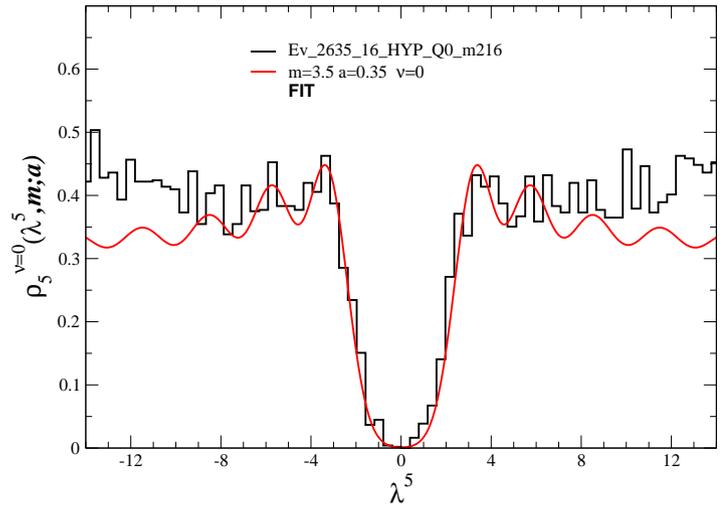
Ens	β_{Iw}	r_0/a	a [fm]	size	L [fm]	$ Q = 0, 1, 2$ cfgs
A	2.635	5.37	0.093	16^4	1.5	1276, 2257, 1518
B	2.635	5.37	0.093	20^4	1.9	379, 641, 610
C	2.79	6.70	0.075	20^4	1.5	1202, 2128, 1408

For ensembles A and B we used $am_0 = -0.216$, and for ensemble C, $am_0 = -0.178$ and -0.184 .

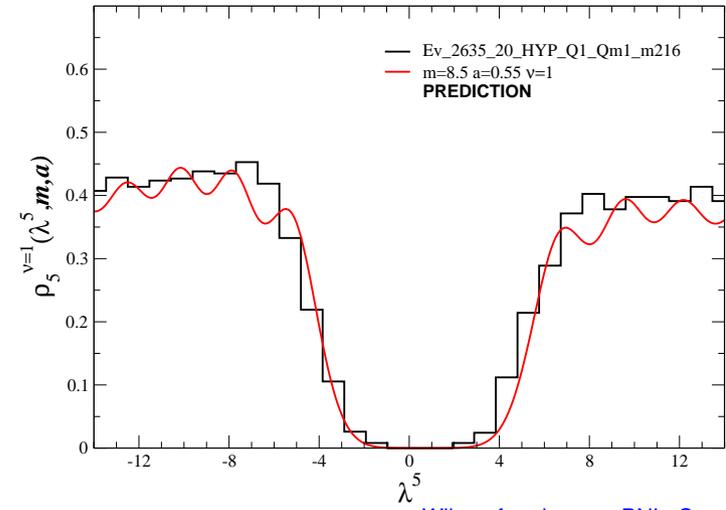
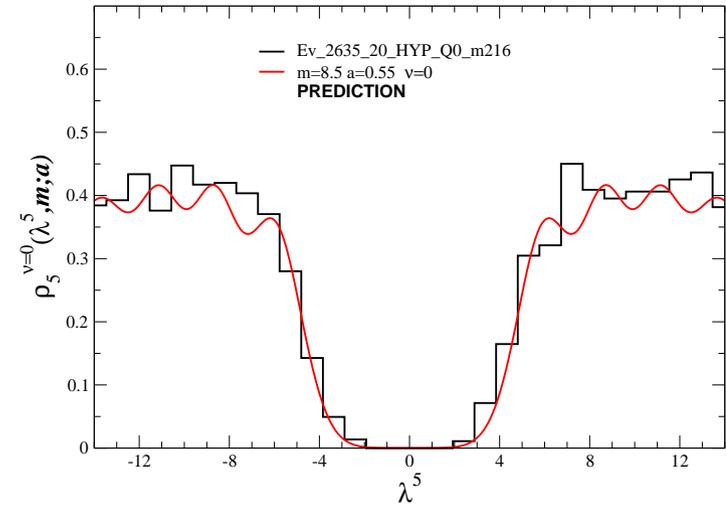
We used the $\nu = 0$ histogrammed distribution of ensemble A and of ensemble C with $am_0 = -0.184$ to obtain the WRMT parameters $\hat{a} = \hat{a}_8$ and \hat{m} , and the eigenvalue rescaling factor ΣV so that WRMT “fits” the distribution “well.”

