Recent Work on Renormalization-Group Properties of Gauge Theories and Connections with Mike Creutz's Contributions

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Creutz Fest 2014, BNL, Sept. 4-5, 2014

Outline

- Some general remarks and reminiscences for Mike Creutz
- Higher-loop calculations of UV to IR evolution in asymptotically free non-abelian gauge theories, including IR zero of β and calculation of anomalous dimension γ_m of fermion bilinear
- Some comparisons with lattice measurements of γ_m
- Higher-loop calculation of structural properties of β
- Study of scheme-dependence
- Other theories
- Ending remarks

Some General Remarks and Reminiscences

It is quite fitting to honor Mike Creutz, one of the great pioneers of lattice gauge theory.

In 1979, by a brilliant use of Monte Carlo simulations, Mike showed confinement for asymptotically free non-abelian Yang-Mills gauge theories, and measured the string tension, demonstrating that it obeyed asymptotic scaling. This work also showed how Wilson's 1974 proof of the area-law behavior of the Wilson loop at strong bare coupling (i.e., small $\beta = 2N_c/g_0^2$) extends to the continuum limit and hence describes actual continuum physics, in, e.g., Creutz, "Monte Carlo Study of Quantized SU(2) Gauge Theory" (Oct. 1979), Phys. Rev. D21, 2308 (1980); Creutz, "Asymptotic Freedom Scales", Phys. Rev. Lett. 45, 313 (1980)...)

This work by Mike is crucial for our understanding of confinement in QCD.

Wilson's proof of area-law behavior also applied for abelian gauge theories, so it was necessary to show that one could not analytically continue this to the continuum limit at $\beta \longrightarrow \infty$ to be consistent with deconfinement of a U(1) gauge theory like QED. Mike also did this, demonstrating evidence for non-analytic behavior in the average plaquette for U(1) gauge theory in Creutz, "Confinement and the Critical Dimensionality of Space-Time" (May, 1979), Phys. Rev. Lett. 43, 53 (1979).

With Jacobs and Rebbi, Mike did pioneering lattice simulations of theories with coupled gauge and scalar fields, in Creutz, Jacobs, and Rebbi, “Experiments with a Gauge-Invariant Ising System” (Mar. 1979), Phys. Rev. Lett. 42, 1390 (1979); Creutz, “Phase Diagrams for Coupled Spin-Gauge Systems” (Aug. 1979), Phys. Rev. D21, 1006 (1980).

Mike had been working on lattice gauge theory several years before the great 1979 breakthrough, with papers dating back to 1976, as well as work on hadron bag models.

I vividly remember the great excitement in 1979 with these breakthroughs. On visits from Stony Brook to Brookhaven I would talk to Mike about this work, which combined the power of Monte Carlo simulations with analytic checks to demonstrate that the results applied to the continuum limit and were not artifacts due to finite lattice spacing or finite lattice volume. Mike’s work has stood the test of time.

Mike built on these very important achievements in 1979-1980 with many valuable subsequent contributions, as discussed by speakers here. He richly deserved the Rahman Prize for Computational Physics which was awarded to him in 2000, as well as other the awards that he has received.

Mike himself has given a brief history of this heroic period, e.g., at Lattice-2000 (in Bangalore), “Lattice Gauge Theory - a Retrospective”, hep-lat/0010047.

In the nearly 40 years that I have known Mike, I have always been very impressed not only by his profoundly important insights and contributions to particle physics, but also by his remarkable friendliness, openness, and modesty.

Lattice gauge theory now provides the most precise quantitative understanding of the hadron spectrum of QCD, as well as other properties, e.g., hadronic matrix elements, finite-temperature behavior, etc. The use of numerical importance-sampling methods revolutionized the study of QCD.

Mike's 1983 review with Jacobs and Rebbi in Physics Reports, and Mike's 1983 book, "Quarks, Gluons, and Lattices" have been of great pedagogical value. Since that time, when I have taught courses on lattice field theory and general particle theory, always including a unit on lattice field theory, I have made use of these, together with other reviews such as those of John Kogut. Although many further advances have been made in the last 30 years, Mike's reviews are still quite valuable.

Mike's work not only had a great influence on the particle physics community in general; it also had a strong influence on my own career. In the 1979-1980 period I was working mainly in flavor, electroweak, and neutrino physics, but I later spent a number of years on research in lattice field theory.

Some of my lattice work involved collaborations with I-Hsiu Lee, Junko Shigemitsu, Sinya Aoki, Nucu Stamatescu, and others. Several of the papers with I-Hsiu and Sinya were written while they were postdocs at Brookhaven (and, for Sinya, later also at Stony Brook). Some of this work included studies of lattice gauge-Higgs theories and lattice Yukawa models. These bring back many good memories.

In 1992, Mike got together a number of us to write review articles on this work for a book he edited, "Quantum Fields on the Computer".

Mike has contributed to many other areas, including statistical mechanics.

Among Mike's many contributions, a major one has been on confinement in QCD. Following this work, one is naturally led to inquire how the IR properties of an asymptotically free non-abelian gauge theory change as one increases the content of fermions in various representations.

For sufficiently many fermions, there is good evidence that these theories are deconfined (at zero temperature) with no spontaneous chiral symmetry breaking. This is an area of considerable activity, both in continuum calculations and lattice simulations at present. We proceed to discuss some of our recent results in this area.

RG Flow from UV to IR; Types of IR Behavior and Role of IR Fixed Point

Consider an asymptotically free, vectorial gauge theory with gauge group G and N_f massless fermions in representation R of G.

Asymptotic freedom $\Rightarrow$ theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale μ in deep ultraviolet (UV).

The question of how this theory flows from large μ in the UV to small μ in the infrared (IR) is of fundamental field-theoretic interest.

For some fermion contents, the theory may have an exact or approximate IR fixed point (zero of β).

