Lattice scales then and now

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Back in 1979....

PHYSICAL REVIEW D VOLUME 21, NUMBER 8 15 APRIL 1980

Monte Carlo study of quantized SU(2) gauge theory

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(Received 24 October 1979)

Using Monte Carlo techniques, we evaluate path integrals for pure SU(2) gauge fields. Wilson's regularization procedure on a lattice of up to 10^4 sites controls ultraviolet divergences. Our renormalization prescription, based on confinement, is to hold fixed the string tension, the coefficient of the asymptotic linear potential between sources in the fundamental representation of the gauge group. Upon reducing the cutoff, we observe a logarithmic decrease of the bare coupling constant in a manner consistent with the perturbative renormalization-group prediction. This supports the coexistence of confinement and asymptotic freedom for quantized non-Abelian gauge fields.

This paper is a real treasure trove

- Monte Carlo algorithm for SU(2)
- Operators - Creutz ratio
- Continuum limit and renormalization group
- Results: string tension!

THE STRING TENSION (ON THE LAST PAGE)

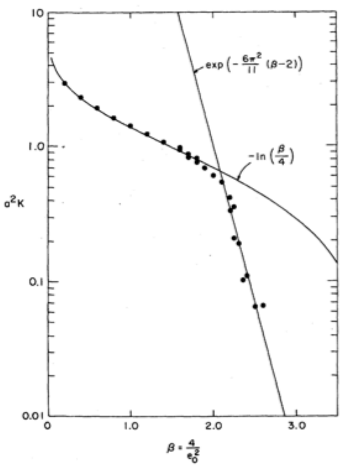

FIG. 6. The cutoff squared times the string tension as a function of β. The solid lines are the strong- and weak-coupling limits.

"We have shown the onset of asymptotic freedom for the bare coupling constant in a renormalization scheme based on confinement.
This is strongly suggestive that SU(2) non-Abelian gauge theory simultaneously exhibits confinement and asymptotic freedom."

On 10^4 volumes - what luck!

THE STRING TENSION (ON THE LAST PAGE)

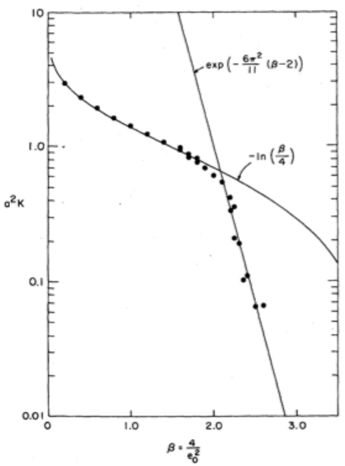

FIG. 6. The cutoff squared times the string tension as a function of β. The solid lines are the strong- and weak-coupling limits.

... where the renormalization scale is

$$\Lambda \approx \sqrt{K}\exp\left(-\frac{6\pi^2}{11}\right) = \frac{1}{200}\sqrt{K},$$

thus we see ... a rather large dimensionless number.

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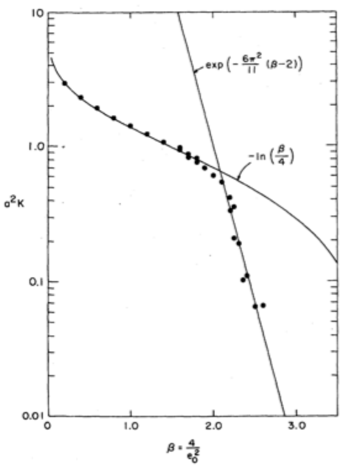

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Is that a problem?

THE Λ PARAMETER

Already in 1979 it was known that

- The continuum Λ_{QCD} depends weakly on the RG scheme (MOM, MS, $\overline{MS}$ are within 10%)
- $\Lambda_{\text{QCD}} \sim \sqrt{K}$ (up to a factor of two or so
- $\sqrt{K}/\Lambda_{\text{latt}} \approx 200$ means
  - either that QCD / SU(N) gauge group does not describe strong interactions
  - or that Λ_{latt} is very different from Λ_{cont} Lattice perturbation theory is strange!

