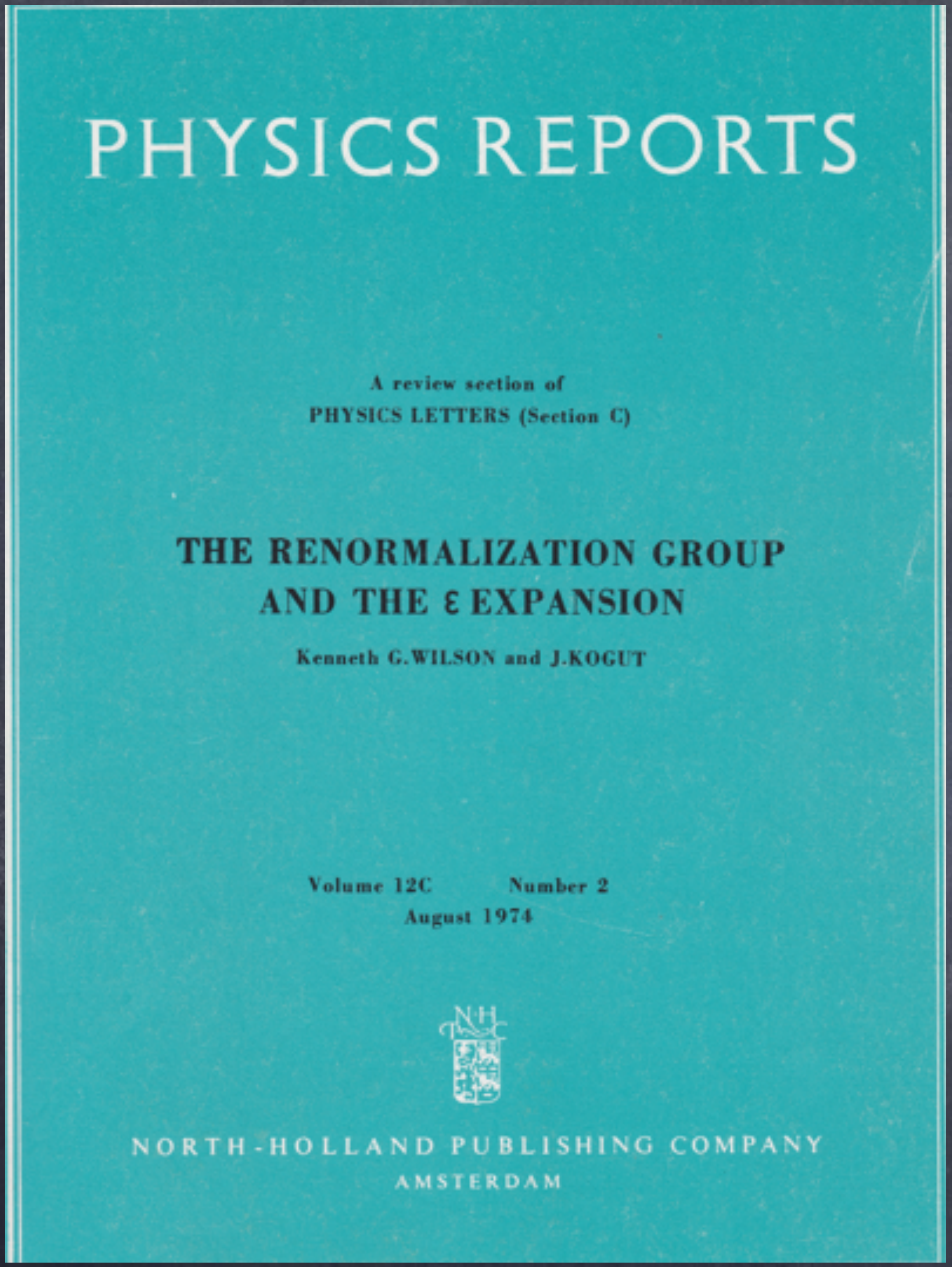
Happy Birthday
to
Mike Creutz

HET-RBRC Symposium
CreutzFest 2014
A Celebration of the Career and Accomplishments of Michael Creutz

