Old lattice guy and youngsters at BNL

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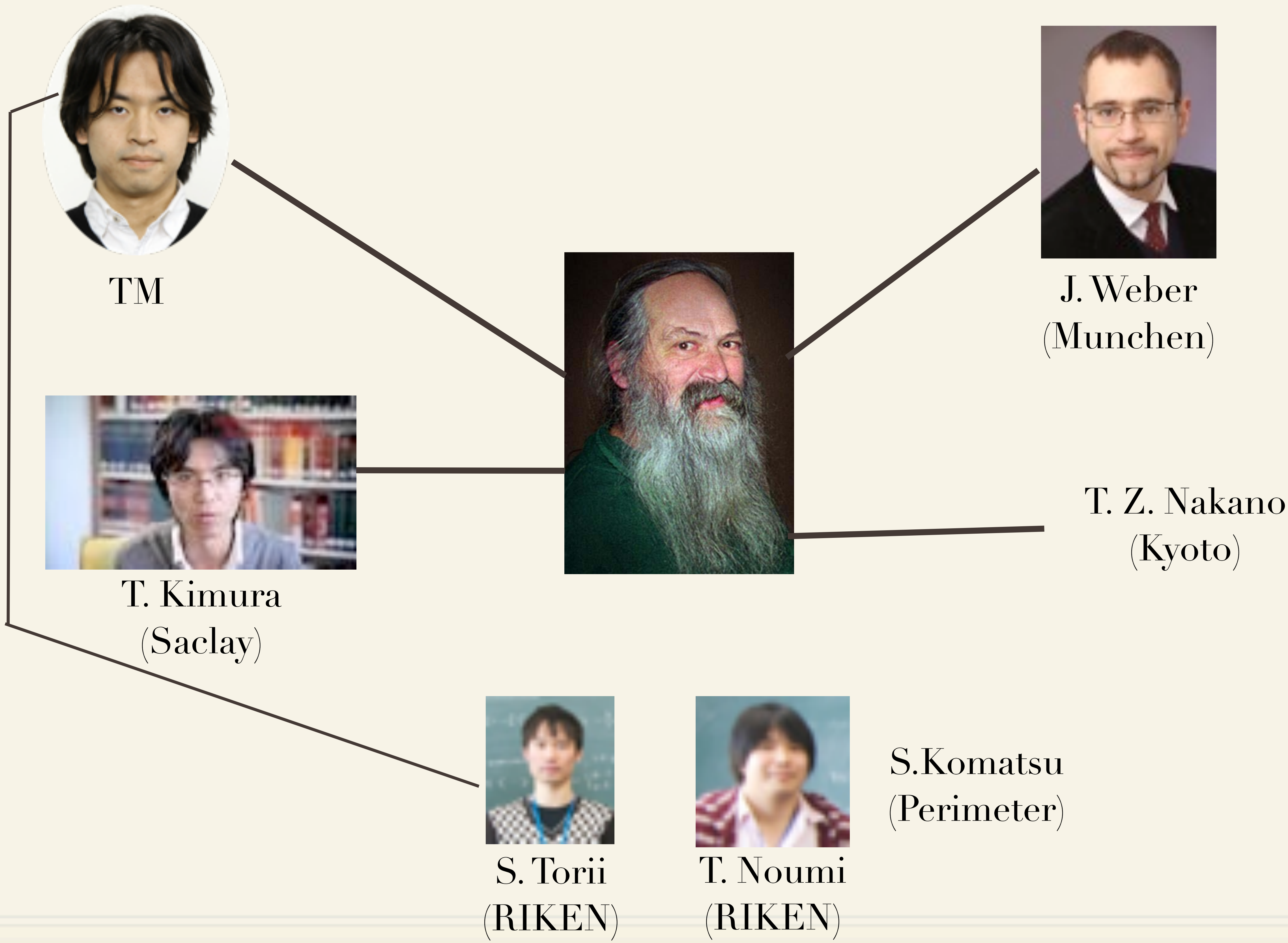
09/05/2014 CreutzFest@BNL

This is a story of the latest days of ours research (2009-2014).

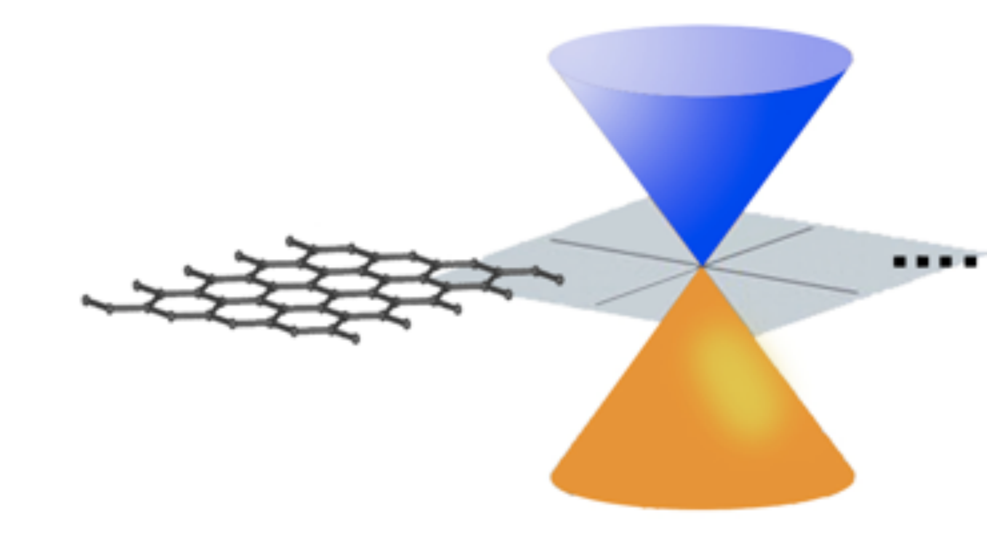
Mike’s web with young researchers

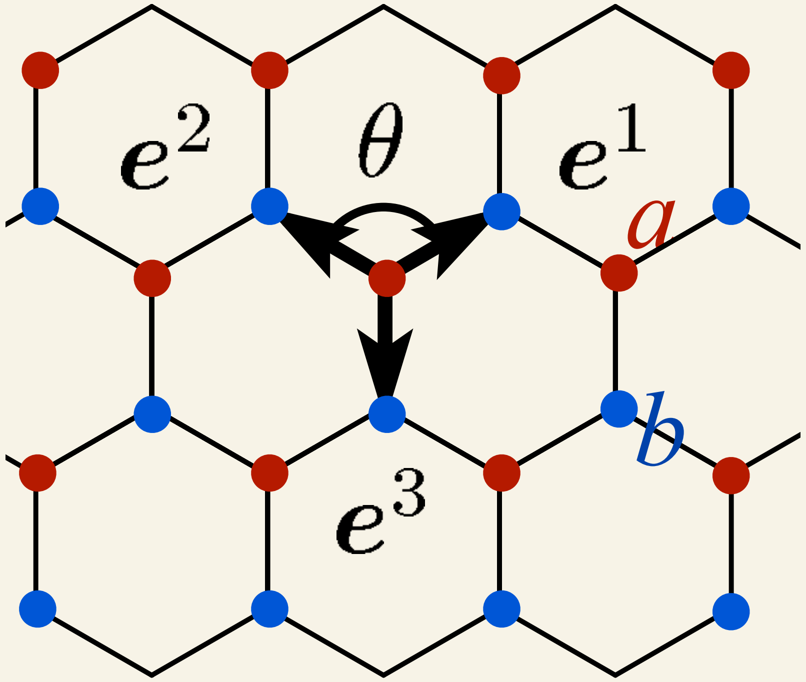


1. Minimal doubling

Graphene electron system

- Unit 2-site → emergent spin
- Bipartite hop → chiral sym
- Linear disp. → relativistic





$$z(\mathbf{k}) = \sum_\mu \exp(i\mathbf{k}\cdot\mathbf{e}^\mu)$$

$$H = \sum_{\mathbf{p}} (c_a^\dagger(\mathbf{k}),\ c_b^\dagger(\mathbf{k})) \begin{pmatrix} 0 & z(\mathbf{k}) \\ z^*(\mathbf{k}) & 0 \end{pmatrix} \begin{pmatrix} c_a(\mathbf{k}) \\ c_b(\mathbf{k}) \end{pmatrix}$$

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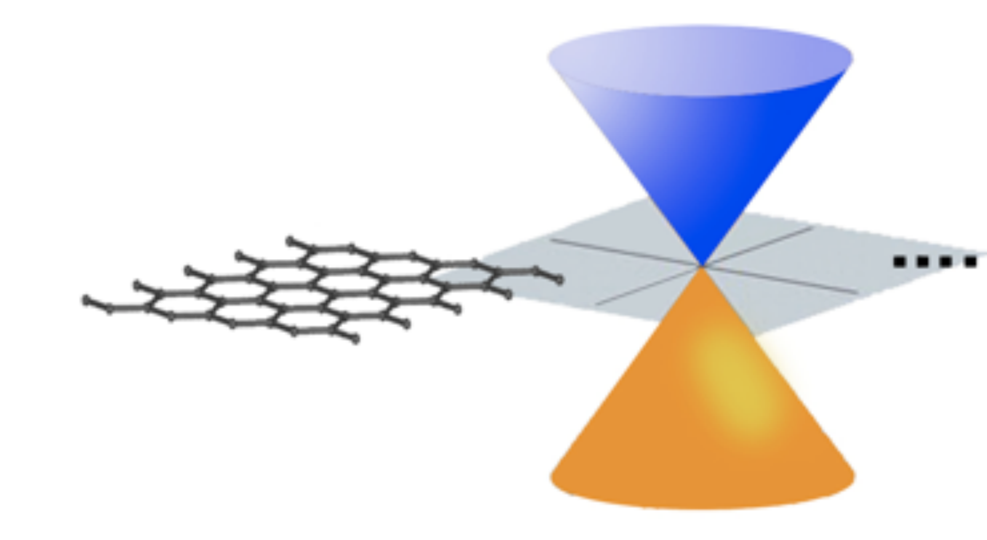

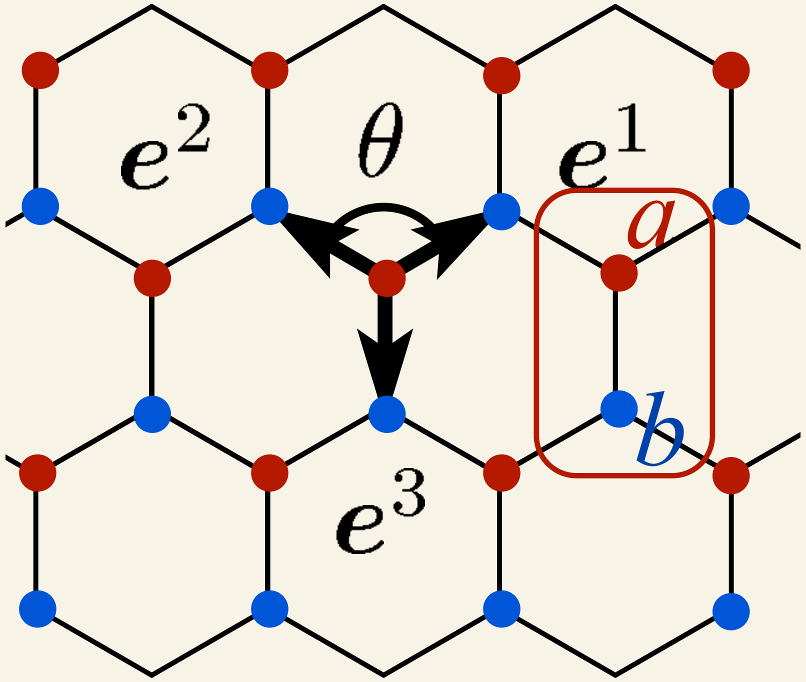

$$z(\mathbf{k}) = \sum_\mu \exp(i\mathbf{k}\cdot\mathbf{e}^\mu)$$

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→ emergence of spin d.o.f.