THE Λ PARAMETER

And so my Master's thesis was born: calculate

$$\Lambda_{\text{latt}}/\Lambda_{\text{MOM}}$$

- Sometimes I even knew what I was doing
- I still have a deep appreciation for lattice PT
- It was all those tadpoles.....

$$\Lambda^{\text{MOM}}_{\text{Feynman gauge}} = 57.5\Lambda_{\text{lattice}} \quad \text{for} \quad SU(2)$$

- an other factor of 3 is due to 2-loop corrections, and everything works out

LATTICE SCALES

- the string tension served for a while as the favored lattice scale
- the Sommer parameter r_0 and its variants took over and are still in play
- fermionic scales (f_π, m_Ω) became popular with reliable dynamical simulations
- there is a new kid on the block : the gradient flow scale

The gradient flow

The gradient flow is a controlled and reversible smoothing transformation [1]

$$\frac{A_\mu(t)}{dt} = -\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu}$$

with smoothing range $\approx \sqrt{8t}$.
If $\sqrt{8t} \gg a$ the UV fluctuations are removed
$\longrightarrow$ renormalized quantities can be defined

[1] Luscher 2009

THE GRADIENT FLOW COUPLING

Define a running coupling

$$g^2_{GF}\left(\mu = \frac{1}{\sqrt{8t}}\right) = \frac{1}{\mathcal{N}} t^2 \langle E(t) \rangle, \qquad E(t) = -\frac{1}{2} G^2_{\mu\nu}$$

- easy to measure with small systematical errors
- can be combined with any boundary conditions
- appropriate both for scale setting and step scaling function

Scale setting with gradient flow

Define a running coupling

$$g^2_{GF}\left(\mu = \frac{1}{\sqrt{8t}}\right) = \frac{1}{\mathcal{N}} t^2 \langle E(t) \rangle, \qquad E(t) = -\frac{1}{2} G^2_{\mu\nu}$$

Fix $g^2_{GF}\left(\mu = \frac{1}{\sqrt{8t_q}}\right) = g^2_q$: defines $\mu \to t_q^{\text{latt}} = t_q/a^2$

Determine t_q in physical units $\to$ a can be calculated

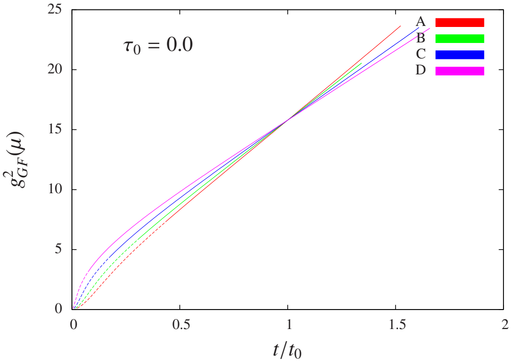
t_0 scale: : $g^2(t_0) = 0.3/\mathcal{N}$

Different choices of g^2_q should predict the same scale :

t_q/t_0 should be independent of the gauge coupling.

SCALE SETTING WITH GRADIENT FLOW

$\tilde{g}^2_{GF}$ vs t/t_0 should be independent of the lattice spacing [2]
Example: HISQ action, $m_s/m_l = 27$, large volume simulations [3]



A: $a \approx 0.15$fm;
B: $a \approx 0.12$fm;
C: $a \approx 0.09$fm;
D: $a \approx 0.06$fm;

[2] Baring cut-off effects and assuming t is large to avoid gradient flow integration artifacts but small enough to minimize finite volume effects

[3] Thanks N. Brown for sharing the MILC gradient flow data

THE GRADIENT FLOW COUPLING

Options to improve: [4]

- w_0 scale [5]
- perturbative improvement (action+flow+operator)
- t-shift improved $\tilde{g}^2_{GF}(\mu)$: simple modification (1404.0984)
  - designed for step scaling but works for scale setting
  - easy to implement with one free parameter
  - can remove most cut-off effects
  - works at strong coupling