September 4-5, 2014

Brookhaven National Laboratory
Upton, NY 11973 USA

Two books that influenced me as a graduate student



PHYSICS REPORTS

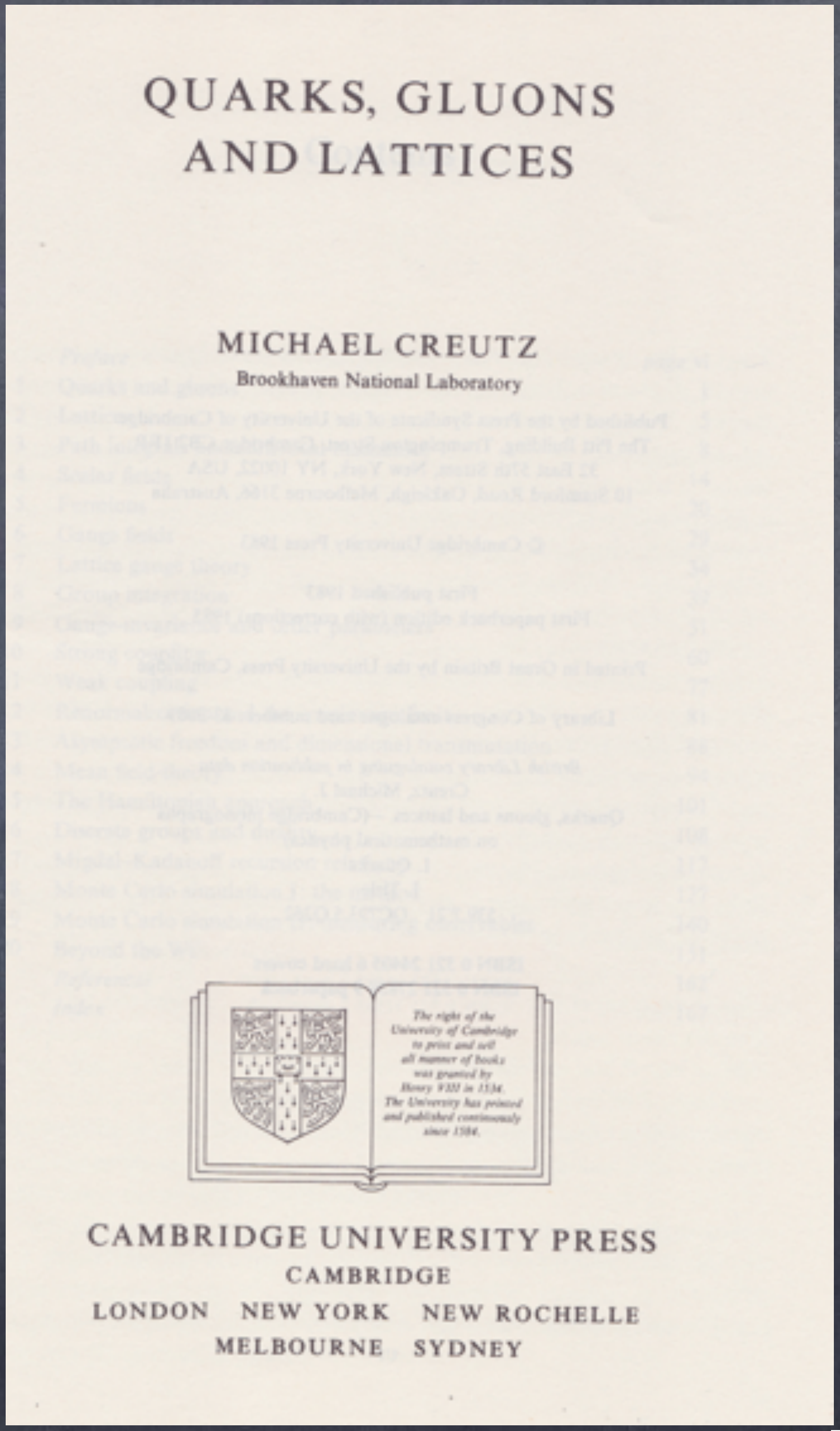
A review section of
PHYSICS LETTERS (Section C)

THE RENORMALIZATION GROUP
AND THE ε EXPANSION

Kenneth G. WILSON and J. KOGUT

Volume 12C Number 2
August 1974

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM



QUARKS, GLUONS
AND LATTICES

MICHAEL CREUTZ
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Mike's book has pages like this which help a student do a calculation

This is evaluated graphically in figure 8.8 Here we first use figure 8.5 to direct all lines upwards, then we use figure 8.6 to eliminate these lines, and finally we use the identity from figure 8.4 to obtain the result

$$I_{ijkl} = (1/n)\delta_{jk}\delta_{il}. \tag{8.58}$$

As a final example consider

$$I = \int dg\, g_{ij}(g^{-1})_{kl}\, g_{mn}(g^{-1})_{pq}. \tag{8.59}$$

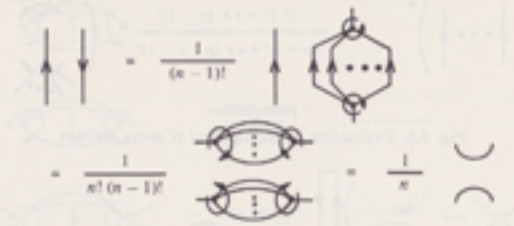

Fig. 8.8. Evaluation of the integral $\int dg\, g_{ij}\, g^{-1}_{kl}$ (Creutz, 1978*b*).

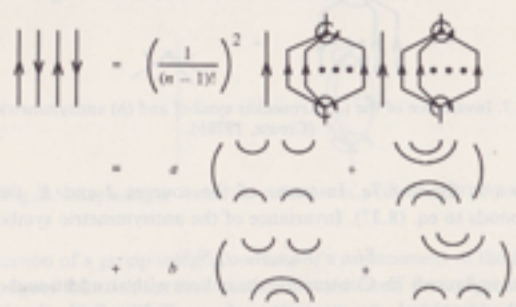

Fig. 8.9. The integral $\int dg\, g_{ij}\, g^{-1}_{kl}\, g_{mn}\, g^{-1}_{pq}$ (Creutz, 1978*b*).