Graphene electron system

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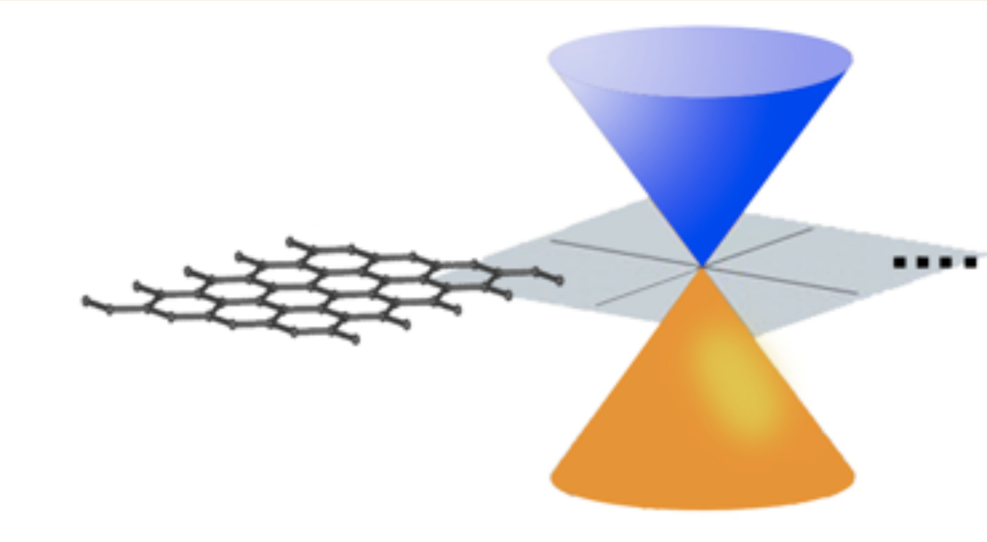

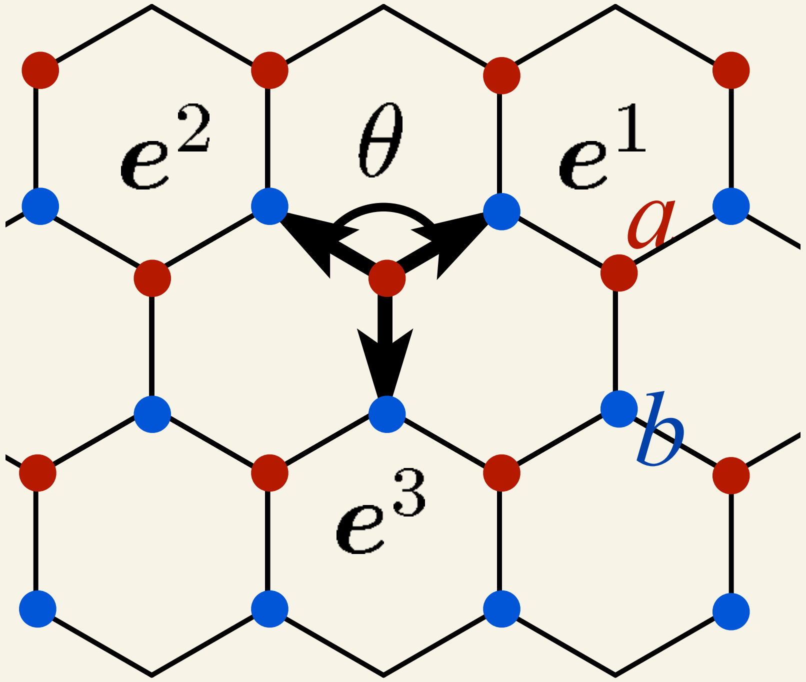

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$$\sigma_3 \mathcal{H} \sigma_3 = -\mathcal{H}$$

$$\sigma_3 D \sigma_3 = -D \quad (\mathcal{H} = \sigma_3 D)$$

subgroup of exact chiral symmetry

Graphene electron system

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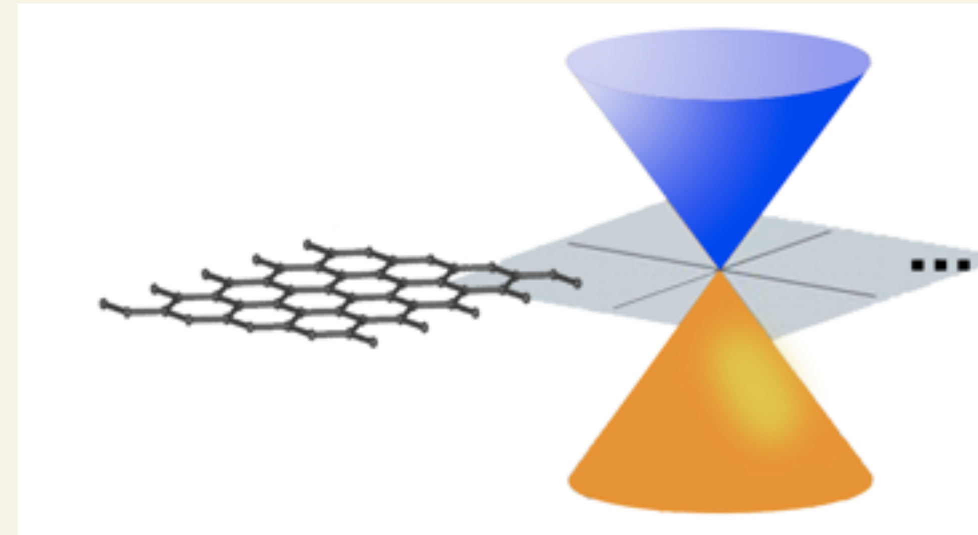

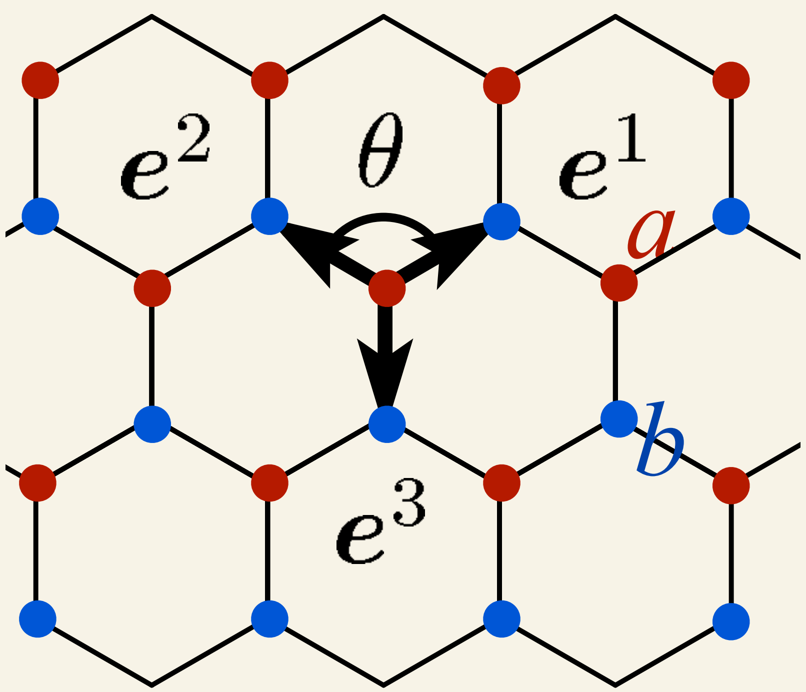

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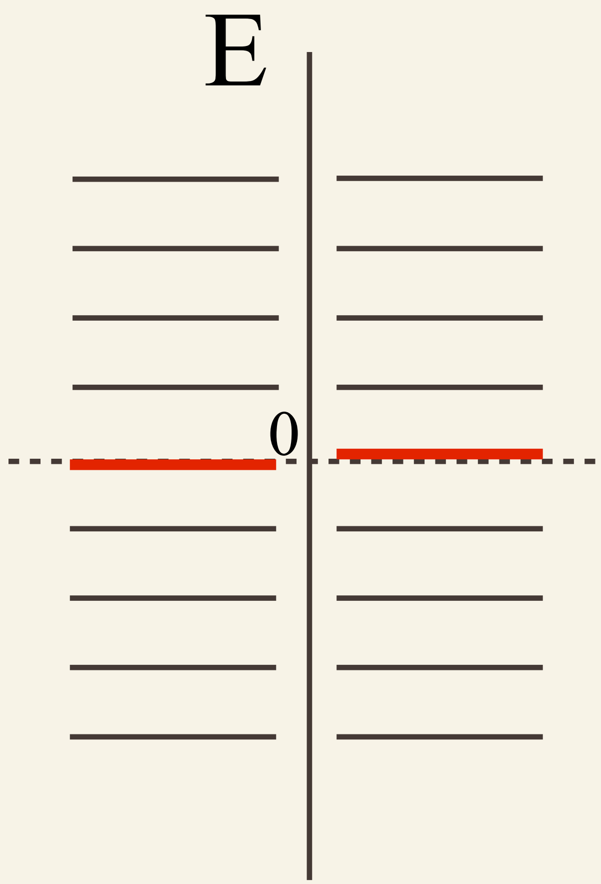
$$H = \sum_{\mathbf{p}} (c_a^\dagger(\mathbf{k}),\ c_b^\dagger(\mathbf{k})) \begin{pmatrix} 0 & z(\mathbf{k}) \\ z^*(\mathbf{k}) & 0 \end{pmatrix} \begin{pmatrix} c_a(\mathbf{k}) \\ c_b(\mathbf{k}) \end{pmatrix}$$

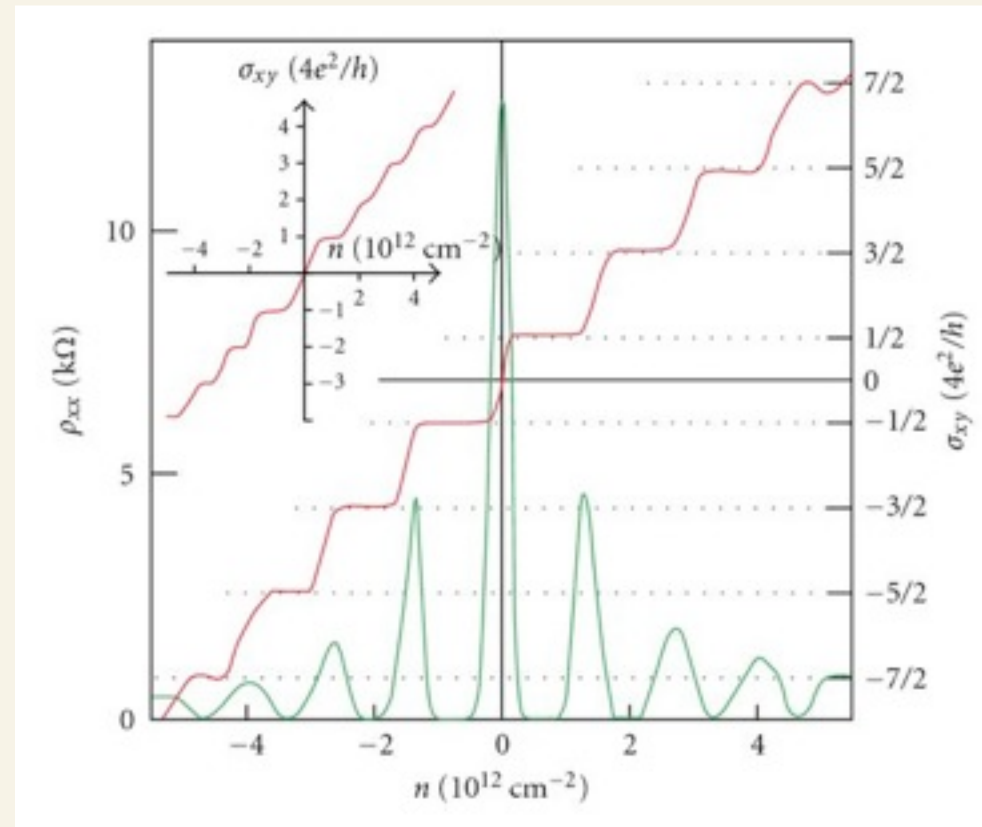
$$\sim \frac{3}{2} \begin{pmatrix} 0 & iq_1 - q_2 \\ -iq_1 - q_2 & 0 \end{pmatrix}$$
$$= \frac{3}{2}\,[-\gamma^0 \vec{\gamma}\cdot\vec{q}]$$

2D Dirac operator emerges

What is observed experimentally ?