Denote running gauge coupling at scale μ as $g = g(\mu)$, and let

$$\alpha(\mu) = \frac{g(\mu)^2}{4\pi}\,, \quad a(\mu) = \frac{g(\mu)^2}{16\pi^2}$$

The dependence of $\alpha(\mu)$ on μ is described by the renormalization group β function

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell\, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell\, \alpha^\ell \ ,$$

where $t = \ln \mu$, $\ell =$ loop order of the coeff. b_ℓ, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$.

Coefficients b_1 and b_2 in β are independent of regularization/renormalization scheme, while b_ℓ for $\ell \geq 3$ are scheme-dependent.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$.

As the scale μ decreases from large values, $\alpha(\mu)$ increases. Denote α_{cr} (depending on G and R) as minimum value for formation of bilinear fermion condensates and resultant spontaneous chiral symmetry breaking (SχSB).

Two generic possibilities for β and resultant UV to IR flow:

- β has no IR zero, so as μ decreases, $\alpha(\mu)$ increases, eventually beyond the perturbatively calculable region. This is the case for QCD.
- β has a IR zero, α_{IR}, so as μ decreases, $\alpha \longrightarrow \alpha_{IR}$. In this class of theories, there are two further generic possibilities: $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$.

If $\alpha_{IR} < \alpha_{cr}$, the zero of β at α_{IR} is an exact IR fixed point (IRFP) of the renorm. group (RG); as $\mu \longrightarrow 0$ and $\alpha \longrightarrow \alpha_{IR}$, $\beta \longrightarrow \beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant with nontrivial anomalous dimensions (Caswell, Banks-Zaks). There is no spontaneous chiral symmetry breaking, and in the IR the theory is expected to be in a deconfined, non-abelian Coulomb phase.

If β has no IR zero, or an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as μ decreases through a scale Λ, $\alpha(\mu)$ exceeds α_{cr} and SχSB occurs, so fermions gain dynamical masses $\sim \Lambda$ and are integrated out of the low-energy effective field theory applicable for $\mu < \Lambda$. In this case, α_{IR} is only approx. IRFP of RG.

If α_{IR} is only slightly greater than α_{cr}, then, as $\alpha(\mu)$ approaches α_{IR}, since $\beta = d\alpha/dt \to 0$, $\alpha(\mu)$ varies very slowly as a function of the scale μ, i.e., there is approximately scale-invariant (= dilatation-invariant) behavior.

SχSB at Λ also breaks the approx. dilatation symmetry, leading to a resultant approx. Nambu-Goldstone boson, the dilaton. This is not massless, since $\beta(\alpha_{cr})$ is nonzero.

Denote the n-loop β fn. as $\beta_{n\ell}$ and the IR zero of $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$.

At the $n = 2$ loop level,

$$\alpha_{IR,2\ell} = -\frac{4\pi b_1}{b_2}$$

which is physical for $b_2 < 0$. One-loop coefficient b_1 (Gross-Wilczek, Politzer,'t Hooft) is

$$b_1 = \frac{1}{3}(11C_A - 4N_f T_f)$$

where $C_A \equiv C_2(G)$ is quadratic Casimir invariant, $T_f \equiv T(R)$ is trace invariant. Focus here on $G = \mathrm{SU}(N_c)$.

Asymptotic freedom requires $N_f < N_{f,b1z}$, where

$$N_{f,b1z} = \frac{11C_A}{4T_f}$$

e.g., for $R =$ fundamental rep., $N_f < (11/2)N_c$.

Two-loop coeff. b_2 is (Caswell, Jones)

$$b_2 = \frac{1}{3}\left[34C_A^2 - 4(5C_A + 3C_f)N_f\,T_f\right]$$

where $C_f \equiv C_2(R)$.

b_2 is positive for small N_f but decreases as fn. of N_f and vanishes with sign reversal at $N_f = N_{f,b2z}$, where

$$N_{f,b2z} = \frac{34C_A^2}{4T_f(5C_A + 3C_f)}$$

For arbitrary G and R, $N_{f,b2z} < N_{f,b1z}$, so there is always an interval in N_f for which β has an IR zero, namely

$$I: \quad N_{f,b2z} < N_f < N_{f,b1z}$$

- for SU(2), I: $\quad 5.55 < N_f < 11$
- for SU(3), I: $\quad 8.05 < N_f < 16.5$
- As $N_c \to \infty$, I: $\quad 2.62 N_c < N_f < 5.5 N_c$.

(expressions given for $N_f \in \mathbb{R}$; of course, physical values of N_f are nonnegative integers)

As N_f decreases from the upper to lower end of interval I, α_{IR} increases. Denote

$$N_f = N_{f,cr} \quad \text{at} \quad \alpha_{IR} = \alpha_{cr}$$

$N_{f,cr}$ separates the (zero-temp.) chirally symmetric and broken IR phases. Old estimate from approx. sol. of Schwinger-Dyson eq. for fermion propagator in SU(N_c): $N_{f,cr} \sim 4N_c$.

As N_f approaches $N_{f,cr}$, α grows to O(1) as theory becomes more strongly coupled.

Higher-Loop Corrections to UV $\rightarrow$ IR Evolution of Gauge Theories

Because of the moderately strongly coupled physics, one should calculate the IR zero in β, α_{IR}, and resultant value of γ_m evaluated at α_{IR}, to higher-loop order. We have done this for arbitrary G and R, in Ryttov and Shrock, PRD83, 056011 (2011) [arXiv:1011.4542] (see also Pica and Sannino, PRD83, 035013 (2011) [arXiv:1011.5917]; results agree).

Although coeffs. in β at $\ell \geq 3$ loop order are scheme-dependent, results give a measure of accuracy of the 2-loop calc. of the IR zero of β, and similarly with γ_m evaluated at this IR zero.