Using the **same** parameter values, WRMT then predicts the $|\nu| = 1$ distribution that can be compared with the measured one.

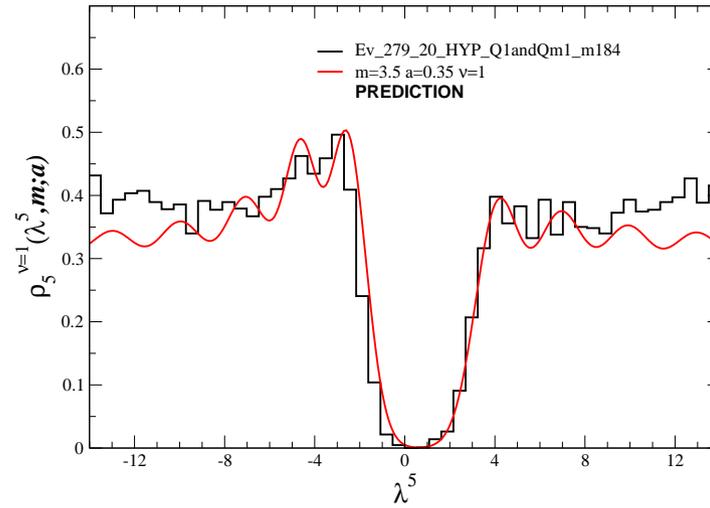
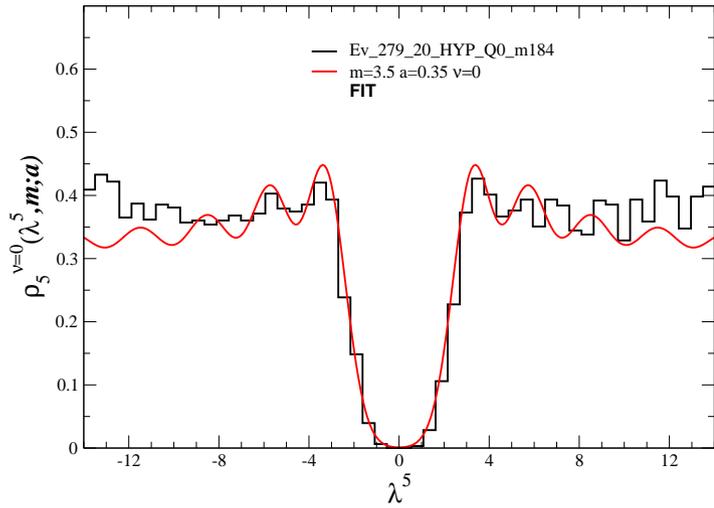
Volume scaling – ensembles A and B



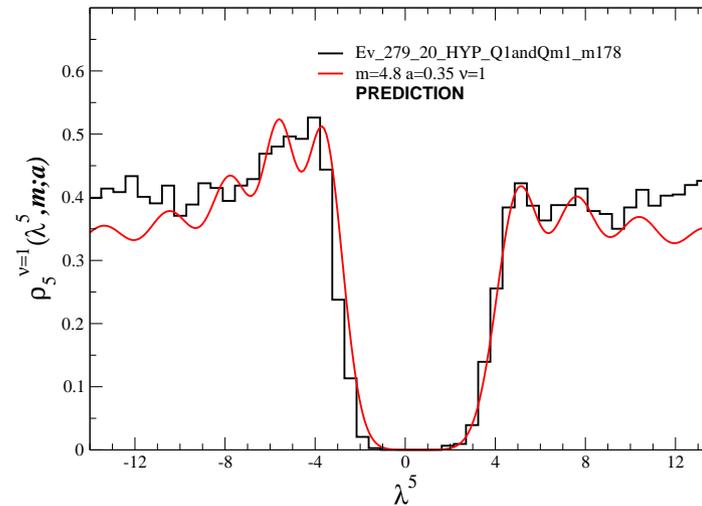
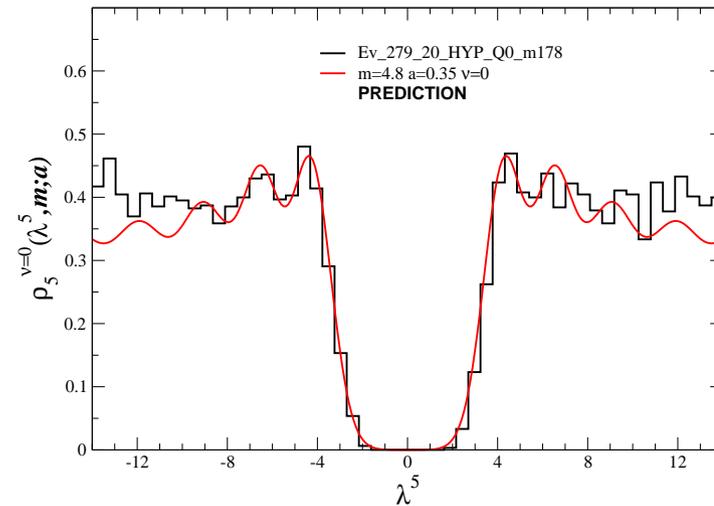
Use volume scaling, $\hat{a}_B = \hat{a}_A \sqrt{V_B/V_A}$ and $\hat{m}_B = \hat{m}_A (V_B/V_A)$:



Mass scaling – ensemble C

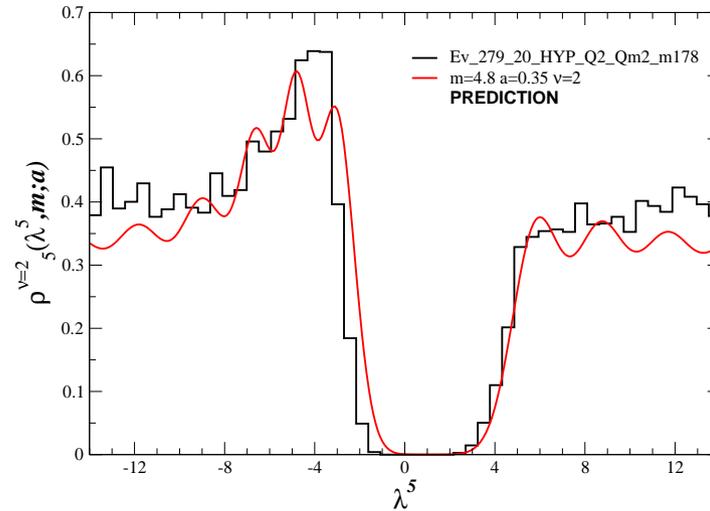
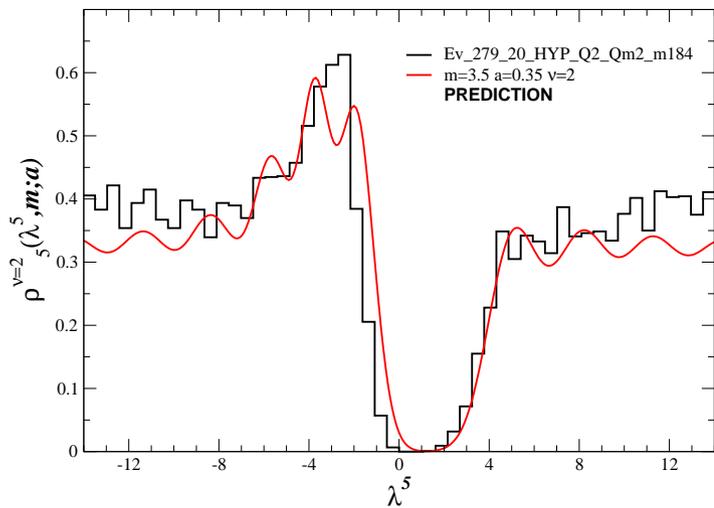


Use mass scaling, *i.e.*, adjust \hat{m} by $\Delta\hat{m} = \Delta m_0 \Sigma V$:



Wilson eigenvalues and WRMT

We can also compare for $|Q| = 2$ ($|\nu| = 2$, but not from flow) for ensemble C, the finer lattice spacing.