Novoselov, Geim,et.al. (05)





Landau zero mode

spectral asymmetry = η-invariant
(parity anomaly)

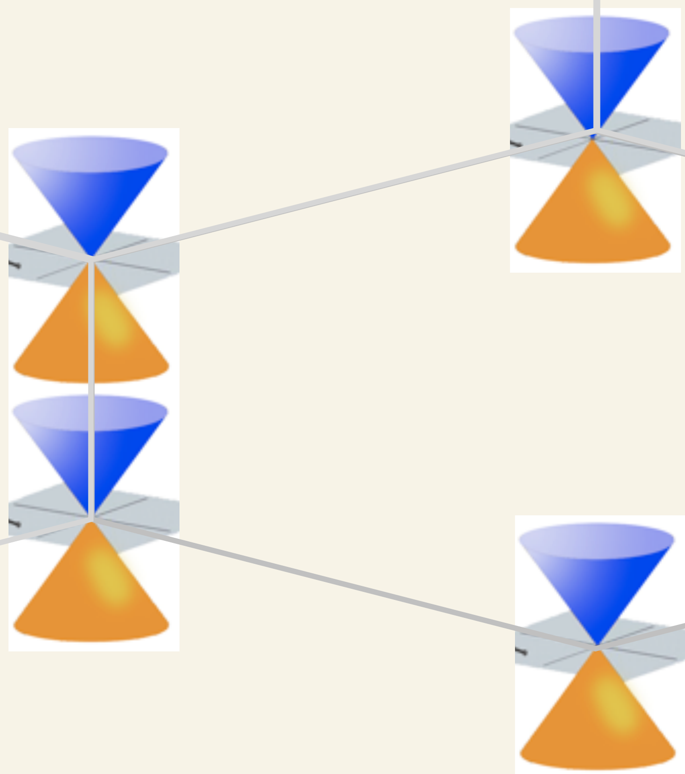
$$\frac{e^2}{8\pi}\int d^3x\, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

Anomalous quantum Hall effect

Outstanding difference from the usual QHE

Particle physics on the condensed matter

Viewpoints of ultralocal chiral fermions



Zeros

$$\hat{\mathbf{k}} = \pm\Big(0, \frac{4\sqrt{3}\pi}{9}\Big), \quad (\pm\frac{2\pi}{3}, \pm\frac{2\sqrt{3}\pi}{9})$$

- Zeros of the Dirac operator also forms *Honeycomb*!
- Unit 2-site in Brillouin zone → 2 species (Minimal-doubling)
- Chiral + sufficient discrete symmetry + ultralocal
- Fermion with ideal properties for two-flavor lattice QCD

Why don't you extend it to 4D lattice ?

2D graphene systems

1. One-spinor on Honeycomb → spin
2. Bipartite hopping → chiral symmetry
3. Linear dispersion → relativistic system
4. Honeycomb zeros → Minimal-doubling

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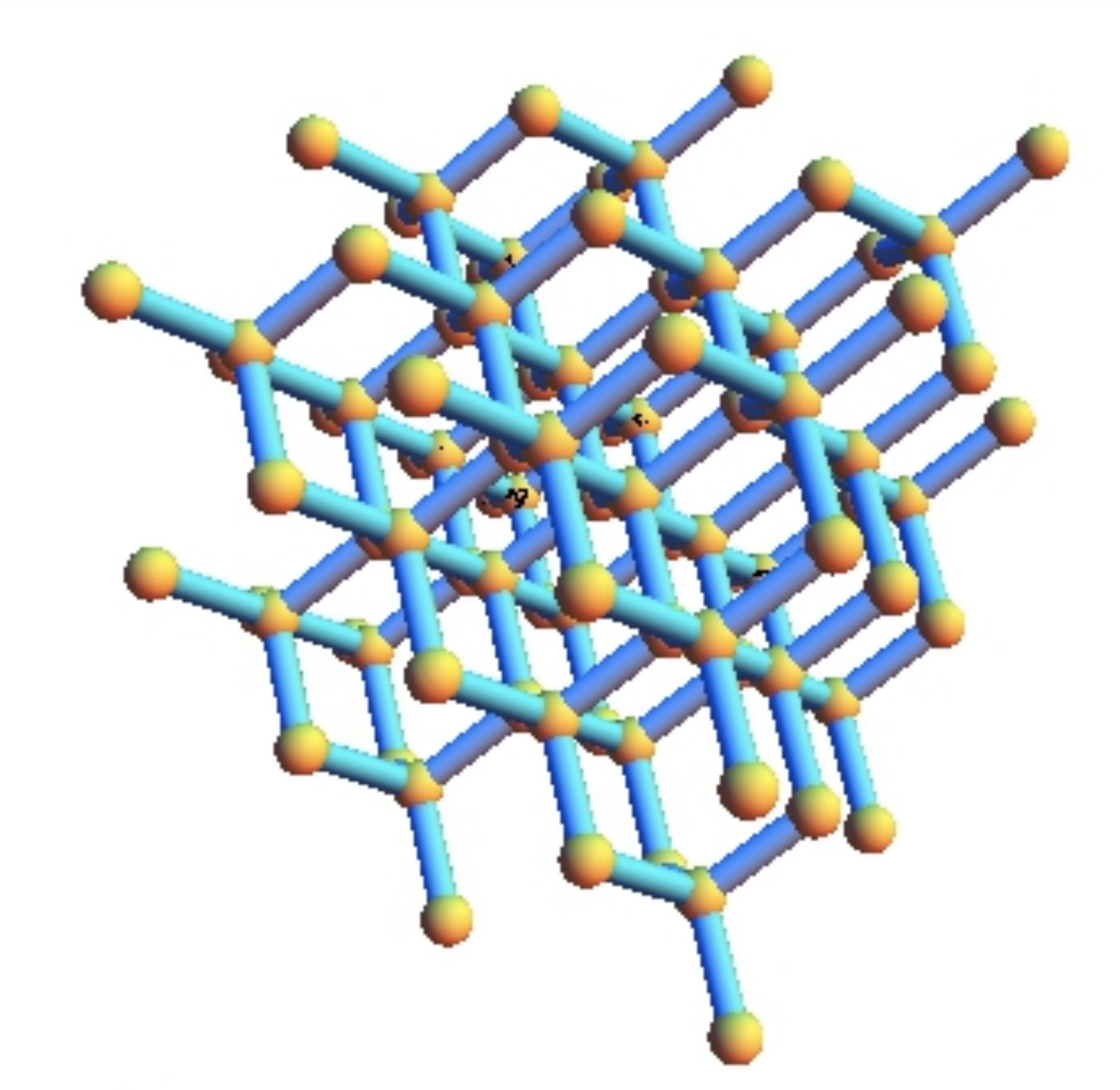
Creutz's minimal-doubling fermion

Creutz (07)

2. chiral symmetry + 4. Hyperdiamond zeros



Creutz's minimal-doubling fermion

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$$H = \begin{pmatrix} 0 & z(\mathbf{p}) \\ z^*(\mathbf{p}) & 0 \end{pmatrix}$$

$$\gamma_5 H \gamma_5 = -H$$

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$$H = \begin{pmatrix} 0 & z(\mathbf{p}) \\ z^*(\mathbf{p}) & 0 \end{pmatrix}$$

$$\gamma_5 H \gamma_5 = -H$$

$$\begin{aligned} z = \ & B[4C - \cos(p_1) - \cos(p_2) - \cos(p_3) - \cos(p_4)] \\ & + i\sigma_x(\sin(p_1) + \sin(p_2) - \sin(p_3) - \sin(p_4)) \\ & + i\sigma_y(\sin(p_1) - \sin(p_2) - \sin(p_3) + \sin(p_4)) \\ & + i\sigma_z(\sin(p_1) - \sin(p_2) + \sin(p_3) - \sin(p_4)) \end{aligned}$$

$$B = \sqrt{5}\cot(\pi/5), \quad C = \cos(\pi/5)$$

Minimal-doubling $\tilde{p} = \pm\cos^{-1} C$

Chiral and Minimal-doubling lattice fermion!

Creutz's minimal-doubling fermion

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$$B = \sqrt{5}\cot(\pi/5), \quad C = \cos(\pi/5)$$

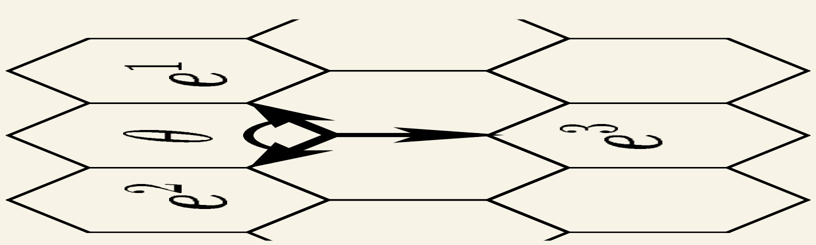
Minimal-doubling $\tilde{p} = \pm \cos^{-1} C$

- Action in real space

$$S_C = \frac{1}{2}\sum_{x,\mu} \bar{\psi}_x \left[(\Xi_\mu + i\,B\gamma_4)\psi_{x+\mu} - (\Xi_\mu - i\,B\gamma_4)\psi_{x-\mu} - 2i\,BC\,\gamma_4\psi_x \right]$$

Bedaque, Buchoff, Tiburzi, Walker-Loud(08)
Capitani, Creutz, Weber, Wittig (09)(10)

→ small discrete symmetry
→ needs parameter tuning



2D graphene systems

1. One-spinor on Honeycomb → spin
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4. Honeycomb zeros → Minimal-doubling

Fermions on hyperdiamond lattice

- Two-spinor on Hyperdiamond lattice   Bedaque, et.al.(08)

$$S_{Be} = \sum_x [(\bar{\phi}_{x-\mu}\sigma \cdot \mathbf{e}^\mu \chi_x - \bar{\chi}_{x+\mu}\bar{\sigma} \cdot \mathbf{e}^\mu \phi_x) + \bar{\phi}_x \sigma \cdot \mathbf{e}^5 \chi_x - \bar{\chi}_x \bar{\sigma} \cdot \mathbf{e}^5 \phi_x]$$

→ Not minimal-doubling

- One-spinor on Hyperdiamond lattice   Kimura, TM (09)

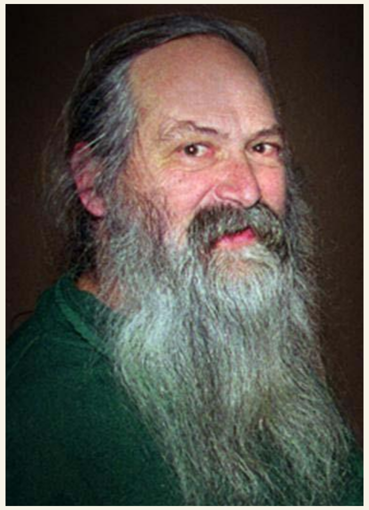
$$H = \sum_x [a_x^\dagger b_x + a_{x+\hat{1}}^\dagger b_x + a_{x-\hat{2}}^\dagger b_x + b_x^\dagger a_x + b_{x-\hat{1}}^\dagger a_x + b_{x+\hat{2}}^\dagger a_x]$$

→ Neither minimal-doubling nor relativistic

Symmetric minimal-doubling is open question

July 2009

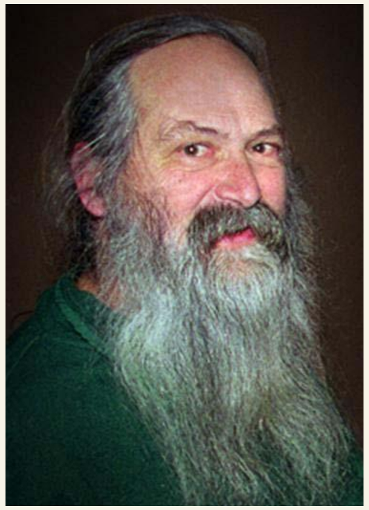
I have things to ask you...
Could you teach me?