Make use of calculation of β and γ_m up to 4-loops in $\overline{MS}$ scheme by Vermaseren, Larin, and van Ritbergen.

The value of higher-loop calculations has been amply shown in comparison of QCD predictions with experimental data, e.g., in $\overline{MS}$ scheme. These are for α near the UVFP at $\alpha = 0$; here we study an IRFP away from $\alpha = 0$.

3-loop coefficient in β function (in $\overline{\mathrm{MS}}$ scheme):

$$b_3 = \frac{2857}{54}C_A^3 + T_f N_f \left[2C_f^2 - \frac{205}{9}C_A C_f - \frac{1415}{27}C_A^2\right]$$
$$+(T_f N_f)^2\left[\frac{44}{9}C_f + \frac{158}{27}C_A\right]$$

Here, $b_3 < 0$ for $N_f \in I$. Since $\beta_{3\ell} = -[\alpha^2/(2\pi)](b_1 + b_2 a + b_3 a^2)$, it follows that $\beta_{3\ell} = 0$ away from $\alpha = 0$ at two values:

$$\alpha = \frac{2\pi}{b_3}\Big(- b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}\ \Big)$$

Since $b_2 < 0$ and $b_3 < 0$, can rewrite as

$$\alpha = \frac{2\pi}{|b_3|}\Big(- |b_2| \mp \sqrt{b_2^2 + 4b_1|b_3|}\ \Big)$$

Soln. with $-$ sqrt is negative, hence unphysical; soln. with $+$ sqrt is $\alpha_{IR,3\ell}$.

We showed analytically and numerically that the value of the IR zero decreases when calculated at the 3-loop level, i.e.,

$$\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$$

In RS, Phys. Rev. D 87, 105005 (2013) [arXiv:1301.3209] we generalized this. If a scheme had $b_3 > 0$ in I, then, since $b_2 \to 0$ at lower end of I, $b_2^2 - 4b_1 b_3 < 0$, so this scheme would not have a physical $\alpha_{IR,3\ell}$ in this region.

Since the existence of the IR zero in β at 2-loop level is scheme-independent, one may require that a scheme should maintain this property to higher-loop order, and hence that $b_3 < 0$ for $N_f \in I$.

With $b_3 < 0$, we proved that the inequality $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ holds in general.

We have given an analysis of the zeros of the four-loop beta function, $\beta_{4\ell}$, in a general scheme. With $\overline{\mathrm{MS}}$, from 3- to 4-loop level, slight increase: $\alpha_{IR,4\ell} \gtrsim \alpha_{IR,3\ell}$; small change, so overall, $\alpha_{IR,4\ell} < \alpha_{IR,2\ell}$.

Our result of smaller fractional change in value of IR zero of β at higher-loop order agrees with expectation that calc. to higher loop order should give more stable result.

Numerical values of $\alpha_{IR,n\ell}$ at the $n = 2,\ 3,\ 4$ loop level for SU(2), SU(3) and N_f fermions in fund. rep.

N_c	N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$
2	6	11.42	1.645	2.395
2	7	2.83	1.05	1.21
2	8	1.26	0.688	0.760
2	9	0.595	0.418	0.444
2	10	0.231	0.196	0.200
3	10	2.21	0.764	0.815
3	11	1.23	0.578	0.626
3	12	0.754	0.435	0.470
3	13	0.468	0.317	0.337
3	14	0.278	0.215	0.224
3	15	0.143	0.123	0.126
3	16	0.0416	0.0397	0.0398

(Perturbative calc. not applicable if $\alpha_{IR,n\ell}$ too large.) We also performed the corresponding higher-loop calculations for SU(N_c) gauge theories with N_f fermions in larger representations.

It is of interest to calculate the anomalous dimension $\gamma_m \equiv \gamma$ for the fermion bilinear.

Denote γ calculated to n-loop ($n\ell$) level as $\gamma_{n\ell}$ and, evaluated at the n-loop value of the IR zero of β, as

$$\gamma_{IR,n\ell} \equiv \gamma_{n\ell} \quad \text{at } \alpha = \alpha_{IR,n\ell}$$

In the IR chirally symmetric phase, an all-order calculation of γ evaluated at an all-order calculation of α_{IR} would be an exact property of the theory.

In the χ bk. phase, just as the IR zero of β is only an approx. IRFP, so also, the γ is only approx., describing the running of $\bar{\psi}\psi$ and the dynamically generated running fermion mass near the zero of β having large-momentum behavior $\Sigma(k) \sim \Lambda(\Lambda/k)^{2-\gamma}$ (with $\gamma < 2$).

For example, at the two-loop level, we calculate

$$\gamma_{IR,2\ell} = \frac{C_f(11C_A - 4T_fN_f)[455C_A^2 + 99C_AC_f + (180C_f - 248C_A)T_fN_f + 80(T_fN_f)^2]}{12[-17C_A^2 + 2(5C_A + 3C_f)T_fN_f]^2}$$

Illustrative numerical values of $\gamma_{IR,n\ell}$ for SU(2) and SU(3) at the $n = 2,\ 3, 4$ loop level and N_f fermions in the fundamental representation:

N_c	N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
2	7	(2.67)	0.457	0.0325
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	10	(4.19)	0.647	0.156
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259

Plots of γ as fn. of N_f for SU(2) and SU(3):