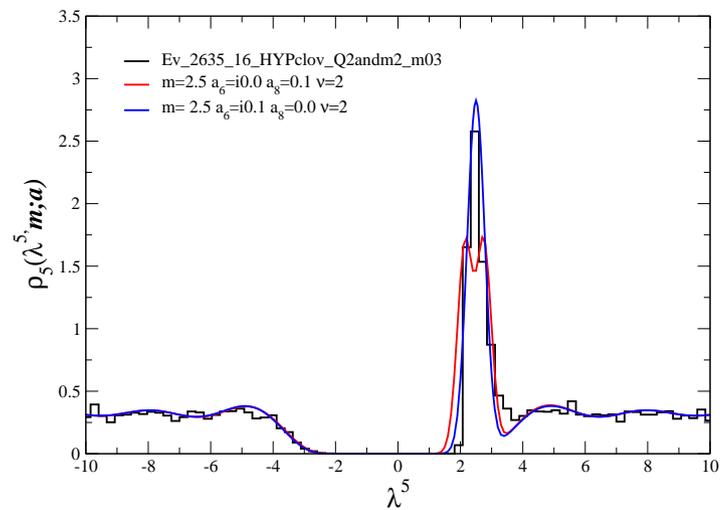
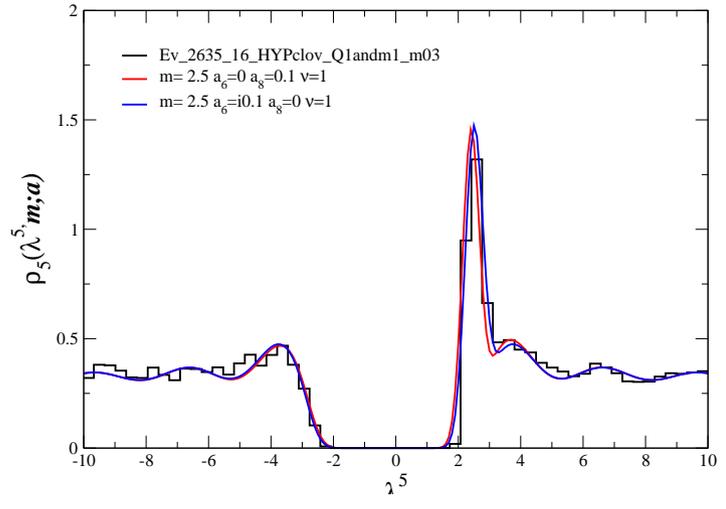
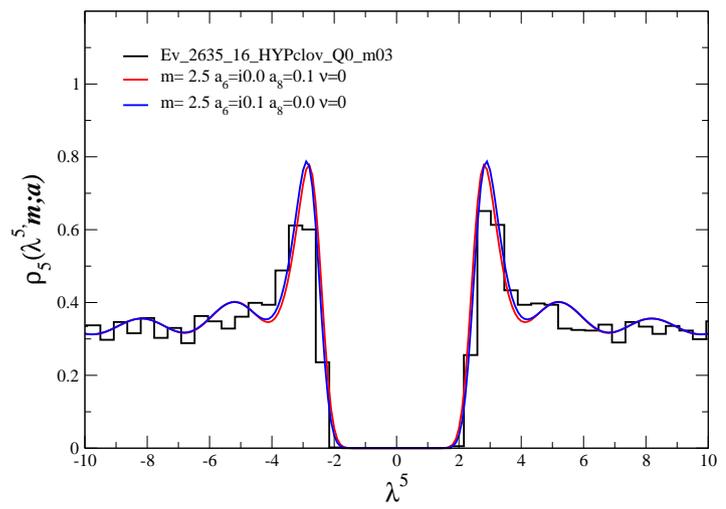


Despite use of smearing and fairly small a , the lattice artifacts are quite substantial, and higher order terms are important, *e.g.*, to explain the asymmetry in the distribution of the real eigenvalues.

A similar study was done by [Deuzeman, Wenger and Wuilloud, JHEP 1112, 109 \(2011\) \[arXiv:1110.4002\]](#).

Clover eigenvalues and WRMT

Use clover term, and determine WRMT parameters from $\nu = 1$ distribution. for ensemble A:



Clover eigenvalues and WRMT

Lattice effects are much smaller – \hat{a}_j decreased by about a factor 3 - 4.