[4] Sommer 2013, Ramos 2014
[5] Borsanyi 2012

T-SHIFT IMPROVED GRADIENT FLOW

Define the t-shifted coupling as

$$\tilde{g}^2_{GF}(\mu = \frac{1}{\sqrt{8t}}, a) = \frac{1}{\mathcal{N}} t^2 \langle E(t + a^2\tau_0) \rangle, \qquad a^2\tau_0 \ll t$$

In the continuum $a \to 0$ limit $\tilde{g}^2_{GF}(\mu) \to g^2_{GF}(\mu)$

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Why would this help? Three ways of looking at it:

1. $\langle E(t)\rangle \to \langle E(t + a^2\tau_0)\rangle$
   replaces $E(t)$ with a smeared operator
   $\to$ smearing tends to remove lattice artifacts
2. $t + a^2\tau_0 \to t$ removes initial flow time artifacts
3. The shift can remove $\mathcal{O}(a^2)$ terms

T-SHIFT IMPROVED GRADIENT FLOW

Expand the t-shifted coupling

$$\tilde{g}^2_{GF}\left(\mu = \frac{1}{\sqrt{8t}}, a\right) = \frac{1}{\mathcal{N}} t^2 \langle E(t + a^2\tau_0)\rangle, \qquad a^2\tau_0 \ll t$$

in $a^2\tau_0$

$$\begin{aligned} \tilde{g}^2_{GF}(\mu, a) &= g^2_{GF}(\mu, a) + a^2\tau_0 \frac{d}{dt}\left(t^2\langle E(t)\rangle\right) + \ldots \\ g^2_{GF}(\mu, a) &= g^2_{GF}(\mu) + a^2\mathcal{C} + \ldots \end{aligned}$$

If $\quad \mathcal{C} = -\tau_0 \frac{d}{dt}\left(t^2\langle E(t)\rangle\right) \quad$ the $\mathcal{O}(a^2)$ corrections are removed

$$\tilde{g}^2_{GF}(\mu, a) = g^2_{GF}(\mu) + \mathcal{O}(a^4, a^2 \log^n(a))$$

T-SHIFT IMPROVED GRADIENT FLOW

It might even help at 1-loop level:

$$\tilde{g}^2_{GF}(t,a) = \frac{1}{\mathcal{N}} t^2 \langle E(t + a^2\tau_0) \rangle = g^2_{GF}(t + a^2\tau_0)\left(1 + \frac{a^2\tau_0}{t}\right)^{-2}$$

$(1 + a^2\tau_0/t)^{-1}$ term gives tree-level corrections while

$$g^2_{GF}(t+a^2\tau_0) = g^2_{GF}(t) + \frac{a^2\tau_0}{t}\, t \frac{dg^2_{GF}}{dt} + \cdots = g^2_{GF}(t) + \frac{a^2\tau_0}{t}\, b_0 g^4_{GF}(t) + \ldots$$

gives 1-loop corrections. If

- the tree level corrections are small
- or removed analytically

the τ_0 shift could give 1-loop improvement!

T-SHIFT IMPROVED GRADIENT FLOW

$$\tilde{g}^2_{GF}\left(\mu = \frac{1}{\sqrt{8t}}, a\right) = \frac{1}{\mathcal{N}} t^2 \langle E(t + a^2\tau_0)\rangle,$$

- Every τ_0 value is correct - some are just better
- If the tree-level corrections are small, $\tau_0 = \text{const}$ can give 1-loop improvement
- For full $\mathcal{O}(a^2)$ improvement τ_{opt} must depend on both the bare and renormalized couplings
  - $\rightarrow$ might mean no predictive power