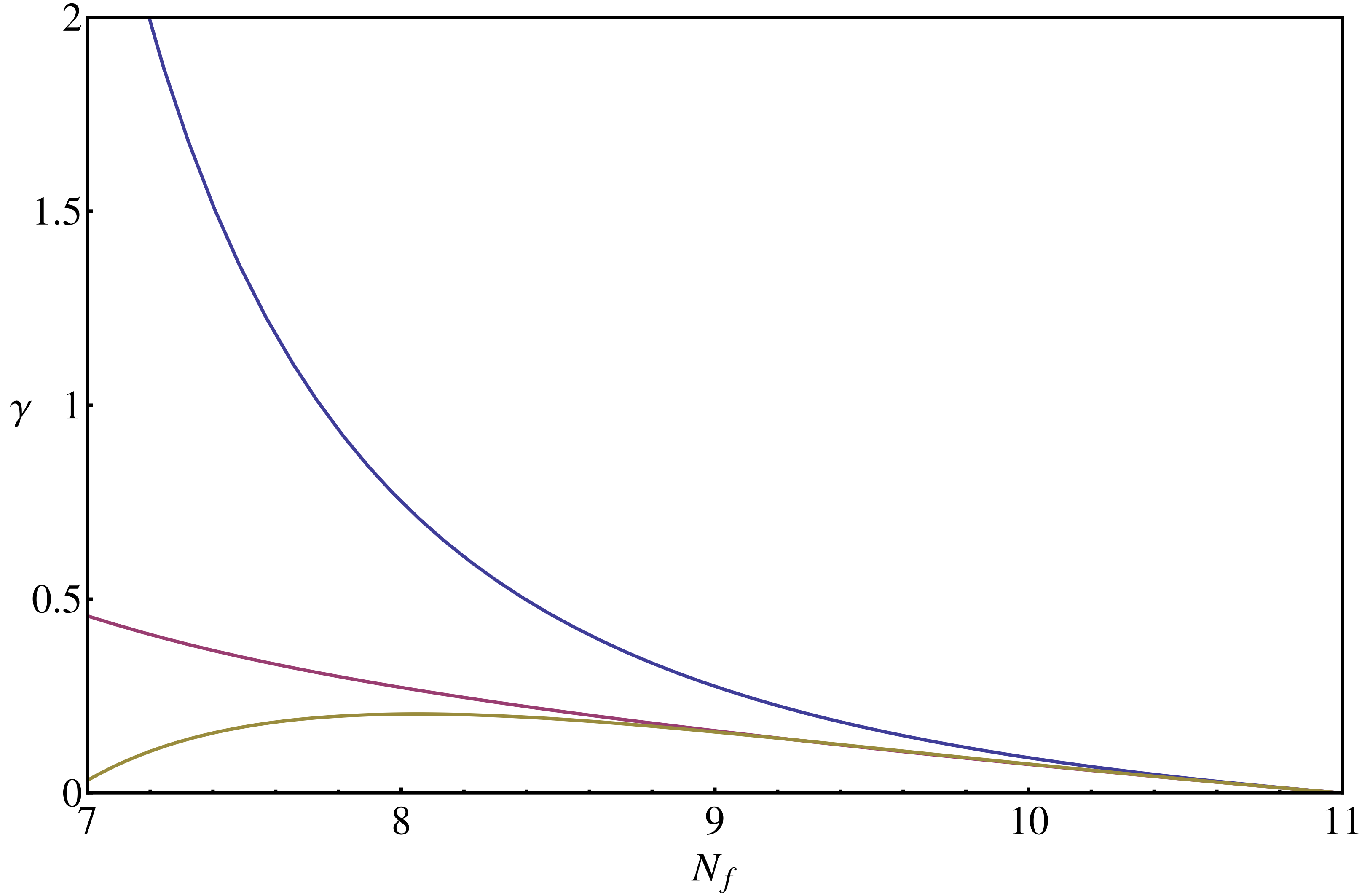

Figure 1: $\boldsymbol{n}$-loop anomalous dimension $\boldsymbol{\gamma_{IR,n\ell}}$ at $\boldsymbol{\alpha_{IR,n\ell}}$ for SU(2) with $\boldsymbol{N_f}$ fermions in fund. rep. (i) blue: $\boldsymbol{\gamma_{IR,2\ell}}$; (ii) red: $\boldsymbol{\gamma_{IR,3\ell}}$; (iii) brown: $\boldsymbol{\gamma_{IR,4\ell}}$.