In figure 8.9 we use figure 8.5 to express I in terms of $2n$ upward lines. Use of figure 8.6 at this point would give an expression with $(2n)!/(2!n!^2)$ terms. Some simple tricks allow us to simplify this expression for general n. All terms in this sum have four, an even number, of ϵ vertices both at the top and at the bottom of the diagram. These can all be eliminated using identities similar to those in figure 8.4. Thus the result must finally appear in terms of sets of Kronecker δ symbols connecting separately indices at the top and bottom of the diagram. Furthermore, note that a Kronecker

δ cannot connect the indices i and m or j and n because they can be initially coupled only through an odd number of ϵ vertices. Thus the final expression for the integral must take the form

$$I = a(\delta_{il}\delta_{mq}\delta_{jk}\delta_{np} + \delta_{iq}\delta_{ml}\delta_{jp}\delta_{nk}) + b(\delta_{il}\delta_{mq}\delta_{jp}\delta_{nk} + \delta_{iq}\delta_{ml}\delta_{jk}\delta_{np}), \tag{8.60}$$

where only two independent coefficients are needed because of the $kl \leftrightarrow pq$ symmetry of the integrand. The coefficients a and b can now be determined by multiplying by δ_{jk} and using figure 8.7*a* to reduce the integral to that

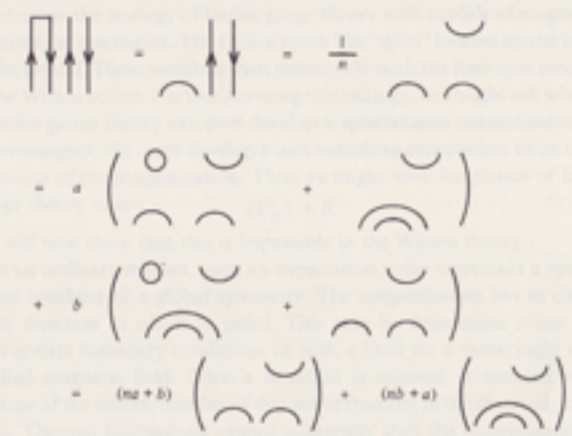

Fig. 8.10. Evaluation of the coefficients a and b. The closed circles represent $\Sigma_i \delta_{ii} = n$ (Creutz, 1978*b*).

already evaluated in figure 8.8. This sequence of steps appears in figure 8.10 and leads to the conclusion

$$a = 1/(n^2-1),$$
$$b = -1/(n(n^2-1)). \tag{8.61}$$

Inserted into eq. (8.60), this gives the desired integral.

Problems

1. Show that for 2-by-2 matrices $\det(A) = \frac{1}{2}((\mathrm{Tr}\,A)^2 - \mathrm{Tr}(A^2))$. What is the corresponding formula for 3-by-3 matrices?

2. For $SU(n)$ evaluate $\int dg\, \mathrm{Tr}(g^n)$.

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Mike's book has elegant pictures like this which help a student understand renormalization group