Yes, of course!

Aug. 2009

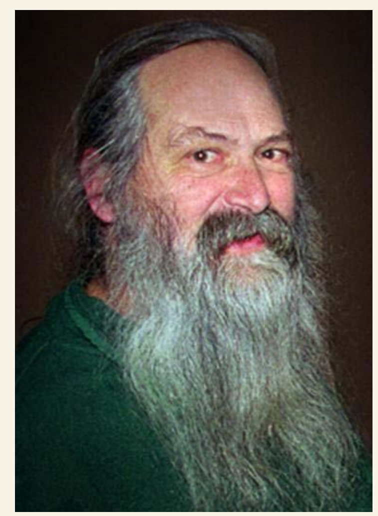
I have a draft I want you to read…
Could you read it?



Yes, sure!
I will comment it.

Sep. 2009

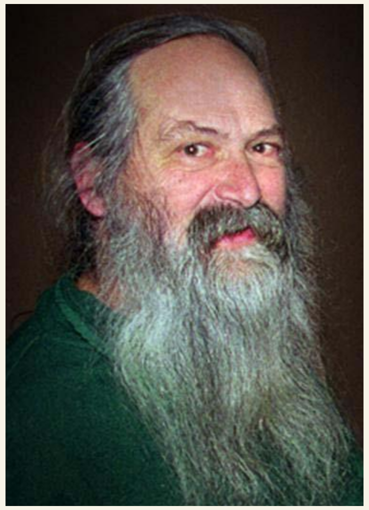
I want to visit you,
and preferably study under you.



OK!
Let us consider what we can do together!

Apr. 2010

I want to visit you,
and preferably study under you.



OK!
Let us consider what we can do together!

Collaboration has started !
How generous he is !

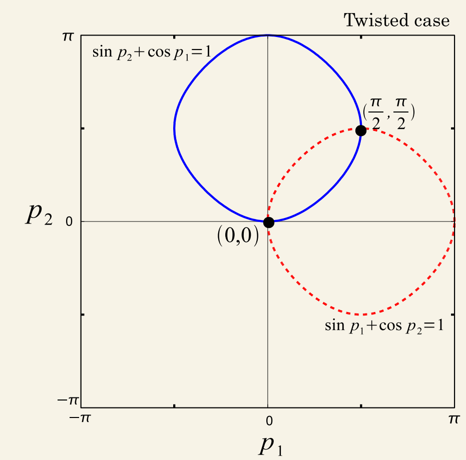
Classification & Index theorem

Creutz,TM(10)
Creutz,Kimura,TM(10)

◆Classification of minimal-doubling

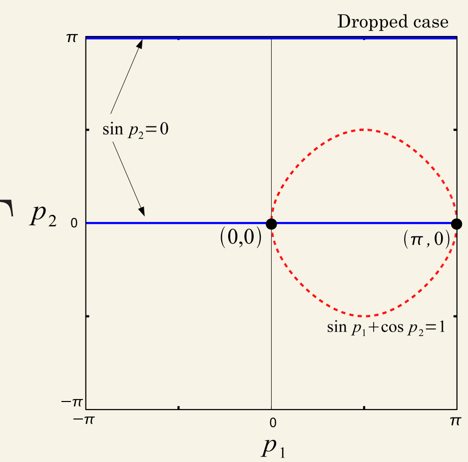
1. Twisted-ordering (includes Karsten-Wilczek)

$$D(p) = \sum_{\mu} i\gamma_\mu \sin p_\mu + \sum_{i,j} i\gamma_i R_{ij}(\cos p_j - 1)$$



2. Dropped twisted-ordering (includes Borici-Creutz)

$$D(p) = i\sum_{\mu}[\gamma_\mu \sin(p_\mu + \beta_\mu) - \gamma'_\mu \sin(p_\mu - \beta_\mu)] - i\Gamma$$



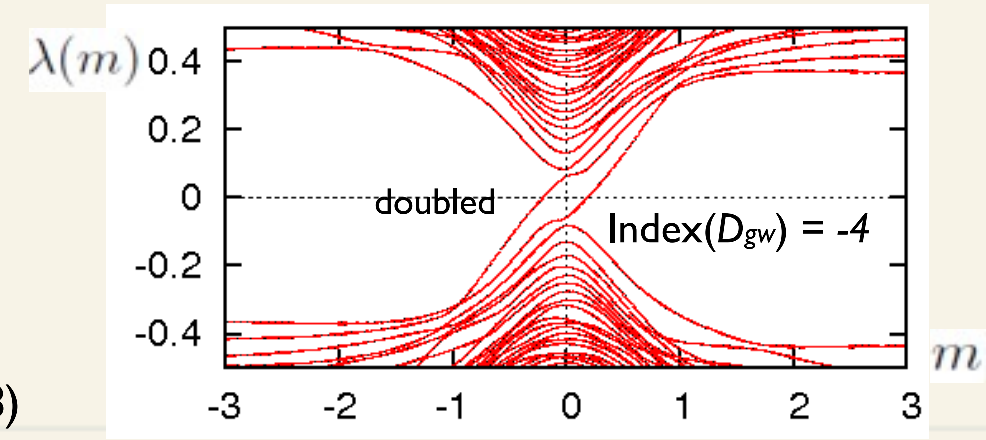
◆Index theorem

Index(D_{MD}) = - Spectral flow(H_{MD})

⇩

$$\mathrm{Index}(D_{MD}) = (-1)^{d/2} Q$$

cf.)Wilson fermion, R.Edwards, U.Heller, R.Narayanan (98)



On-going projects and future

- Simulations with tuning parameters Capitani, Creutz, Weber, Wittig (09)(10)
  Capitani (12)(13)

  Development of efficient ways of tuning + Reduction of parameters

- Relation to imaginary chemical potential TM (12)
  TM, Kimura, Ohnishi (12)

  $$\sum_{\mu}(1-\cos p_{\mu}) \qquad \text{(i)}\,\gamma_4\sum_{j=1}^{3}(1-\cos p_j)$$

  Finite-mass (Wilson) v.s. Finite-density (MD)

- SUSY lattice-inspired chiral fermions TM (13)

  Embed (D+2)-dim gamma matrices in D-dim lattice fermions

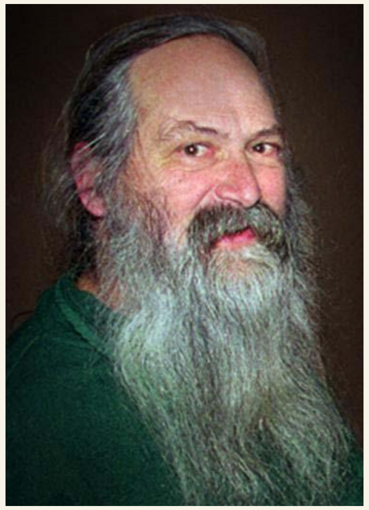
  $$D(p) = i\Gamma_{\mu}\sin p_{\mu} + iP_{\mu}(1-\cos p_{\mu}) \qquad \{\Gamma_{\mu},\Gamma_{\nu}\} = 2\delta_{\mu\nu}\,, \quad \{P_{\mu},P_{\nu}\} = 2\delta_{\mu\nu}\,, \quad \{\Gamma_{\mu},P_{\nu}\} = 0$$

  Invariant under D+2 chiral rotation → Chiral symmetry with no more doublers

II. Flavored mass

Oct 2010

I want my collaborator to visit us,
and start collaboration.



OK, good!
Let us discuss something new!

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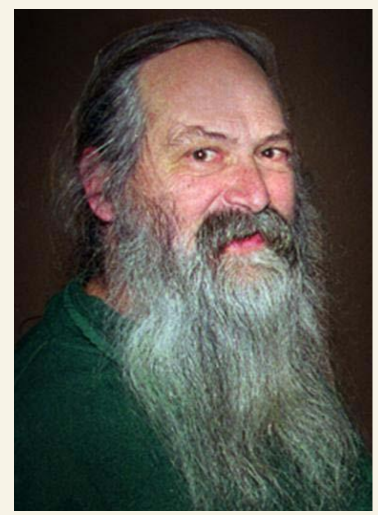


OK, good!
Let us discuss something new!

T. Kimura
(Saclay)

Another collaboration has started !

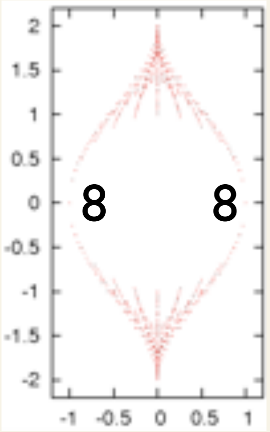
◆ **Flavored mass** Creutz, Kimura, TM (10)(11) inspired by works on staggered, Adams (09) Hoelbling (10)

$$M_{\mathrm{V}} = \sum_{\mu} C_{\mu},$$ *Vector (1-link)*

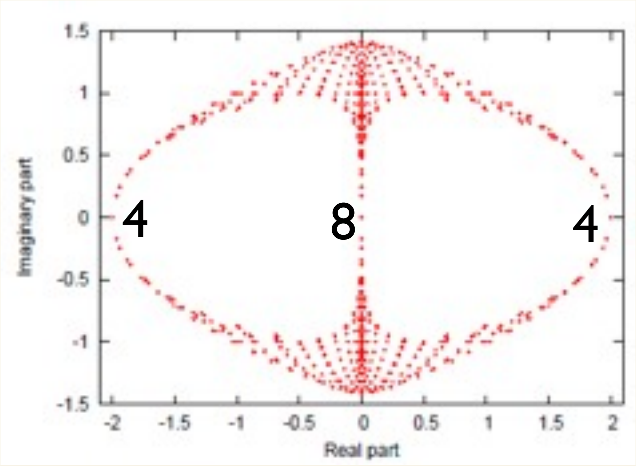
$$M_{\mathrm{T}} = \sum_{perm.} \sum_{sym.} C_{\mu} C_{\nu},$$ *Tensor (2-link)*

$$M_{\mathrm{A}} = \sum_{perm.} \sum_{sym.} \prod_{\nu} C_{\nu},$$ *Axial-V (3-link)*