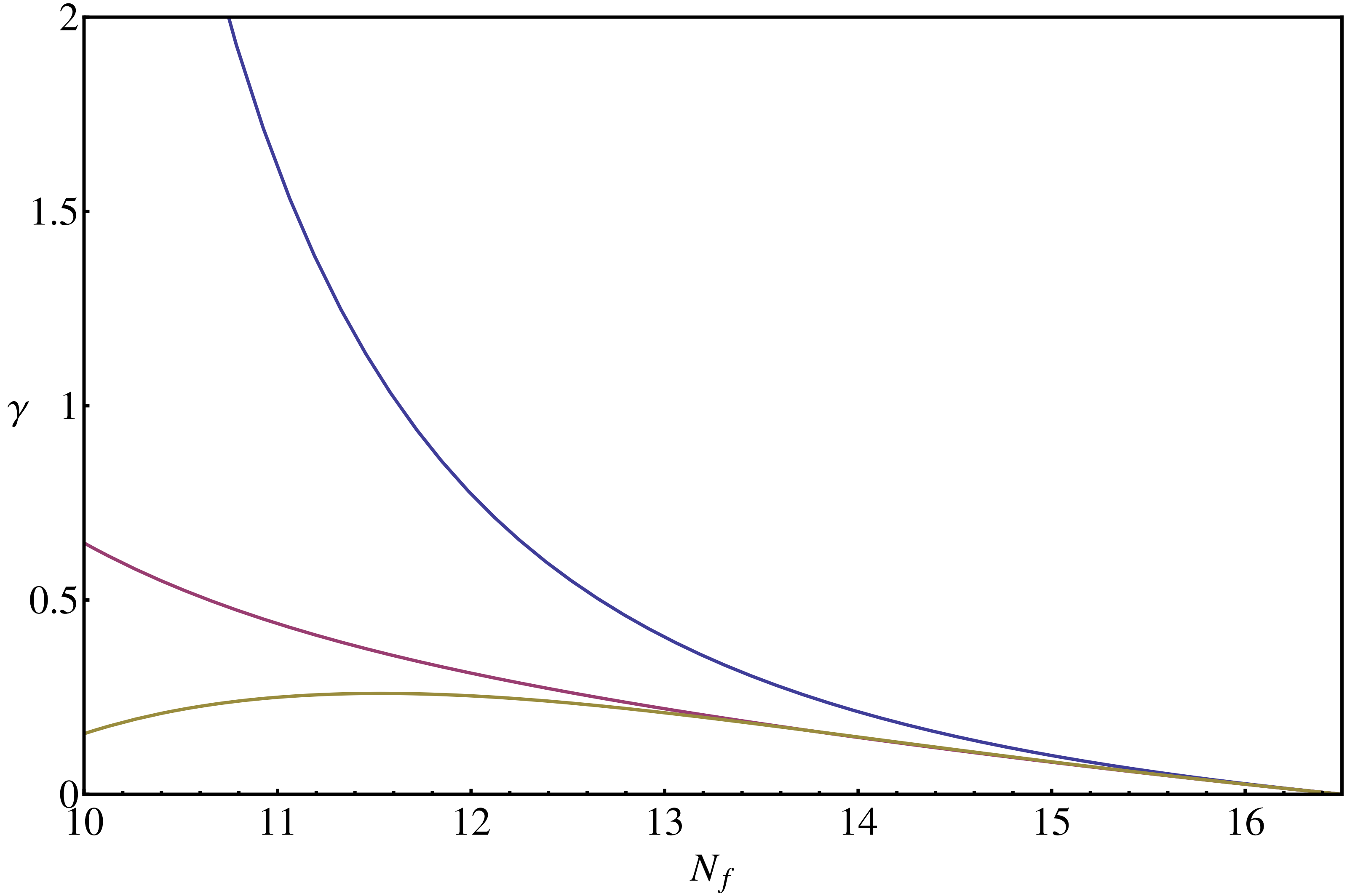

Figure 2: $\boldsymbol{n}$-loop anomalous dimension $\boldsymbol{\gamma_{IR,n\ell}}$ at $\boldsymbol{\alpha_{IR,n\ell}}$ for SU(3) with $\boldsymbol{N_f}$ fermions in fund. rep: (i) blue: $\boldsymbol{\gamma_{IR,2\ell}}$; (ii) red: $\boldsymbol{\gamma_{IR,3\ell}}$; (iii) brown: $\boldsymbol{\gamma_{IR,4\ell}}$.

A necessary condition for a perturbative calculation to be reliable is that higher-order contributions do not modify the result too much. We find that the 3-loop and 4-loop results are closer to each other for a larger range of N_f than the 2-loop and 3-loop results.

So our higher-loop calcs. of α_{IR} and γ allow us to probe the theory reliably down to smaller values of N_f and thus stronger couplings, closer to $N_{f,cr}$.

We find that, for a given N_c, R, and N_f, the values of $\gamma_{IR,n\ell}$ calculated to 3-loop and 4-loop order are smaller than the 2-loop value.

We compare these calculations with lattice measurements next.

Some Comparisons with Lattice Measurements

For example, for SU(3) with $N_f = 12$ fermions in fund. rep., we calculate

$$\gamma_{IR,2\ell} = 0.77, \qquad \gamma_{IR,3\ell} = 0.31, \qquad \gamma_{IR,4\ell} = 0.25$$

some lattice results (N.B.: error estimates do not always include all systematic uncertainties)

$\gamma = 0.414 \pm 0.016$ (Appelquist et al. (LSD Collab.), PRD 84, 054501 (2011))

$\gamma \sim 0.35$ (DeGrand, PRD 84, 116901 (2011))

$0.2 \lesssim \gamma \lesssim 0.4$ (Kuti et al. (method-dep.) arXiv:1205.1878, arXiv:1211.3548, 1211.6164)

$\gamma = 0.4 - 0.5$ (Y. Aoki et al., (LatKMI) PRD 86, 054506 (2012))

$\gamma = 0.27(3)$ (Hasenfratz et al., arXiv:1207.7162; $\gamma \simeq 0.25$ (Hasenfratz et al., arXiv:1310.1124).

So here the 2-loop value is larger than, and the 3-loop and 4-loop values closer to, these lattice measurements.

Thus, as expected, our higher-loop calculations of γ yield better agreement with these lattice measurements than two-loop calculations.

For lower N_f, α_{IR} is larger and γ_m can also be larger. For sufficiently small N_f the coupling becomes too large for perturbative methods to be reliable.

N.B.: for some theories with given gauge group G and fermion content, there is not yet a consensus as to whether the theory is chirally symmetric or chirally broken in the IR; e.g., for SU(3), $N_f = 12$, Appelquist et al. (LSD); Deuzeman et al; Hasenfratz et al.; DeGrand et al.; Y. Aoki et al. find IR-χ sym. while Jin and Mawhinney and Kuti et al. find SχSB.

We have also performed calculations for higher-dimensional representations, e.g.:

SU(N_c) with N_f fermions in symmetric rank-2 (S2) tensor representation - we find:

N_c	N_f	$\gamma_{IR,2\ell,S2}$	$\gamma_{IR,3\ell,S2}$	$\gamma_{IR,4\ell,S2}$
3	2	(2.44)	1.28	1.12
3	3	0.144	0.133	0.133
4	2	(4.82)	(2.08)	1.79
4	3	0.381	0.313	0.315

Some lattice results for $N_f = 2$ fermions in symmetric rank-2 tensor rep.:

e.g., SU(3) (sextet rep.), $N_f = 2$

$\gamma \lesssim 0.45$ (Degrand, Shamir, Svetitsky, arXiv:1201.0935, find IR-conformality)

$\gamma \sim 1.5$ (Kuti et al., arXiv:1205.1878, PTP, find SχSB)

Also phase structure studies by Kogut, Sinclair.