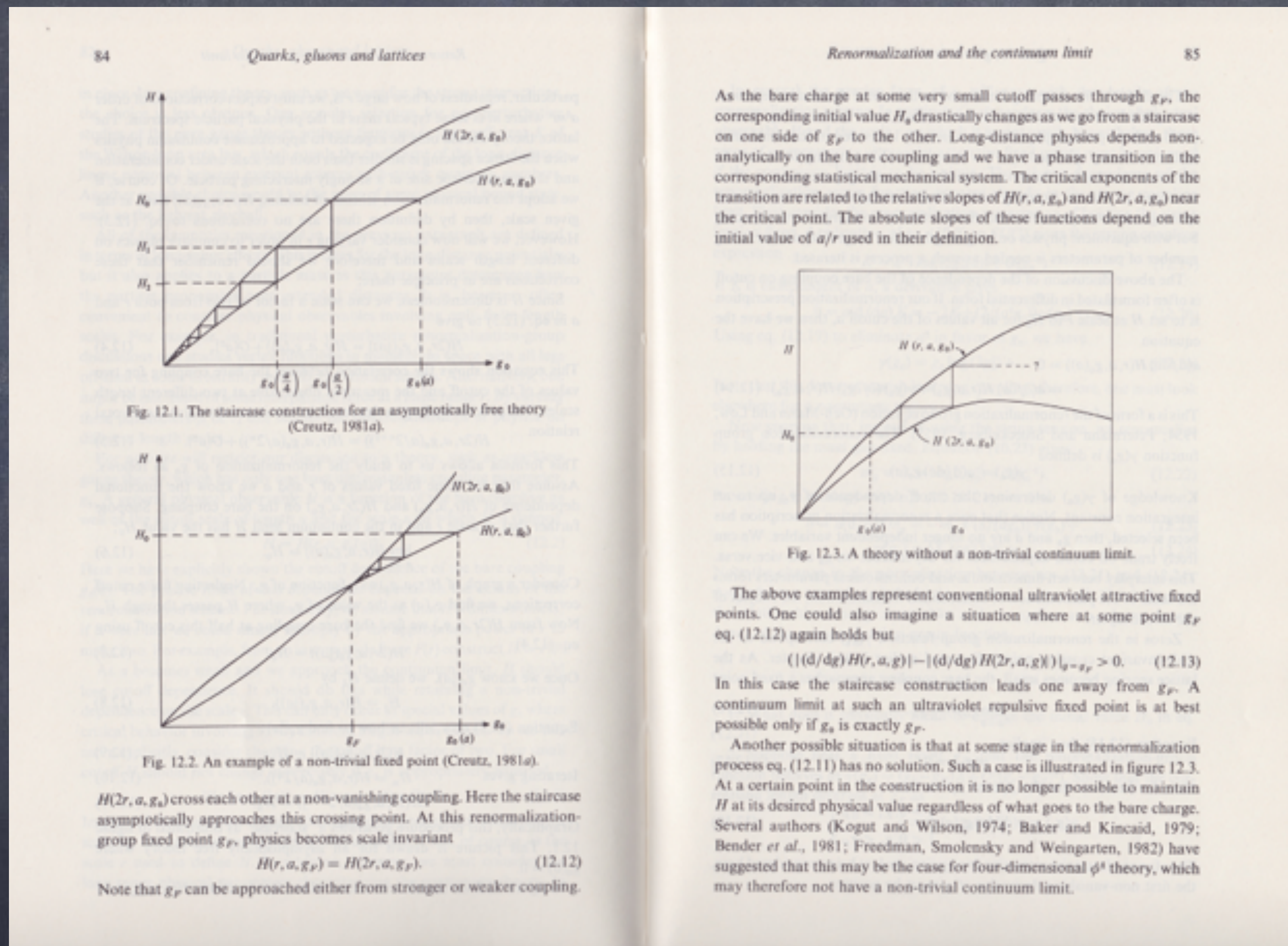


Fig. 12.1. The staircase construction for an asymptotically free theory (Creutz, 1981*a*).

Fig. 12.2. An example of a non-trivial fixed point (Creutz, 1981*a*).

$H(2r, a, g_0)$ cross each other at a non-vanishing coupling. Here the staircase asymptotically approaches this crossing point. At this renormalization-group fixed point g_F, physics becomes scale invariant

$$H(r, a, g_F) = H(2r, a, g_F). \qquad (12.12)$$

Note that g_F can be approached either from stronger or weaker coupling.

As the bare charge at some very small cutoff passes through g_F, the corresponding initial value H_0 drastically changes as we go from a staircase on one side of g_F to the other. Long-distance physics depends non-analytically on the bare coupling and we have a phase transition in the corresponding statistical mechanical system. The critical exponents of the transition are related to the relative slopes of $H(r, a, g_0)$ and $H(2r, a, g_0)$ near the critical point. The absolute slopes of these functions depend on the initial value of a/r used in their definition.

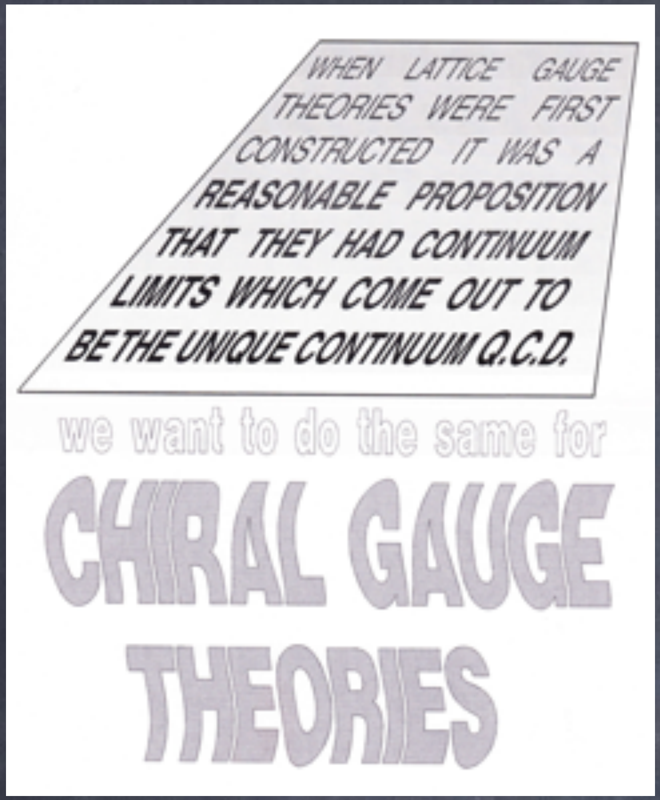
Fig. 12.3. A theory without a non-trivial continuum limit.

The above examples represent conventional ultraviolet attractive fixed points. One could also imagine a situation where at some point g_F eq. (12.12) again holds but

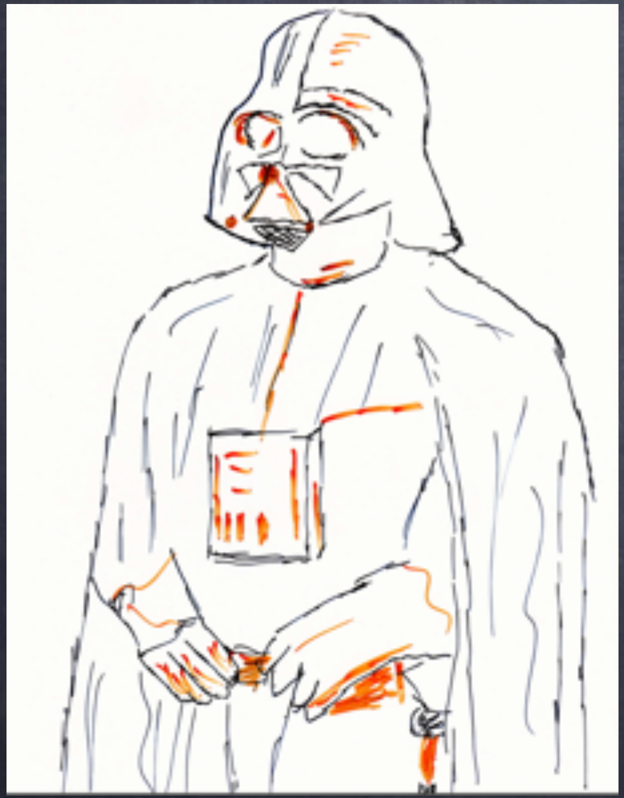
$$\left(|(\mathrm{d}/\mathrm{d}g)\, H(r, a, g)| - |(\mathrm{d}/\mathrm{d}g)\, H(2r, a, g)|\right)\big|_{g=g_F} > 0. \qquad (12.13)$$

In this case the staircase construction leads one away from g_F. A continuum limit at such an ultraviolet repulsive fixed point is at best possible only if g_0 is exactly g_F.