$\rightarrow$ comparing different τ_0 values is a good consistency check

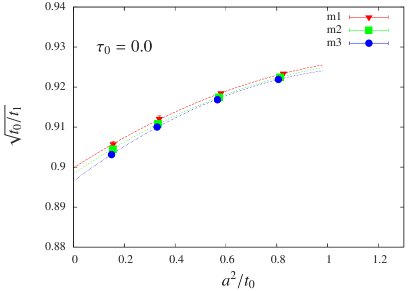
HISQ 2+1+1

Quantify the cut-off dependence: define 2 scales

$$t^2\langle E(t)\rangle|_{t=t_0} = 0.3, \quad t^2\langle E(t)\rangle|_{t=t_1} = 0.35$$

and compare $\sqrt{t_0/t_1}$ vs a^2/t_0 for $m_s/m_l = 5, 10, 27$

without t-shift



Quadratic + quartic a^2 dependence?

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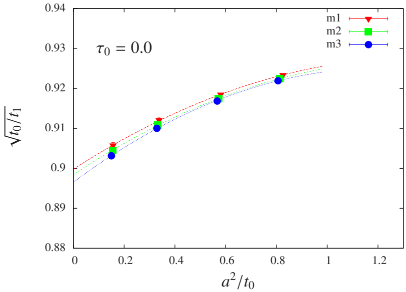

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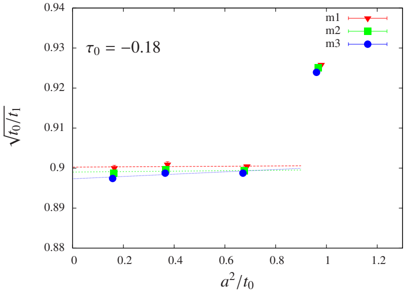

Quadratic + quartic a^2 dependence?

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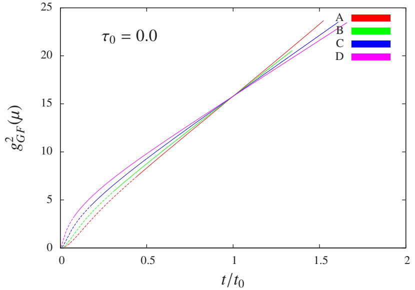
The coarsest $a \approx 0.15$fm set is (probably) not in the $\mathcal{O}(a^2)$ scaling regime!

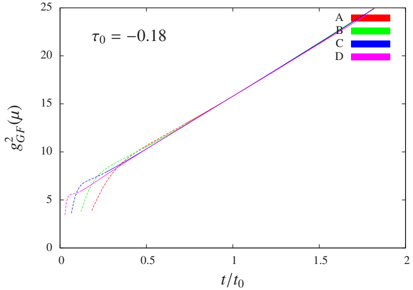
HISQ 2+1+1

There is nothing special about t_0 or t_1:
$\tilde{g}^2_{GF}$ vs t/t_0 should be independent of the lattice spacing if there are no cut-off effects [6]

$\tau_0 = 0.0$



$\tau_0 = -0.18$



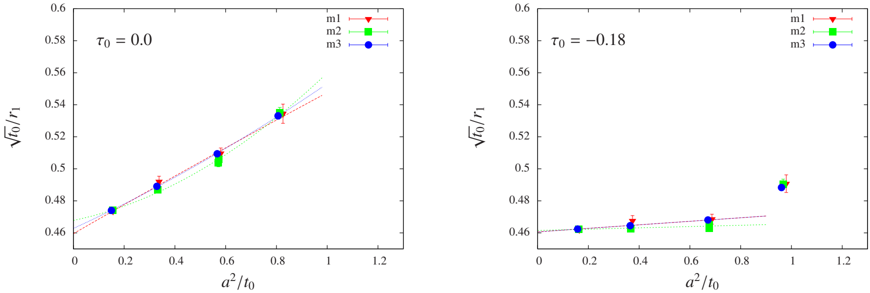
A: $a \approx 0.15$fm; B: $a \approx 0.12$fm; C: $a \approx 0.06$fm; D: $a \approx 0.06$fm;