Further Higher-Loop Structural Properties of β

In addition to $\alpha_{IR,n\ell}$, further interesting structural properties of the n-loop beta fn. $\beta_{n\ell}$ include

- the derivative $\beta'_{IR,n\ell} \equiv \frac{d\beta_{n\ell}}{d\alpha}$ evaluated at $\alpha_{IR,n\ell}$.
- the magnitude and location of the minimum in $\beta_{n\ell}$

In quasi-scale-invariant case where $\alpha_{IR} \gtrsim \alpha_{cr}$, dilaton mass depends on how small β is for α near to α_{IR} and hence, at n-loop order, on $\beta'_{IR,n\ell}$, via the series expansion of $\beta_{n\ell}$ around $\alpha_{IR,n\ell}$,

$$\beta_{n\ell}(\alpha) = \beta'_{IR,n\ell}\,(\alpha - \alpha_{IR,n\ell}) + O\Big((\alpha - \alpha_{IR,n\ell})^2\Big)$$

We have calculated these structural properties analytically and numerically in RS, Phys. Rev. D87, 105005 (2013) [arXiv:1301.3209].

We prove a general inequality: for a given gauge group G, fermion rep. R, and $N_f \in I$ (in a scheme with $b_3 < 0$, which thus preserves the existence of the 2-loop IR zero in β at 3-loop level),

$$\beta'_{IR,3\ell} < \beta'_{IR,2\ell}$$

We carry out a similar analysis of the derivative of the 4-loop β function evaluated at $\alpha_{IR,4\ell}$, denoted $\beta'_{IR,4\ell}$, and find a similar decrease from 3-loop to 4-loop order. Some numerical values:

N_c	N_f	$\beta'_{IR,2\ell}$	$\beta'_{IR,3\ell}$	$\beta'_{IR,4\ell}$
2	7	1.20	0.728	0.677
2	8	0.400	0.318	0.300
2	9	0.126	0.115	0.110
2	10	0.0245	0.0239	0.0235
3	10	1.52	0.872	0.853
3	11	0.720	0.517	0.498
3	12	0.360	0.2955	0.282
3	13	0.174	0.156	0.149
3	14	0.0737	0.0699	0.0678
3	15	0.0227	0.0223	0.0220
3	16	0.00221	0.00220	0.00220

Illustrative figures for SU(2) with $N_f = 8$ fermions and SU(3) with $N_f = 12$ fermions:

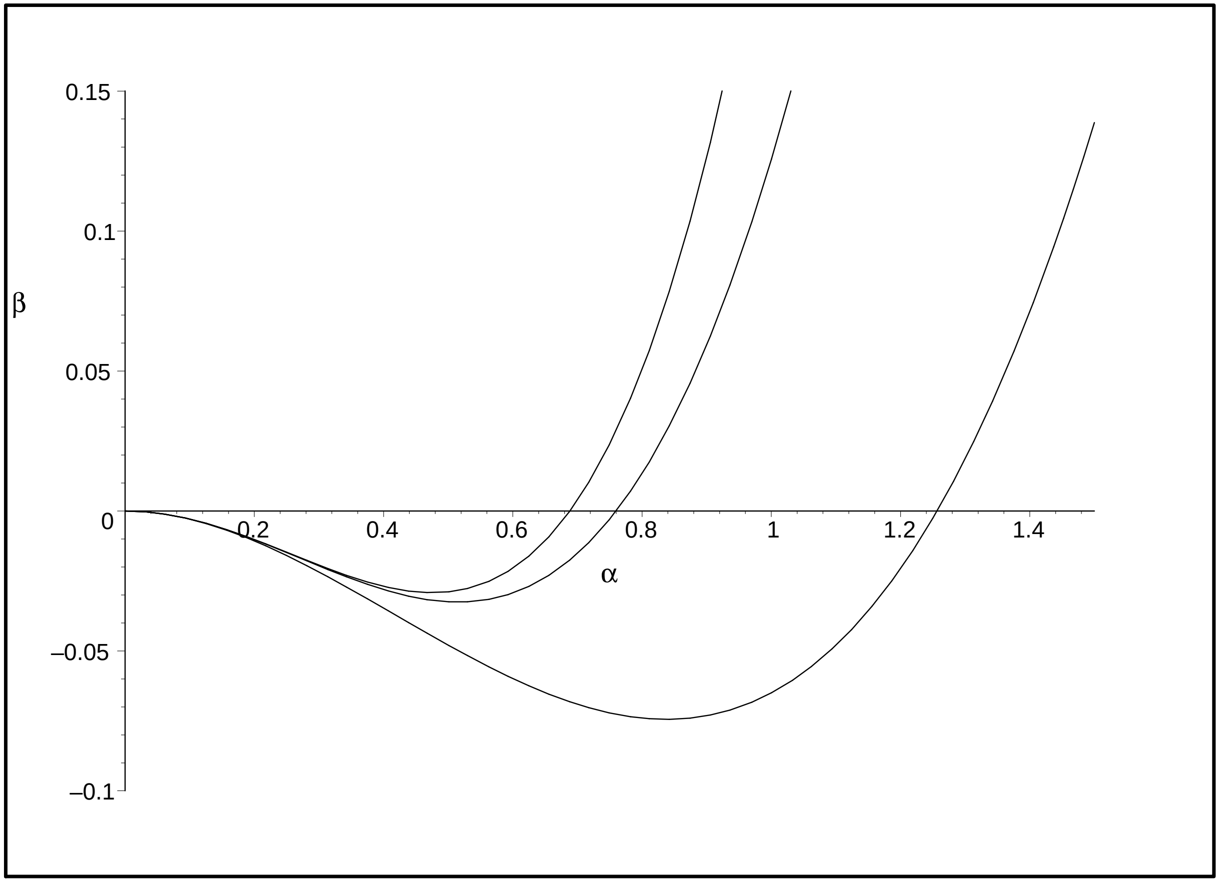

Figure 3: $\boldsymbol{\beta_{n\ell}}$ for SU(2), $\boldsymbol{N_f = 8}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