Another possible situation is that at some stage in the renormalization process eq. (12.11) has no solution. Such a case is illustrated in figure 12.3. At a certain point in the construction it is no longer possible to maintain H at its desired physical value regardless of what goes to the bare charge. Several authors (Kogut and Wilson, 1974; Baker and Kincaid, 1979; Bender *et al.*, 1981; Freedman, Smolensky and Weingarten, 1982) have suggested that this may be the case for four-dimensional ϕ^4 theory, which may therefore not have a non-trivial continuum limit.



Some slides from a talk by Herbert Neuberger at the January 1995 Santa Barabara chiral fermion workshop organized by Mike



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Mike among kangaroos at the Tidbinbilla Nature Preserve near Canberra in June 1995

Two Dimensional QCD at Finite Temperature and Chemical Potential

Rajamani Narayanan
Florida International University

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I will go back to a relatively old paper to study the theory at finite temperature and chemical potential

Canonical ensembles and nonzero density quantum chromodynamics
A. Hasenfratz, D. Toussaint (Arizona U.). Jun 10, 1991. 15 pp.
Published in **Nucl.Phys. B371 (1992) 539-549**
AZPH-TH-91-21
DOI: 10.1016/0550-3213(92)90247-9
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
Science Direct
Detailed record - Cited by 76 records 50+

- To the best of my knowledge, much of the work following this paper has been in the numerical evaluation of the fourier components of the partition function with imaginary chemical potential.
- I will follow a slightly different route. A recent reference close to what I will talk about here is

A property of fermions at finite density by a reduction formula of fermion determinant
K. Nagata. 2013. 7 pp.
Published in **PoS LATTICE2013 (2013) 207**
Conference: C13-07-29.1 Proceedings
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
Link to Fulltext

Close to a Hamiltonian formalism in a fixed gauge field background

Partially gauge fix so that all temporal links are set to unity except the one set of links that closes the periodic lattice

$$Z(\mu) = e^{-\mu\beta n_s N V N_f} \int dU_i(\mathbf{x},k) dU_d(\mathbf{x}) e^{S_g} \left(\prod_{j=1}^{n} \det(-B_j)\right)^{N_f} \det\left[\mathbf{1} - \mathcal{T} e^{\mu\beta}\right]^{N_f}$$

Spatial links

Labels the spatial point

Labels the time slice

Temporal links that close the periodic lattice

$$S_g = bN \sum_{\mathbf{x}} \sum_{k=1}^{n} (S_g^m(\mathbf{x},k) + S_g^e(\mathbf{x},k))$$

Magnetic

$$S_g^m(\mathbf{x},k) = \sum_{i,j=1}^{d-1} \mathrm{Tr}\left[U_i(\mathbf{x},k)U_j(\mathbf{x}+\hat{i},k)\left\{U_j(\mathbf{x},k)U_i(\mathbf{x}+\hat{j},k)\right\}^\dagger\right]$$

Electric

$$S_g^e(\mathbf{x},k) = \sum_{i=1}^{d-1} \mathrm{Tr}\left[U_i(\mathbf{x},k)U_i^\dagger(\mathbf{x},k+1) + U_i(\mathbf{x},k+1)U_i^\dagger(\mathbf{x},k)\right]; \quad k \in [1, n-1]$$

$$S_g^e(\mathbf{x},n) = \sum_{i=1}^{d-1} \mathrm{Tr}\left[U_i(\mathbf{x},n)U_d(\mathbf{x}+\hat{i})\left\{U_d(\mathbf{x})U_i(\mathbf{x},1)\right\}^\dagger + U_d(\mathbf{x})U_i(\mathbf{x},1)\left\{U_i(\mathbf{x},n)U_d(\mathbf{x}+\hat{i})\right\}^\dagger\right]$$

Hamiltonian integrated over Euclidean time in a fixed gauge field background

$$\mathcal{T} = \left(\prod_{k=1}^{n} \mathcal{T}_k\right) T_d$$

This formula can be found in the original paper by Hasenfratz and Toussaint

$$(T_d\psi)(\mathbf{x},k) = U_d(\mathbf{x},k)\psi(\mathbf{x},k+1)$$

This factor arises due to the presence of the temporal links that close the periodic lattice

$$\begin{aligned}
B_k &= d + M - \frac{1}{2}\sum_{j=1}^{d-1}\left(T_j(k) + T_j^\dagger(k)\right)\\
C_k &= \frac{1}{2}\sum_{j=1}^{d-1}\sigma_j\left(T_j(k) - T_j^\dagger(k)\right); \quad C_k^\dagger = -C_k.\\
(T_j(k)\psi)(\mathbf{x}) &= U_j(\mathbf{x},k)\psi(\mathbf{x}+\hat{j})\\
(T_j^\dagger(k)\psi)(\mathbf{x}) &= U_j^\dagger(\mathbf{x}-\hat{j},k)\psi(\mathbf{x}-\hat{j}); \quad T_j^\dagger T_j = \mathbf{1};
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_k &= \begin{pmatrix} B_k^{-1} & -B_k^{-1}C_k \\ C_kB_k^{-1} & B_k - C_kB_k^{-1}C_k \end{pmatrix};\\
\mathcal{T}_k^{-1} &= \begin{pmatrix} B_k - C_kB_k^{-1}C_k & C_kB_k^{-1} \\ -B_k^{-1}C_k & B_k^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{T}_k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Lambda\mathcal{T}_k\Lambda
\end{aligned}$$

$$\Lambda = \Lambda^\dagger; \quad \Lambda^2 = 1; \quad \det\mathcal{T}_k = 1; \quad \mathcal{T}_k^\dagger = \mathcal{T}_k.$$