[6] Assuming t is large to avoid gradient flow integration artifacts but small enough to minimize finite volume effects

HISQ 2+1+1

How robust is τ_{opt}? Compare t_0 and r_1:

$\tau_0 = 0.0$ $\tau_0 = -0.18$

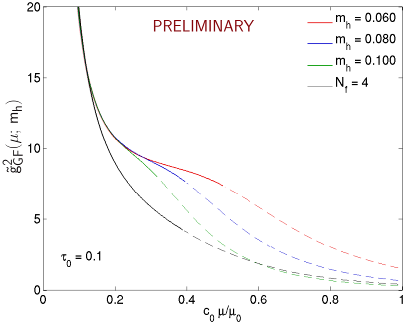


Just like before:

- without t-shift improvement lattice artifacts mask that the coarsest set is not in the $\mathcal{O}(a^2)$ scaling regime
- With t-shift the lattice scale is predicted better than 1% (τ_{opt} is predicted using t_0/t_1)

$4 + 8$ FLAVOR RUNNING COUPLING

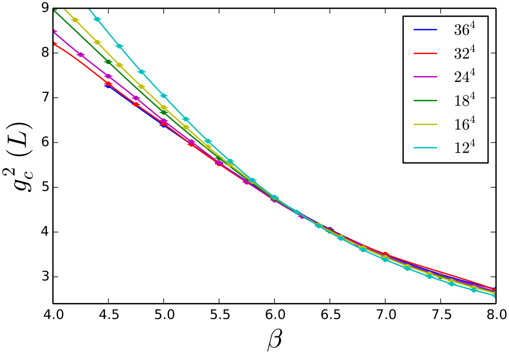
The energy dependence of the running coupling with 4 chiral ($m_l = 0$) and 8 heavy flavors:



interpolates between 12 flavors (conformal) and 4 flavors (chirally broken) : observe the walking!

STEP SCALING: $N_f = 12$

$g^2_{GF}(L)$ versus β bare coupling shows crossings
($g^2_{GF}(L)$ is independent of the scale)
- does that imply an IRFP?



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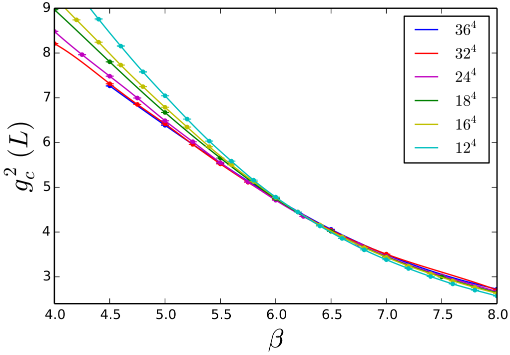
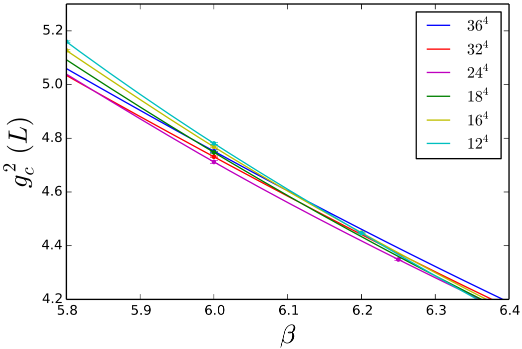


Only if the crossings survive the continuum limit!

$N_f = 12$

Zoom in:

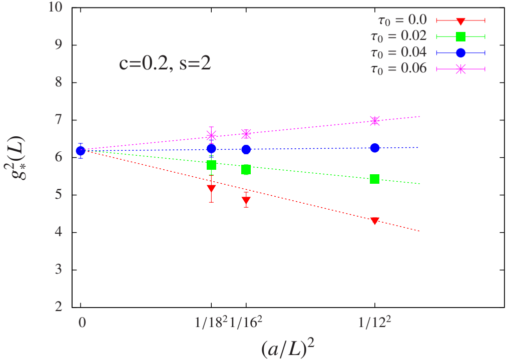


Only if the crossings survive the continuum limit!