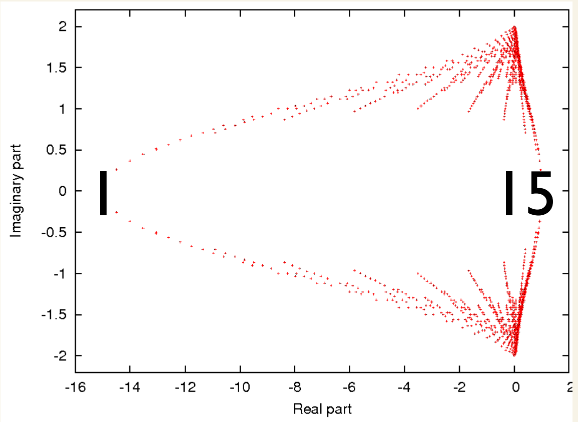
$$M_{\mathrm{P}} = \sum_{sym.} \prod_{\mu=1}^{4} C_{\mu},$$ *Pseudo-S (4-link)*



*M*P



*M*T



*M*V+*M*T+*M*A+*M*P

- Extension of Wilson fermion
- Sufficient discrete symmetry
- Index theorem holds within admissibility

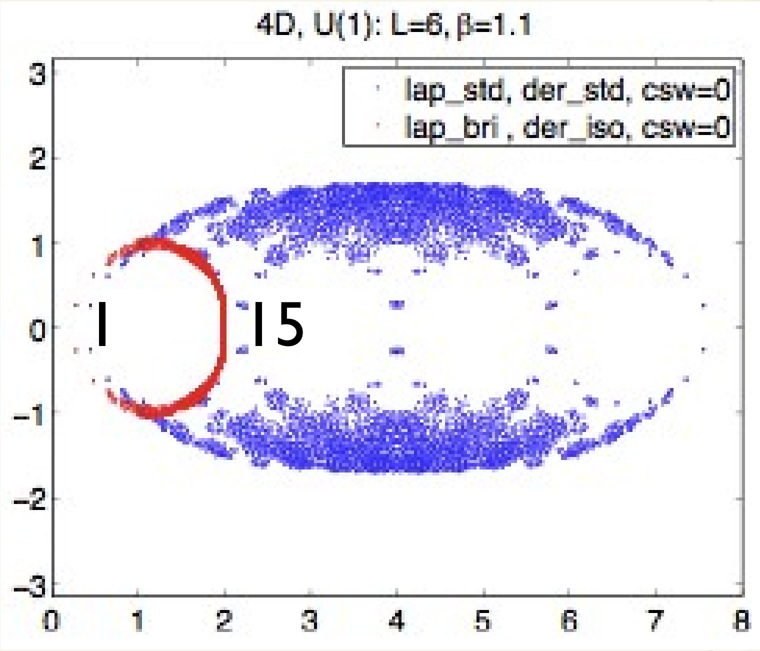
◆Brillouin fermion

Durr, Koutsou (11)(12)　　Creutz, Kimura, TM (10)

Brillouin laplacian　$M_{\rm V} \Rightarrow M_{\rm V} + M_{\rm T} + M_{\rm A} + M_{\rm P}$

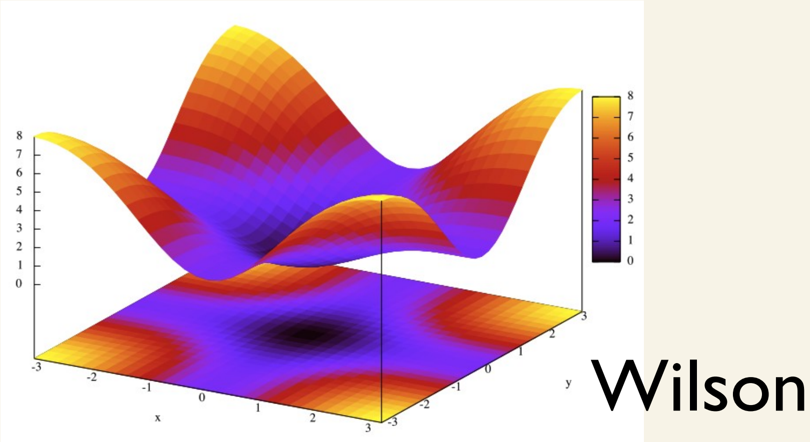
$$\triangle = 4 - \sum_\mu C_\mu \Rightarrow \triangle^{bri} = 2 - 2\prod_\mu \frac{1+C_\mu}{2}$$

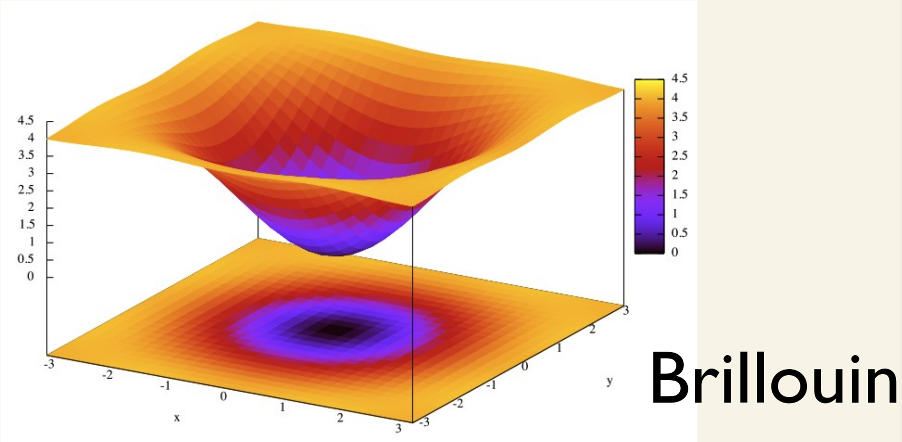
$$\left(\nabla_\mu = i\gamma_\mu S_\mu \Rightarrow \nabla_\mu^{iso} = i\gamma_\mu S_\mu \prod_{\nu\neq\mu} \frac{2+C_\nu}{3} \right)$$



Good isotropy

$|D|^2$





$$|D_W|^2 = (E^2 - \mathbf{p}^2) + \frac{1}{12}(E^4 + \mathbf{p}^4 + 6E^2\mathbf{p}^2)a^2 + O(a^4) = (E^2 - \mathbf{p}^2) + O(a^2)$$

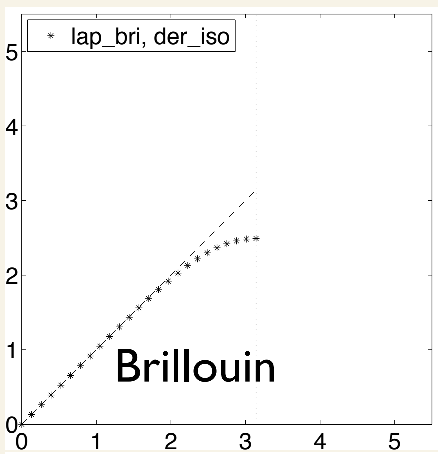
$$|D_B|^2 = (E^2 - \mathbf{p}^2) + \frac{1}{12}(E^2 - \mathbf{p}^2)^2 a^2 + \frac{1}{360}(E^2 - \mathbf{p}^2)(E^4 + E^2\mathbf{p}^2 + \mathbf{p}^4)a^4 + O(a^6)$$

Better dispersion relation !

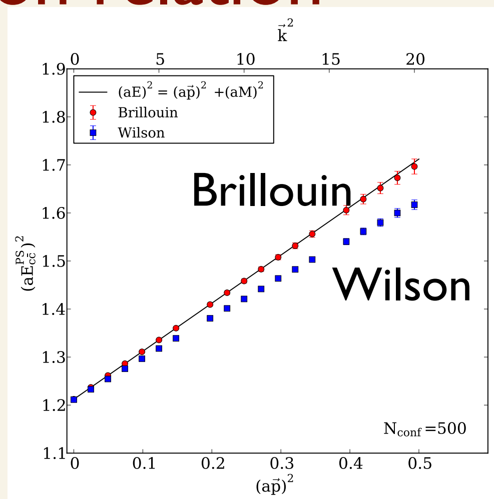
Advantages

Durr, Koutsou (11)(12)

- Better dispersion relation

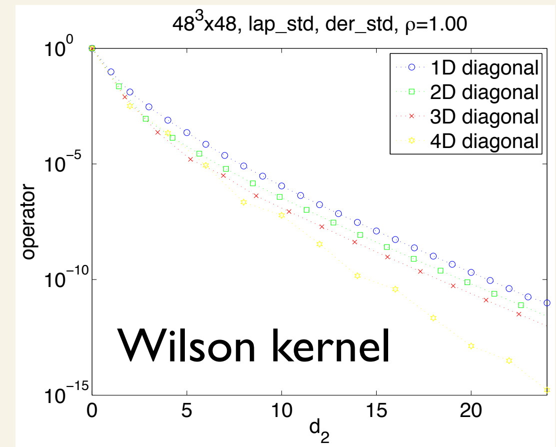


free, m=0

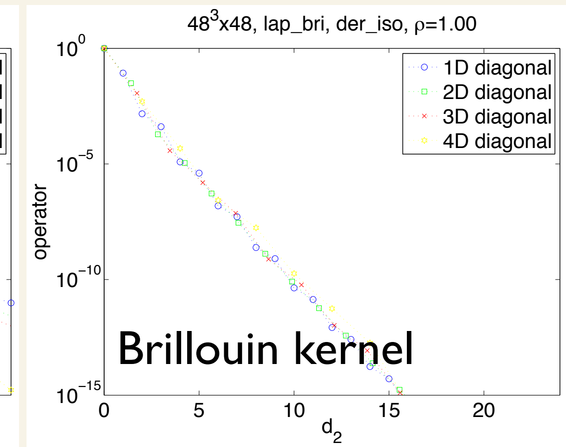


heavy-heavy
PS meson

- Locality of overlap operator

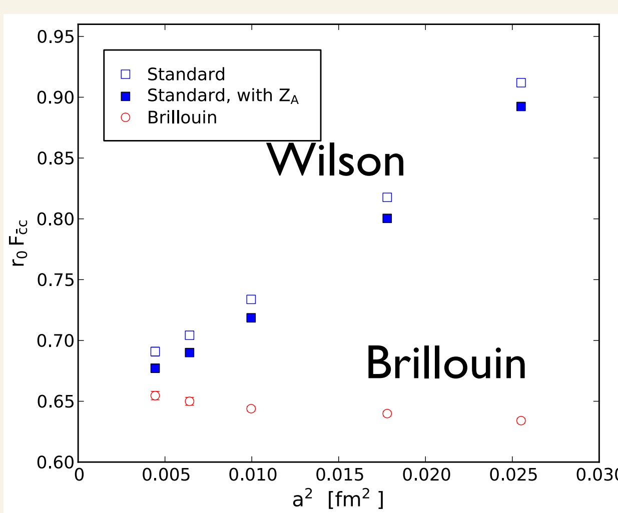


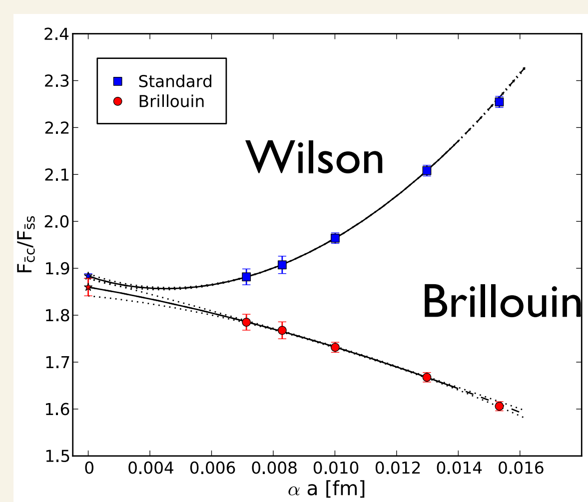
$\lambda_{max}/\lambda_{min} \sim 50$



$\lambda_{max}/\lambda_{min} \sim 1$

- Good scaling in charm mass region





Heavy PS meson decay constant
(Quench, Clover csw=1,
APE-smear α=0.72)

suits heavy-quark system

Relation to overlap fermions

◆Difference from GW fermion

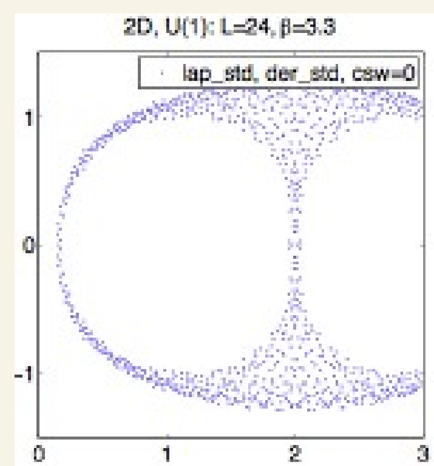
$$D\gamma_5 + \gamma_5 D = R D \gamma_5 D \quad \Rightarrow \quad R\Big(\sum_\mu \nabla_\mu^2 + \triangle^2\Big) = 2\triangle$$

$\left| \nabla^2 + \triangle^2 - \frac{2}{R}\triangle \right|$ indicates difference from GW relation.