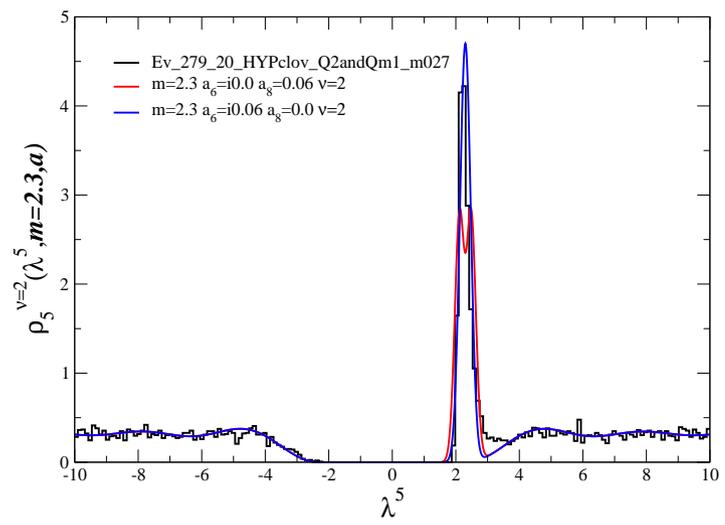
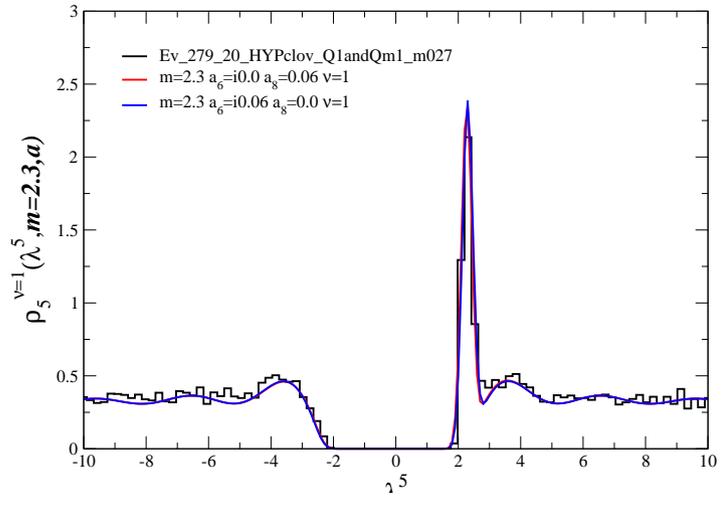
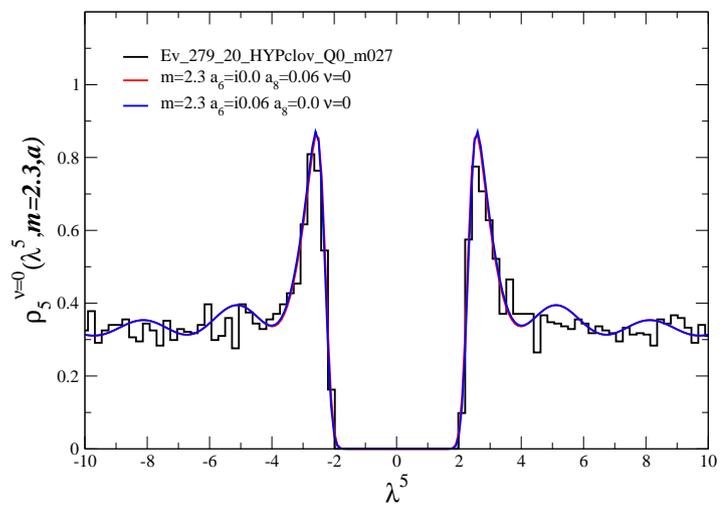
Lattice effects affect mostly the “topological peak,” the delta-function peak in the continuum at $\lambda_5 = m$ from the zeromodes of the massless Dirac operator.

For small \hat{a}_j , W_6 and W_7 contribute as $|W_6 + W_7|$. They smear out the delta-function peak, as does W_8 .

For $\nu > 1$, W_8 , as part of the WRMT Dirac operator, induces level repulsion in the topological peak. W_6 does not, so we can differentiate the contributions: W_6 describes the observed distribution.

Clover eigenvalues and WRMT

This is confirmed at smaller a , ensemble C:



Summary

- Like χ PT before, RMT has been recently extended to include lattice effects for (staggered and) Wilson fermions.
- We have used eigenvalues of the Hermitian Wilson-Dirac operator to test predictions of WRMT, or ϵ -regime $W\chi$ PT.
- For the Wilson-Dirac operator with one HYP smearing, we find nice agreement of the predicted low-lying eigenvalue distributions and the measured ones.
- Parameters obtained from the distribution of one ν -sector give predictions for other ν -sectors that work well.
- Volume and mass scaling of the WRMT parameters holds.
- Clover improvement makes the lattice effects much smaller, and allows to distinguish between W_8 and W_6 .
- The combination of smearing and $\mathcal{O}(a)$ improvement is important to decrease the lattice artifacts.