Properties of $\mathcal{T}$

$$\mathcal{T} = \left(\prod_{k=1}^{n} \mathcal{T}_k\right) \mathcal{T}_d$$

$\det \mathcal{T}_d = 1$ and $\det \mathcal{T}_k = 1$ for all k $\longrightarrow$ $\det \mathcal{T} = 1$

right eigenvectors $\mathcal{T}\mathcal{R} = \mathcal{R}\mathcal{D}$

we can show that

$$\mathcal{D}^{-1\dagger}\mathcal{R}^{\dagger}\Lambda = \mathcal{R}^{\dagger}\Lambda\mathcal{T}$$

for every eigenvalue $\mathcal{D}_i$, we have another eigenvalue $\frac{1}{\mathcal{D}_i^*}$

$$\mathcal{D} = \begin{pmatrix} \mathcal{C} & 0 \\ 0 & \frac{1}{\mathcal{C}^\dagger} \end{pmatrix}$$

Using $\det \mathcal{T} = 1$, it follows that

$$1 = \det \mathcal{D} = \frac{\det \mathcal{C}}{\det \mathcal{C}^\dagger} \Rightarrow \det \mathcal{C} = \det \mathcal{C}^\dagger$$

Euclidean time reversal

- Gauge action remains the same
- Wilson term remains the same

$$\mathcal{T} \Rightarrow T_d \mathcal{T}^\dagger T_d^\dagger$$

$$\det\left[\mathbf{1} - \mathcal{T} e^{\mu\beta}\right] \Rightarrow \det\left[\mathbf{1} - \mathcal{T}^\dagger e^{\mu\beta}\right]$$

the partition function is real and an even function of $\mu\beta$.

Global Z_N transformations

Partition function can be rewritten as

$$Z(\mu) = \left(\prod_{i=1}^{K} \sum_{s_i, r_i = 0}^{N_f}\right) e^{\mu\beta\left[\sum_{i=1}^{K}(r_i - s_i)\right]} Z_{\{s_i, r_i\}}$$

$K = n_s N V$

where

$$Z_{\{s_i,r_i\}} = \int dU_i(\mathbf{x},k)\, dU_d(\mathbf{x})\, e^{S_g} \left(\prod_{j=1}^{n} \det(-B_j)\right)^{N_f} (\det(-\mathcal{C}))^{N_f} \prod_{i=1}^{K} \frac{1}{(-\mathcal{D}_i)^{s_i}\,(-\mathcal{D}_i^*)^{r_i}}$$

Under a global Z_N transformation, $U_d(x) \to e^{i\frac{2\pi k}{N}} U_d(x)$; $k = 0, \cdots N-1$

- Gauge action remains the same
- Wilson term remains the same

$T_d \to e^{i\frac{2\pi k}{N}} T_d$ $\qquad$ $\mathcal{D}_i \to e^{i\frac{2\pi k}{N}} \mathcal{D}_i$ $\qquad$ $\det(-\mathcal{C}) \to \det(-\mathcal{C})$

$$Z_{\{s_i,r_i\}} \to e^{i\frac{2\pi k}{N}\left[\sum_{i=1}^{K}(r_i - s_i)\right]} Z_{\{s_i,r_i\}}$$

$\sum_{i=1}^{K}(r_i - s_i)$ has to be a multiple of N

A double expansion

$$Z(\mu) = \sum_{Q=-n_s V N_f}^{n_s V N_f} e^{\mu\beta N Q} \sum_{P=0}^{2KN_f} Z_{PQ}; \qquad K = n_s N V$$

$$Z_{PQ} = \left(\prod_{i=1}^{K} \sum_{s_i, r_i = 0}^{N_f}\right) \delta\left(\sum_{i=1}^{K} (r_i - s_i) - NQ\right) \delta\left(\sum_{i=1}^{K} (r_i + s_i) - P\right) Z_{\{s_i, r_i\}}$$

$$Z_{\{s_i, r_i\}} = \int dU_i(\mathbf{x}, k) dU_d(\mathbf{x}) e^{S_g} \left(\prod_{j=1}^{n} \det(-B_j)\right)^{N_f} (\det(-\mathcal{C}))^{N_f} \prod_{i=1}^{K} \frac{1}{(-\mathcal{D}_i)^{s_i} (-\mathcal{D}_i^*)^{r_i}}$$

Z_{00} is positive definite for even number of flavors and is the partition function at zero temperature and chemical potential.

We expect $\frac{Z_{PQ}}{Z_{00}}$ to lead off as $e^{-m_{PQ}\beta}$ with $m_{PQ} < m_{(P+1)Q}$.