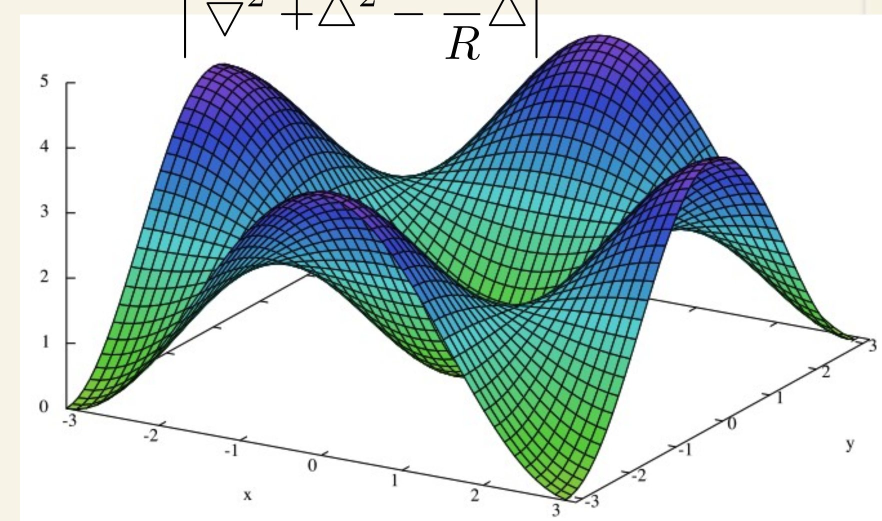
・Wilson (2D)

$$\nabla_\mu = i\gamma_\mu \sin p_\mu$$

$$\triangle = 2 - \cos p_1 - \cos p_2$$



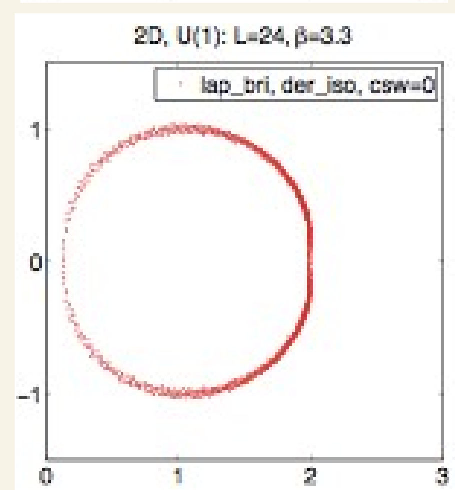
$\left| \nabla^2 + \triangle^2 - \frac{2}{R}\triangle \right|$

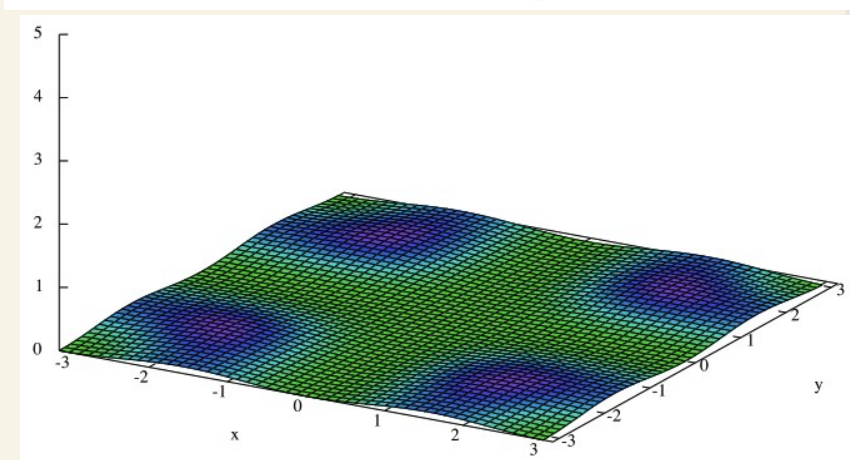


・Brillouin (2D)

$$\nabla_\mu^{iso} = i\gamma_\mu \sin p_\mu (2 + \cos p_\nu)/3$$

$$\triangle^{bri} = 2 - 2\cos^2\frac{p_1}{2}\cos^2\frac{p_2}{2}$$





Ultra-local fermion mimicking GW relation !

cf.)Hyepercubic fermion
Bietenholtz (00)(02)

Further improvement

(i) Better dispersion relation

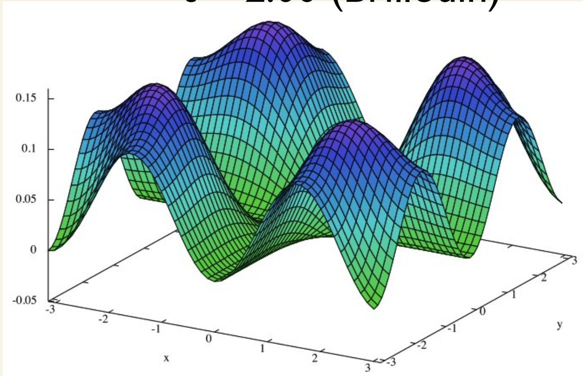
Cho, Hashimoto, Noaki, Juttner, Marinkovic (13)

(ii) More GW-like fermion

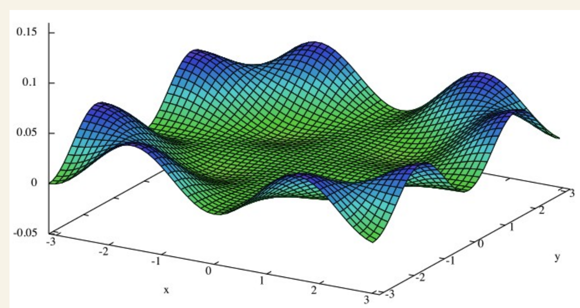
TM (14)

- 1-parameter deformation → $\nabla_\mu(t) = i\gamma_\mu S_\mu \prod_{\nu\neq\mu} \frac{t+C_\nu}{1+t}$

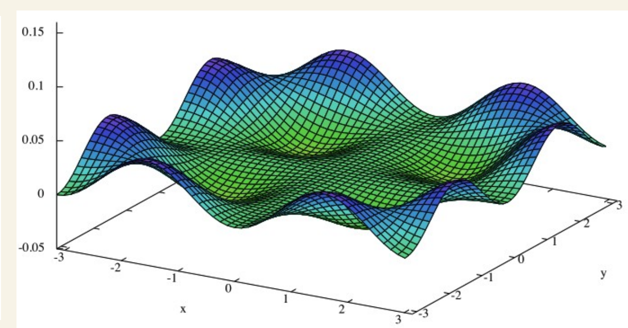
t = 2.00 (Brillouin)



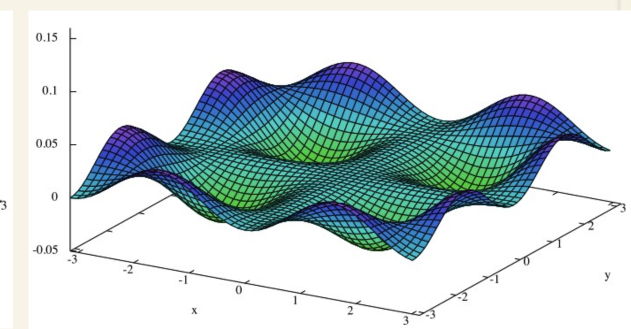
t = 1.60



t = 1.55



t = 1.50



$\left|\nabla^2 + \triangle^2 - \frac{2}{R}\triangle\right|$

More GW-like and suitable to overlap kernel

- Exponential locality bound for overlap

$$||(A^\dagger A)^{-1/2}(x,y)|| \leq K e^{-\theta|x-y|/a} \qquad \theta = \frac{1}{2l}\log\left(\frac{\sqrt{C}+1}{\sqrt{C}-1}\right) \qquad C = \frac{\lambda_{max}(A^\dagger A)}{\lambda_{min}(A^\dagger A)}$$

→ $$A^\dagger A = \nabla^2 + (\triangle - \frac{1}{R})^2$$

Θ=0.141 for Wilson
Θ=0.415 for Brillouin
Θ=0.555 for t=1.55

t=1.55 is best

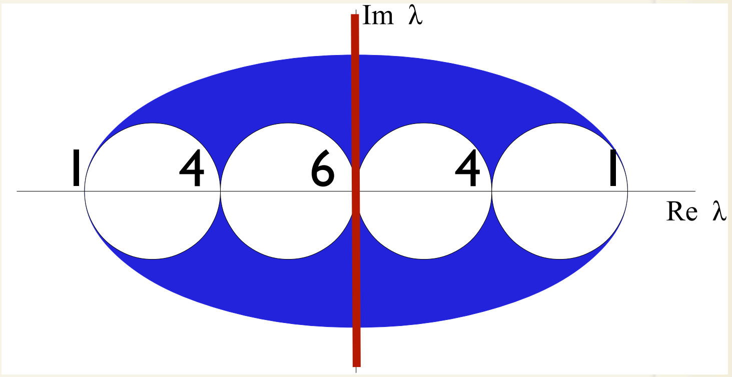
III. Central branch

III. Central-branch

Creutz, Kimura, TM (11)
Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)

◆Wilson fermion without onsite term $M_W \equiv m + 4r = 0$

$$S = \frac{1}{2}\sum_{x,\mu}\bar{\psi}_x[\gamma_\mu(U_{x,\mu}\psi_{x+\mu} - U_{x,-\mu}\psi_{x-\mu}) - (U_{x,\mu}\psi_{x+\mu} + U_{x,-\mu}\psi_{x-\mu})]$$



➡ Extra U(1) symmetry! →prohibits usual mass

$$\psi_x \to e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}\psi_x,\ \ \bar{\psi}_x \to \bar{\psi}_x e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}$$

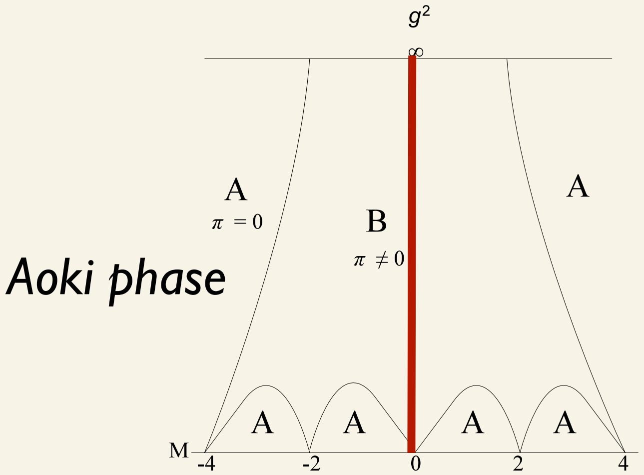
- Strong-coupling QCD Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)