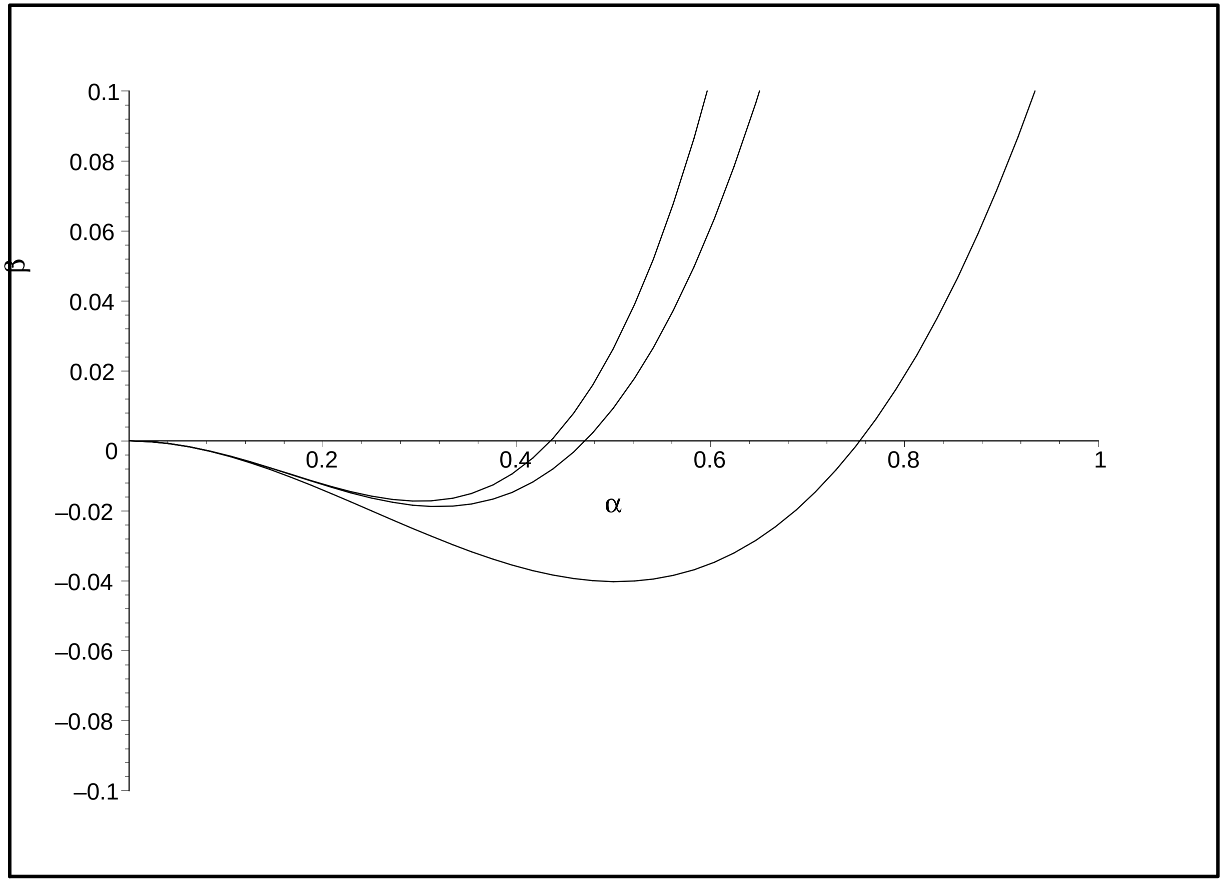

Figure 4: $\boldsymbol{\beta_{n\ell}}$ for SU(3), $\boldsymbol{N_f = 12}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

Interesting property: for $R =$ fund. rep., $\alpha_{IR,n\ell} N_c$, $\gamma_{IR,n\ell}$, and other structural properties of $\beta_{n\ell}$ are similar in theories with different values of N_c and N_f if they have equal or similar values of

$$r \equiv \frac{N_f}{N_c}$$

This motivates a study of the UV to IR evolution of an SU(N_c) gauge theory with N_f fermions in the fundamental rep. in the 't Hooft-Veneziano limit $N_c \to \infty$, $N_f \to \infty$ with r fixed and $\alpha(\mu) N_c \equiv \xi(\mu)$ independent of N_c.

We have carried out this study in RS, Phys. Rev. D87, 116007 (2013) [arXiv:1302.5434].

We show that the approach to this limit is quite rapid, with leading correction terms suppressed by $1/N_c^2$. For example, at the two-loop level, the IRFP is

$$\xi_{IR,2\ell} = \frac{4\pi(11-2r)}{13r-34} + \frac{12\pi r(11-2r)}{(34-13r)^2 N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

and the anomalous dimension is

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2}$$

$$+ \frac{(11-2r)(18836-5331r+648r^2-140r^3)}{(13r-34)^3 N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

These results provide an understanding of the approximate universality that is exhibited in calculations of these quantities for different (finite) values of N_c and N_f with similar or identical values of r.

Study of Scheme Dependence in Calculation of IR Fixed Point

Since coeffs. b_n in $\beta_{n\ell}$, and hence also $\alpha_{IR,n\ell}$, are scheme-dependent for $n \geq 3$, it is important to assess the effects of this scheme dependence. We have studied this in RS, Phys. Rev. D88, 036003 (2013) [arXiv:1305.6524] and RS, Phys. Rev. D90, 045011 (2014) [arXiv:1405.6244], extending our earlier work in Ryttov and RS, PRD 86, 065032 (2012) [arXiv:1206.2366] and PRD 86, 085005 (2012) [arXiv:1206.6895].