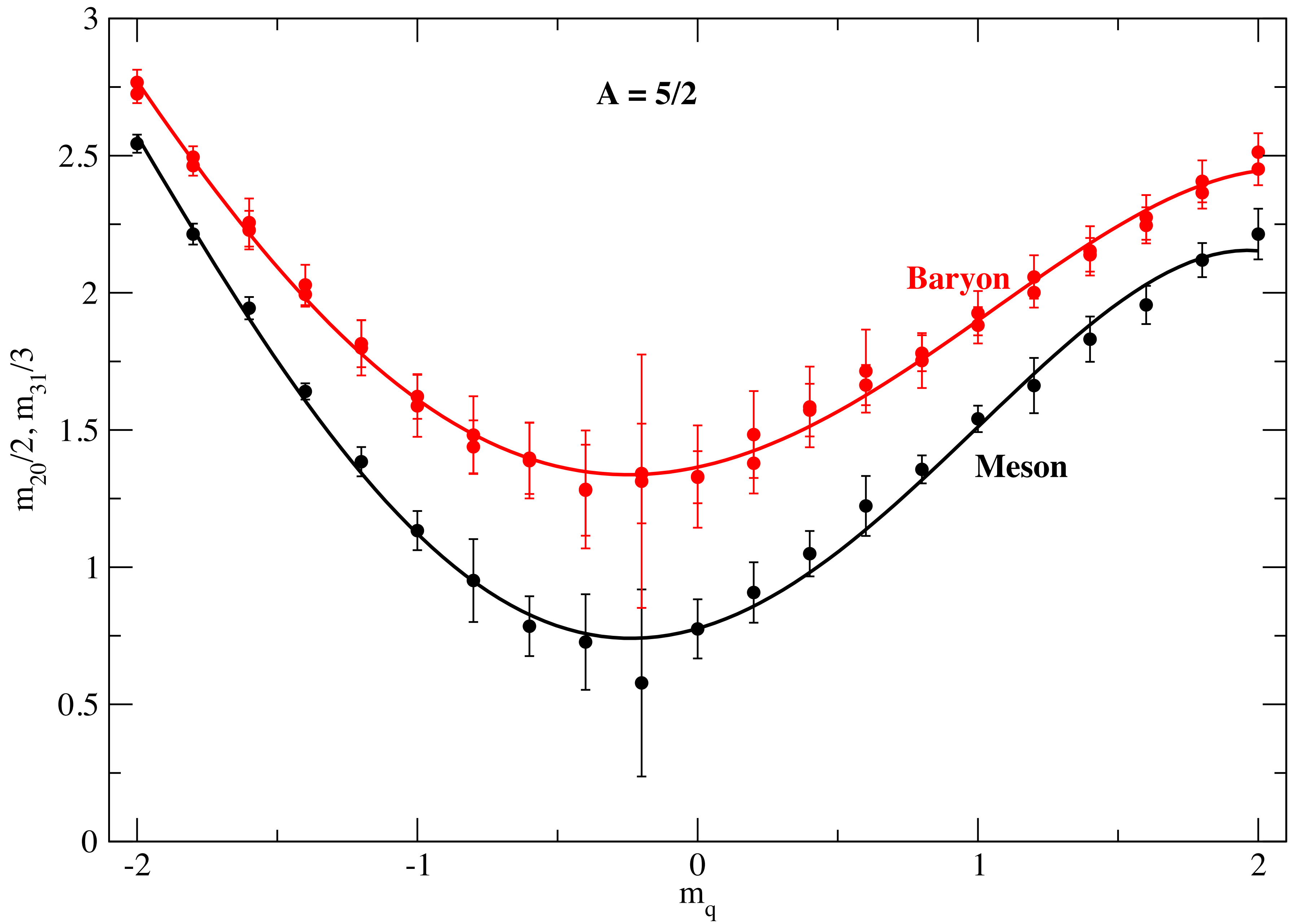
- m_{20} : Lowest mesonic excitation
- m_{N1} : Lowest baryonic excitation