η-condensate

$$\langle\bar{\psi}\gamma_5\psi\rangle \neq 0 \quad \langle\bar{\psi}\psi\rangle = 0$$

NG boson with U(1) breaking

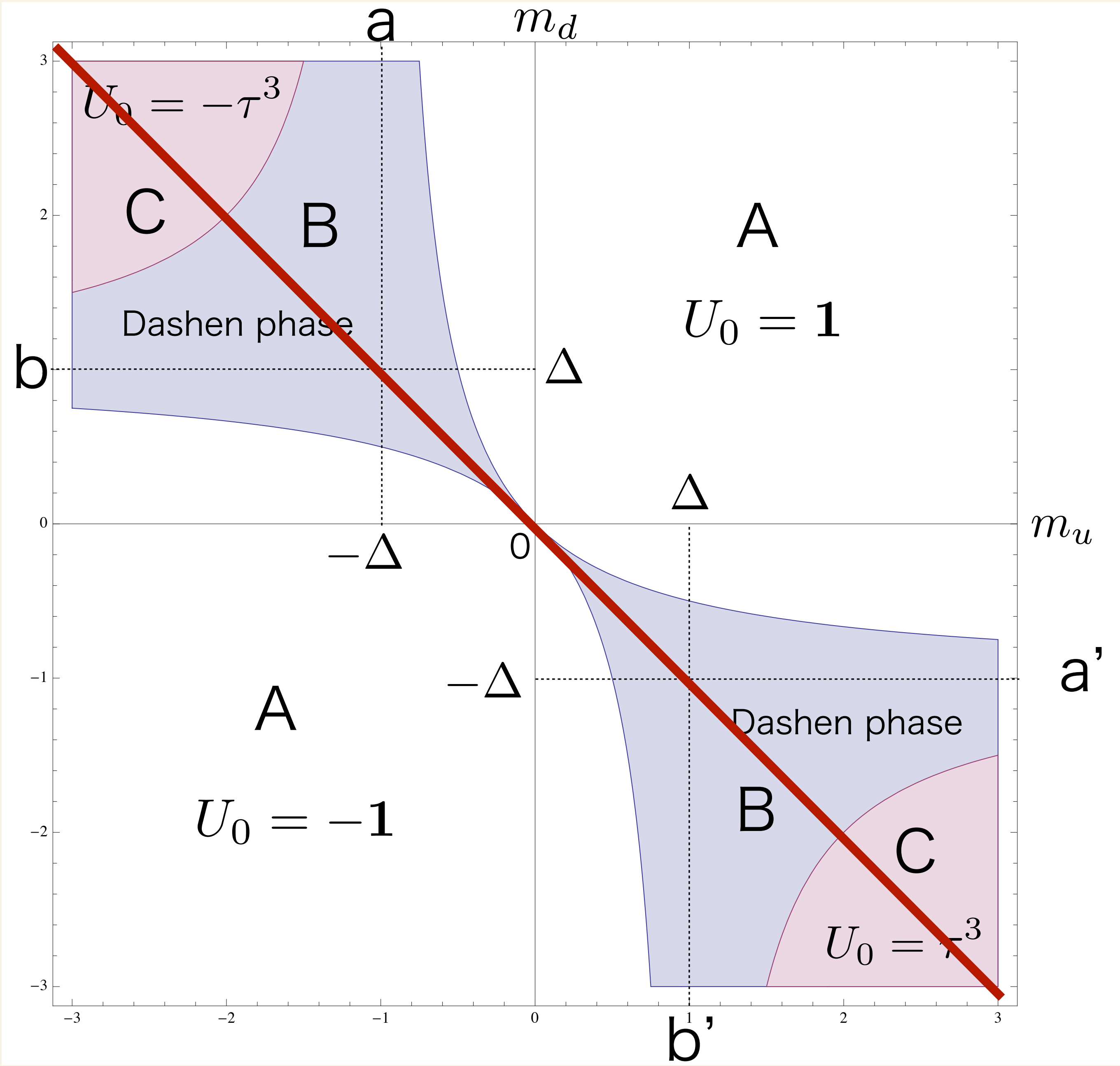
$$\cosh(m_\pi) = 1 + \frac{2M_W^2(16 + M_W^2)}{16 - 15M_W^2} \ \to\ m_\pi = 0$$



◆Relation to Dashen-Creutz phase

Creutz(03)(04)
Aoki, Creutz(14)

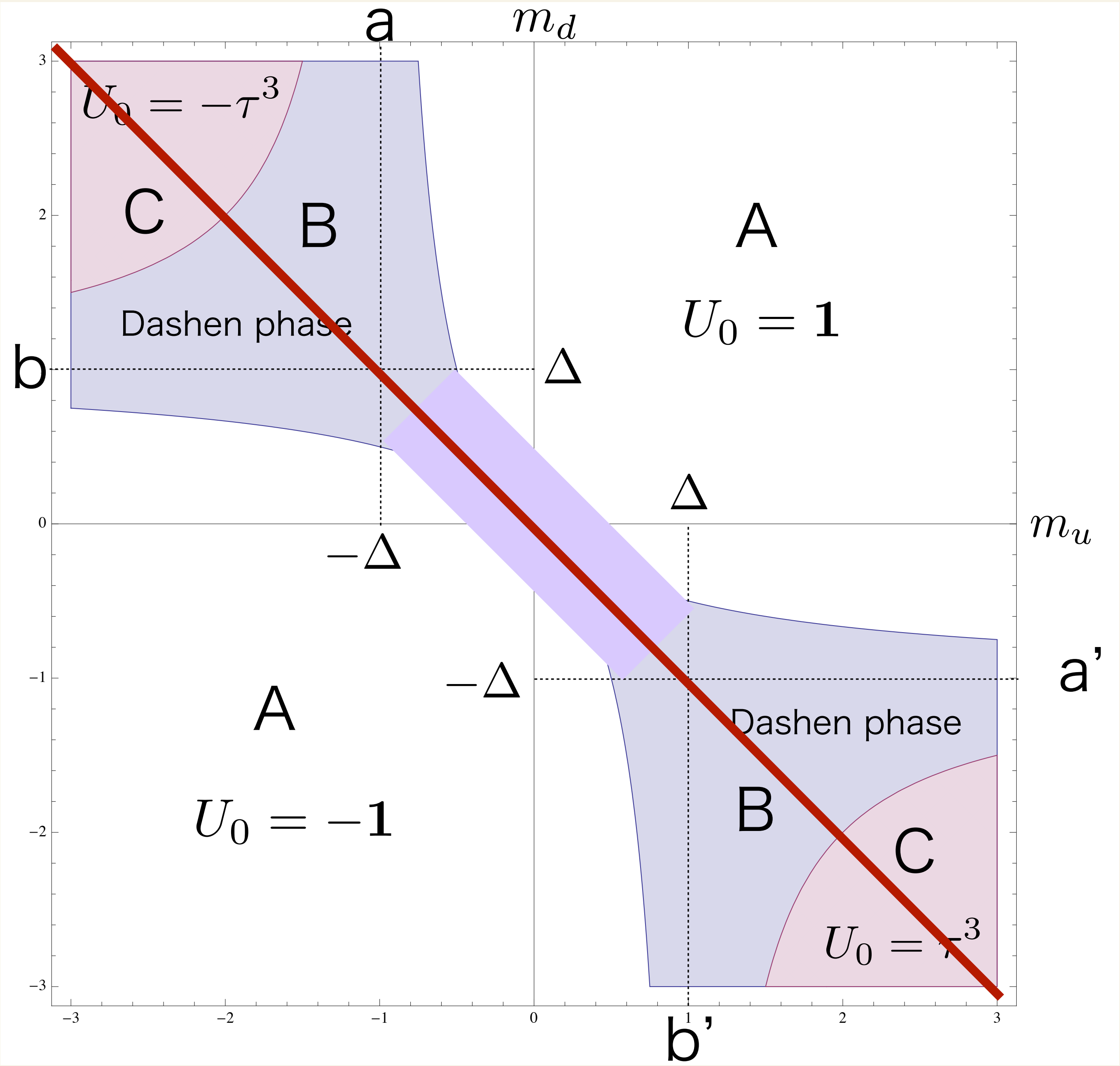
$$m_u = -m_d$$



◆Relation to Dashen-Creutz phase

Creutz(03)(04)
Aoki, Creutz(14)

$m_u = -m_d$

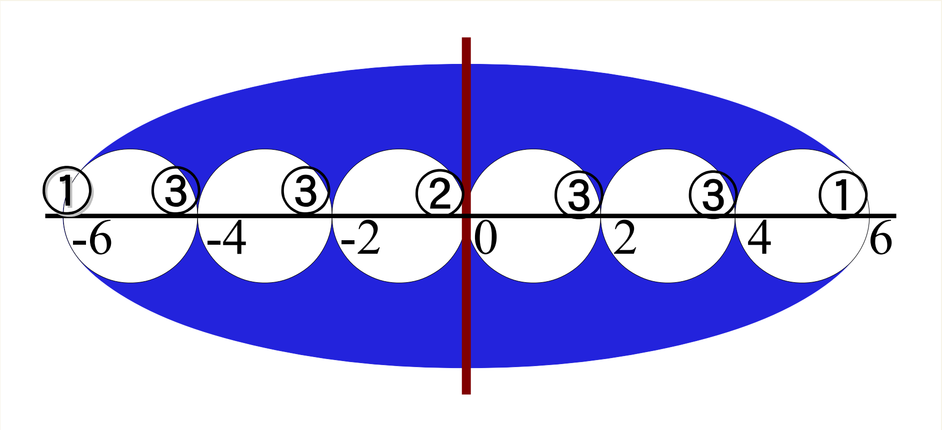


on the lattice ?