A scheme transformation (ST) is a map between α and α' or equivalently, a and a', where $a = \alpha/(4\pi)$ of the form

$$a = a' f(a')$$

with $f(0) = 1$ since for $\alpha = 0$, a ST has no effect. Write

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s = 1 + \sum_{s=1}^{s_{max}} \bar{k}_s (\alpha')^s ,$$

where $\bar{k}_s = k_s/(4\pi)^s$, and s_{max} may be finite or infinite. Jacobian $J = da/da' = d\alpha/d\alpha'$.

We calculate the coefficients in the transformed scheme. In addition to the well-known results $b'_1 = b_1$ and $b'_2 = b_2$, we find

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1 \ ,$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1$$

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2) b_3 + (-k_1^3 + 3k_1 k_2 - k_3) b_2$$
$$+(4k_1^4 - 11k_1^2 k_2 + 6k_1 k_3 + 4k_2^2 - 3k_4) b_1$$

etc. at higher-loop order.

A physically acceptable scheme transformation must satisfy several conditions:

- C_1: it should map a (real positive) α to a real positive α'
- C_2: it should map a moderate value of α, where perturbation theory is applicable, to a value of α' that is not too large.
- C_3: it must have a nonzero Jacobian to be invertible.
- C_4: Existence of an IR zero of β is a scheme-independent property, so if it is present in one scheme, it should also be present in the transformed scheme.

These conditions are easily satisfied when one applies a scheme transformation to a fixed point at the origin $\alpha = \alpha' = 0$ (UVFP for asymptotically free theory), but they are not automatic, and can be quite restrictive at an IRFP.

For example, consider the ST (dependent on a parameter r)

$$a = \frac{\tanh(ra')}{r}$$

with inverse

$$a' = \frac{1}{2r} \ln\left(\frac{1 + ra}{1 - ra}\right)$$

This is acceptable for small a, but if $a > 1/r$, i.e., $\alpha > 4\pi/r$, it maps a real α to a complex α' and hence is physically unacceptable. For, say, $r = 8\pi$, this pathology can occur at the moderate value $\alpha = 0.5$.

We have constructed several STs that are acceptable at an IRFP and have studied scheme dependence of the IR zero of $\beta_{n\ell}$ using these. For example,

$$a = \frac{\sinh(ra')}{r}$$

with inverse

$$a' = \frac{1}{r} \ln\left[ra + \sqrt{1 + (ra)^2}\,\right]$$

We find reasonably small scheme-dependence for moderate α_{IR}.

For example, for $N_c = 3$, $N_f = 12$, $\alpha_{IR,2\ell} = 0.754$, and:

$$\alpha_{IR,3\ell,\overline{\rm MS}} = 0.435, \quad \alpha'_{IR,3\ell,r=3} = 0.434, \quad \alpha'_{IR,3\ell,r=6} = 0.433,$$

$$\alpha_{IR,4\ell,\overline{\rm MS}} = 0.470, \quad \alpha'_{IR,4\ell,r=3} = 0.470, \quad \alpha'_{IR,4\ell,r=6} = 0.467,$$

Since the b_n with $n \geq 3$ are scheme-dependent, one might expect that it would be possible, at least in the vicinity of the origin, $\alpha = \alpha' = 0$, to construct a scheme transformation that would set $b'_n = 0$ for some range of $n \geq 3$, and, indeed a ST that would do this for all $n \geq 3$, so that $\beta_{\alpha'}$ would consist only of the 1-loop and 2-loop terms ('t Hooft scheme).

We have constructed an explicit scheme transformation that does this in the vicinity of the origin, $\alpha = \alpha' = 0$, and have also investigated the limited range in α over which this can be done away from the origin for which the ST is physically acceptable, satisfying conditions C_1-C_4.

Other Theories

We have studied UV to IR evolution in asymptotically free supersymmetric gauge theories with T. Ryttov and in chiral gauge theories with T. Appelquist.

It is also of interest to study RG flows in theories that are IR-free. Early example: exact solution of the O(N) nonlinear σ model in $d = 2 + \epsilon$ in the large-N limit in W. Bardeen, B. W. Lee, and RS, Phys. Rev. D14, 985 (1976); E. Brézin and J. Zinn-Justin, Phys. Rev. B 14, 3110 (1976), yielded the beta function (for small ϵ)

$$\beta(\lambda) = \epsilon\lambda\Big(1 - \frac{\lambda}{\lambda_c}\Big)$$

where λ is the effective coupling and $\lambda_c = 2\pi\epsilon/N$. So this theory has an IRFP at $\lambda = 0$ and a UVFP at $\lambda = \lambda_c$. This is an example of Weinberg's notion of "asymptotic safety".

There has long been interest in RG properties of $d = 4$ QED and, more generally, U(1) gauge theory.

We carried out a study of possible zeros of the beta function for U(1) gauge theory (in $d = 4$) up to the five-loop order in RS, Phys. Rev. D89, 045019 (2014) [arXiv:1311.5268]. We find evidence against a UV zero in this beta function.

Ending Remarks

- Let us celebrate and honor Mike Creutz for his pioneering and crucial contributions to lattice gauge theory and QCD. It is a privilege to be part of this celebration.
- We hope that Mike will stay around for many more years, as a colleague and friend.
- The study of the UV to IR evolution of an asymptotically free gauge theory with various fermion contents continues to be of fundamental field-theoretic interest.
- Higher-loop calculations give further information on this UV to IR flow and on determination of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$.
- These higher-loop calculations improve agreement with lattice measurements.