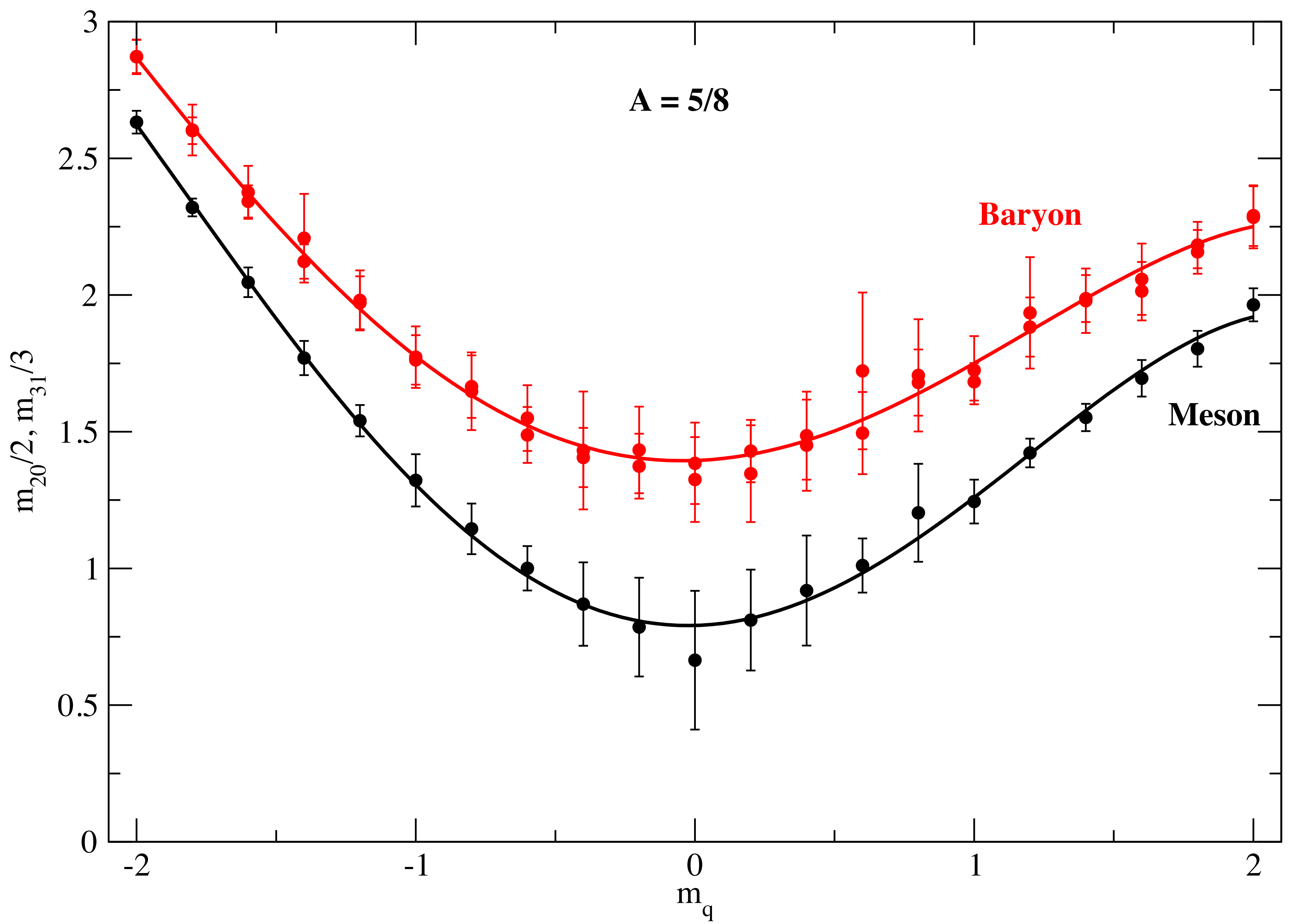
Numerical details in two dimensions

- We set N=3 and N_f=2.
- We use the HMC algorithm.
- We work on a L x L_t lattice.
- The lattice gauge coupling is set to b= $L^2/(4A)$ where A is the physical area of the continuum torus.
- The Wilson mass is set to m_q/L where m_q is the quark mass.
- Keeping, A, m_q, L fixed, we measure m_{20} and m_{31} by taking L_t to infinity.
- The continuum limit is taken by sending L to infinity at fixed A and m_q.

We fit the mass as a function of the quark mass to

$$\frac{m_{PQ}}{P} = \sum_{k=0}^{\infty} \left(a_{2k} m_q^{2k} + b_{2k+1} m_q^{2k+1} \right)$$


A = 5/2
Baryon
Meson
$m_{20}/2,\ m_{31}/3$
m_q
0
0.5
1
1.5
2
2.5
3
-2
-1
0
1
2


A = 5/8
Baryon
Meson
m20/2, m31/3
mq
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2

Is negative quark mass different from positive quark mass?

A	a_0	b_1	a_2	b_3	a_4
5/2	0.78	0.29	0.59	-0.10	-0.05
5/4	0.80	0.11	0.52	-0.07	-0.04
1	0.80	0.08	0.53	-0.06	-0.04
5/6	0.82	Meson	0.50	-0.05	-0.03
5/8	0.79	0.03	0.53	-0.05	-0.04
5/2	1.36	0.22	0.42	-0.08	-0.03
5/4	1.37	0.08	0.40	-0.05	-0.02
1	1.38	Baryon [illegible]	0.38	-0.05	-0.02
5/6	1.38	0.04	0.40	-0.05	-0.03
5/8	1.39	0.03	0.39	-0.05	-0.03

$$\frac{m_{PQ}}{P} = \sum_{k=0}^{\infty} \left(a_{2k} m_q^{2k} + b_{2k+1} m_q^{2k+1} \right)$$

- Since the fit seems to need b_1 and b_3, the meson and baryon masses are not even functions of the quark mass.
- Since b_1 seems to go down with decreasing A, the minimum mass occurs closer to zero quark mass.
- The asymmetry tends to increase with decreasing A, since there is only one term (b_3) that matters.
- $m_{31}/3 > m_{20}/2$.
- The value at zero quark mass does not seem to depend on the size of the torus.
- The quadratic dependence on the quark mass also does not seem to depend on the size of the torus.

Why is negative quark mass different from positive quark mass in two dimensional QCD?

We do not have global topology in two dimensions.

Wilson term:

$$B_k = d + M - \frac{1}{2}\sum_{j=1}^{d-1}\left(T_j(k) + T_j^\dagger(k)\right)$$

M -> -M is not the same as Wilson term going to its own negative

As we take the continuum limit, the laplacian in the Wilson term needs to become irrelevant compared to M.

Our results for the meson and baryon excitation masses suggest that the laplacian has an effect in the continuum limit. It is possible this is special to two dimensions.

What about overlap fermions?

- By construction, the fermion operator is an even function of the quark mass at zero topology.
- We expect the meson and baryon excitation masses to be an even function of the quark masses.
- But, as Mike has suggested in his papers, there is still a Wilson mass parameter in the overlap formalism. The onus is on the overlap formalism to show that physics does not depend on the Wilson mass parameter.

Some work is needed to work out the details for the overlap formalism to obtain the results obtained here with Wilson fermions.

Prior to working with overlap fermions, it will be interesting to study Wilson fermions for $N > 3$