◆ 2-flavor central branch TM(14)

- 4D 2-flavor CB

$$\sum_{\mu=1}^{4} C_\mu \rightarrow \sum_{j=1}^{3} C_j + 3C_4.$$

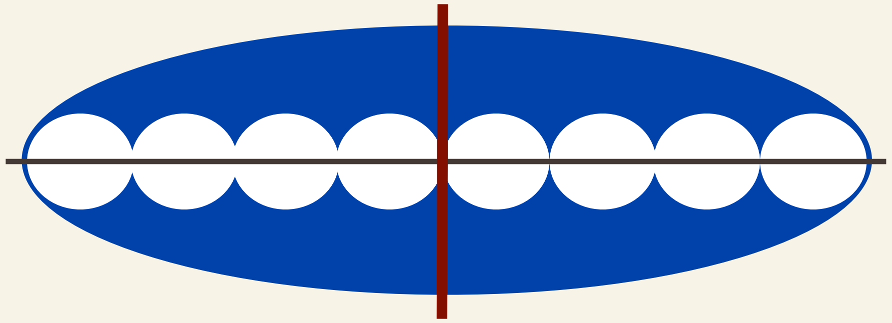


Hypercubic symmetry → Cubic symmetry

(1,3,3,2,3,3,1) splitting

- 5D 2-flavor CB

$$\sum_{\mu=1}^{5} C_\mu \rightarrow \sum_{j=1}^{4} C_j + 4C_5$$



5D hypercubic → 4D hypercubic

(1,4,6,4,2,4,6,4,1) splitting

Other types of Minimal-doubling

What I want to emphasize

1. The latest of Mike's contribution includes broad subjects of lattice fermions and QCD.

2. All of them are on-going hot topics.

3. Mike encourages lots of young researchers.

4. I hope Mike will continue contributing the community directly and indirectly !

SUSY-based chiral fermion

TM (13)

Extended SUSY lattice : doubling problem is more serious

1. #boson = #fermion
2. R symmetry ~ chiral symmetry

"Well-defined SUSY lattice" $\leftrightarrows$ "Successful doubling bypass"

◆2D N=(2,2) SUSY lattice Sugino (03)

- 4D N=1 SYM → 2D N=(2,2) SYM
- 4 SUSY $Q_\pm \quad \bar{Q}_\pm$ 4 real spinor $\lambda_\pm \quad \bar{\lambda}_\pm$ 2 U(1)R (flavor sym)
- Topological twist → scalar supercharge (BRST)

$$SO(2)_{E'} = \mathrm{diag}[SO(2)_E \otimes SO(2)_R]$$

Euclidian R-Flavor

$$Q_\pm,\ \bar{Q}_\pm \quad \rightarrow \quad Q,\ Q_\mu\ Q_{12} \quad \longrightarrow \quad Q^2 = 0$$

No relation to translation

→Scalar SUSY can survive on the lattice $S = QV(U, \phi, \psi)$

2D N=(2,2) fermion part

$$S_F^{\mathcal{N}=(2,2)} = -\frac{a^3}{4g_0^2}\sum_{n,\mu} \mathrm{tr}\, \Psi_n^T [\Gamma_\mu (U_{n,\mu}\Psi_{n+\mu} - U_{n,-\mu}\Psi_{n-\mu})$$ Kinetic term

$$+ P_\mu (2 - U_{n,\mu}\Psi_{n+\mu} - U_{n,-\mu}\Psi_{n-\mu})]$$ Wilson-like term

- 2-flavor × 2-spinor → 4D 4-spinor $\Psi^T = (\psi_1, \psi_2, \chi, \frac{1}{2}\eta)$

$$\Gamma_1 = -i\begin{pmatrix} & \sigma_1 \\ \sigma_1 & \end{pmatrix}, \quad \Gamma_2 = i\begin{pmatrix} & \sigma_3 \\ \sigma_3 & \end{pmatrix}, \quad P_1 = \begin{pmatrix} & \sigma_2 \\ \sigma_2 & \end{pmatrix}, \quad P_2 = i\begin{pmatrix} & -\mathbf{1} \\ \mathbf{1} & \end{pmatrix}$$

$$\{\Gamma_\mu, \Gamma_\nu\} = -2\delta_{\mu\nu}, \quad \{P_\mu, P_\nu\} = 2\delta_{\mu\nu}, \quad \{\Gamma_\mu, P_\nu\} = 0$$ 4D clifford algebra !

1. No more species doubling (never conflicts with no-go theorem)

$$aD(p) = \sum_{\mu=1}^{2}[-i\,\Gamma_\mu \sin ap_\mu + P_\mu (1 - \cos ap_\mu)] \Rightarrow |a^2 D^2(p)| = \sum_{\mu=1}^{2}[\sin^2 ap_\mu + (1-\cos ap_\mu)^2]$$

with only zero at $p = (0,0,0,0)$

2. U(1)R invariance (flavor-chiral invariance)

$$\Gamma_5 = \Gamma_1\Gamma_2 P_1 P_2 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}$$

$\{D(p), \Gamma_5\} = 0$ → prohibits additive mass

Main points

1. D-dim Two-flavor → (D+2)-dim fermion
   D-dim Four-flavor → (D+4)-dim fermion

2. D+2 (D+4)-dim clifford algebra
   (Two or four sets of D-dim gamma matrices)
   → No further doubling

3. D+2 (D+4)-dim chiral symmetry
   → R invariance (chiral invariance)

Let us construct chiral 2-flavor setup inspired by SUSY !

◆2D two-flavor chiral fermion $\Psi = (\psi_A,\ \psi_B)^T$

$$D(p) = i\Gamma_\mu \sin p_\mu + iP_\mu(1 - \cos p_\mu) \qquad \mu = 1, 2$$

- 4D clifford algebra

$$\Gamma_1 = \mathbf{1} \otimes \sigma_1 = \begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix}, \qquad \Gamma_2 = \mathbf{1} \otimes \sigma_2 = \begin{pmatrix} \sigma_2 & \\ & \sigma_2 \end{pmatrix}$$

$$P_1 = -\sigma_2 \otimes \sigma_3 = \begin{pmatrix} & i\sigma_3 \\ -i\sigma_3 & \end{pmatrix}, \quad P_2 = \sigma_1 \otimes \sigma_3 = \begin{pmatrix} & \sigma_3 \\ \sigma_3 & \end{pmatrix}$$

⇒

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$$
$$\{P_\mu, P_\nu\} = 2\delta_{\mu\nu}$$
$$\{\Gamma_\mu, P_\nu\} = 0$$

1. No further species doubling $D^2 = \sum_{\mu=1}^{2} \left[\sin^2 p_\mu + (1 - \cos p_\mu)^2\right] \qquad p = (0,0,0,0)$

2. Chiral invariance $\Gamma_5 = \Gamma_1\Gamma_2 P_1 P_2 = (\sigma_3 \otimes \sigma_3) = \begin{pmatrix} \gamma_5 & \\ & -\gamma_5 \end{pmatrix}, \qquad \{D(p), \Gamma_5\} = 0$

3. O(a) SU(2) flavor breaking → SW_2 of $SO(2)_E \times SU(2)_F$

$$D = i(\mathbf{1} \otimes \sigma_\mu) \sin p_\mu + (\sigma_\mu \sigma_3 \otimes \sigma_3)(1 - \cos p_\mu)$$

Wilson-like term
→ Flavor-Lorentz mixing

◆Extension to 4D fermions

6D clifford algebra : only 6 gamma matrices

Short of 8 matrices for kinetic & Wilson terms.

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$$
$$\{P_\mu, P_\nu\} = 2\delta_{\mu\nu} \qquad \mu = 1,2,3,4$$
$$\{\Gamma_\mu, P_\nu\} = 0$$

We need tricks !

(1) 8D clifford algebra

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$$
$$\{P_\mu, P_\nu\} = 2\delta_{\mu\nu} \qquad \mu = 1,2,3,4$$
$$\{\Gamma_\mu, P_\nu\} = 0$$

16×16 matrices → 16-spinor
→ 4-flavor

(2) Common P for 4 directions

$$D(p) = i\Gamma_\mu \sin p_\mu + iP\sum_{\mu=1}^{4}(1-\cos p_\mu)$$

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}\,, \qquad \{\Gamma_\mu, P\} = 0$$

8×8 matrices → 8-spinor
→ 2-flavor

(1) 4D four-flavor chiral fermion $\Psi = (\psi_A, \psi_B, \psi_C, \psi_D)^T$

$$D(p) = i\Gamma_\mu \sin p_\mu + iP_\mu(1 - \cos p_\mu) \qquad \mu = 1,2,3,4$$

with 8D gamma matrices

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$$
$$\{P_\mu, P_\nu\} = 2\delta_{\mu\nu}$$
$$\{\Gamma_\mu, P_\nu\} = 0$$

8 elements are sufficient !

1. No more species doubling $D^2 = \sum_{\mu=1}^{2} \left[\sin^2 p_\mu + (1 - \cos p_\mu)^2\right] \qquad p = (0,0,0,0)$

2. Chiral invariance $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4 P_1 P_2 P_3 P_4 = (\gamma_5 \otimes \gamma_5) = \begin{pmatrix} \gamma_5 & \\ & -\gamma_5 \end{pmatrix}$

3. O(a) SU(4) flavor breaking → SW_4 of $SO(4)_E \times SU(4)_F$

$$D(p) = i(\mathbf{1} \otimes \gamma_\mu) \sin p_\mu + (\gamma_\mu\gamma_5 \otimes \gamma_5)(1 - \cos p_\mu)$$

Wilson-like term
→ Flavor-Lorentz mixing

Note it is not equivalent to staggered fermion !

(O(a^2) flavor breaking)

(2) 4D two-flavor chiral fermion

$$\Psi = (\psi_A,\ \psi_B)^T$$

$$D(p) = i\Gamma_\mu \sin p_\mu + iP\sum_{\mu=1}^{4}(1-\cos p_\mu)$$

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}\,, \qquad \{\Gamma_\mu, P\} = 0$$

$$\Gamma_j = \mathbf{1}\otimes\sigma_1\otimes\sigma_j = \mathbf{1}\otimes\gamma_j = \begin{pmatrix}\gamma_j & \\ & \gamma_j\end{pmatrix},$$

$$\Gamma_4 = \mathbf{1}\otimes\sigma_2\otimes\mathbf{1} = \mathbf{1}\otimes\gamma_4 = \begin{pmatrix}\gamma_4 & \\ & \gamma_4\end{pmatrix},$$

$$P = \sigma_3\otimes\sigma_3\otimes\mathbf{1} = \sigma_3\otimes\gamma_5 = \begin{pmatrix}\gamma_5 & \\ & -\gamma_5\end{pmatrix}$$

Two more anti-commuting elements $\{\Gamma_\mu, \Gamma_5^{\mathrm{A,B}}\} = 0\,, \qquad \{P, \Gamma_5^{\mathrm{A,B}}\} = 0$

1. No more species doubling $$|D(p)|^2 = \sum_\mu^4 \sin^2 p_\mu + \left[\sum_\mu^4 (1-\cos p_\mu)\right]^2$$

2. Chiral invariance

$$\{D(p), \Gamma_5^{\mathrm{A,B}}\} = 0$$

$$\Gamma_5^{\mathrm{A}} = \sigma_1\otimes\sigma_3\otimes\mathbf{1} = \sigma_1\otimes\gamma_5 = \begin{pmatrix} & \gamma_5 \\ \gamma_5 & \end{pmatrix},$$

$$\Gamma_5^{\mathrm{B}} = \sigma_2\otimes\sigma_3\otimes\mathbf{1} = \sigma_2\otimes\gamma_5 = \begin{pmatrix} & -i\gamma_5 \\ i\gamma_5 & \end{pmatrix}.$$

3. Flavored-P,T → reduces to true P,T.

$$\Psi_{n_0,n_j} \ \to\ i(\sigma_{1,2}\otimes\gamma_4)\Psi_{n_0,-n_j}\,,$$
$$\bar\Psi_{n_0,n_j} \ \to\ -\bar\Psi_{n_0,-n_j}\, i(\sigma_{1,2}\otimes\gamma_4)$$

It is unitary equivalent to Twisted-mass Wilson.

Change of variables as

$$\Psi_n \;\to\; \Psi'_n \equiv \exp\left[i\frac{\pi}{4}(\sigma_3 \otimes \gamma_5)\right]\Psi_n \,,$$
$$\bar\Psi_n \;\to\; \bar\Psi'_n \equiv \bar\Psi_n \exp\left[i\frac{\pi}{4}(\sigma_3 \otimes \gamma_5)\right] .$$

$$\Rightarrow \quad D(p) = i(\mathbf{1} \otimes \gamma_\mu)\sin p_\mu - \sum_\mu (1 - \cos p_\mu) + m_0(\sigma_3 \otimes \gamma_5)$$

1. Flavor-non-singlet chiral symmetry with Γ_5^{A} and Γ_5^{B},
2. Flavor symmetry associated with $i\Gamma_5^{\mathrm{A}}\Gamma_5^{\mathrm{B}}$,
3. Hypercubic symmetry,
4. Flavored-parity and flavored-time-reversal associated with σ_2 and σ_3,