This is special: other published step scaling function studies of $N_f = 12$ do not see crossings, they identify an IRFP by extrapolating from the weak coupling side.

$N_f = 12$

Take the continuum limit of the crossings:
$g^2_{GF}(L) = g^2_{GF}(sL) \implies g^2_\star(L;s) = g^2_{GF}(L)$

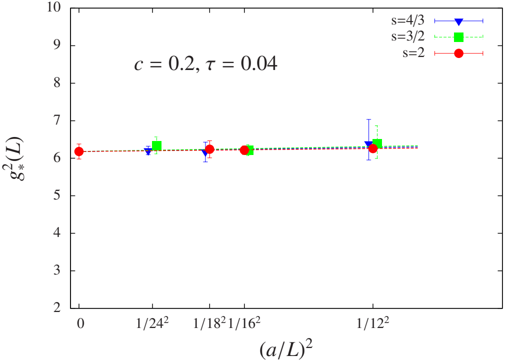


$c = 0.2, s = 2$

optimization is essential,
$\tau_{\mathrm{opt}} \approx 0.04$

$N_f = 12$

Combine $s = 4/3, 3/2$ and 2 with common $\tau_0 = 0.04$

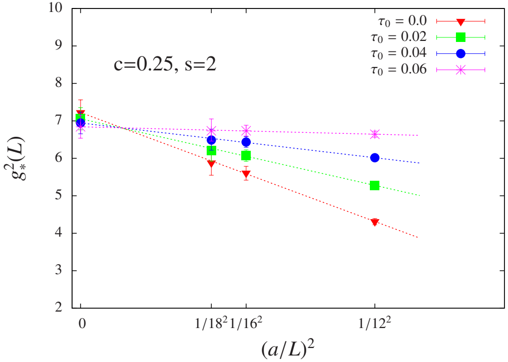


All scale factors predict $g_\star^2(L) \approx 6.2$ with no (apparent) dependence on the lattice spacing

Extrapolating $g_\star^2(L)$ is more reliable than the β function

$N_f = 12$

Results are similar with $c = 0.25, 0.3$ Larger c gives stronger $g_\star^2(L)$ and has increased statistical errors, but t-shift improvement works the same



$c = 0.25, s = 2$

$\tau_{opt} \approx 0.06$

Preliminary

Conclusion

t-shift gradient flow improvement is a simple yet powerful method

- It is easy to implement and can give 1-loop improvement
- In step scaling function studies extrapolation to the continuum limit is possible even at strong running coupling
- In scale setting the optimal τ_0 parameter can be found by comparing two configuration sets
- t-shift improved coupling can reveal non $\mathcal{O}(a^2)$ scaling violations that are hidden otherwise

THANK YOU MIKE FOR LEADING THE WAY!

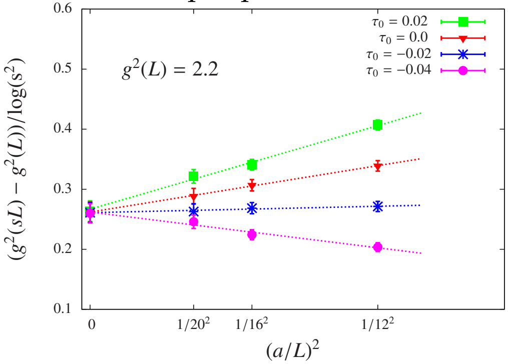
$N_f = 4$

Test case: step scaling function with 4 flavor staggered fermions

- Set $\mu = (cL)^{-1}$, $c = 0.25$
- Define discrete β function with scale change $s = 1.5$

$$\beta_{\text{lat}}(g^2_{GF}; s; a) = \frac{\tilde{g}^2_{GF}(L; a) - \tilde{g}^2_{GF}(sL; a)}{\log(s^2)}$$

Continuum exptrapolation:

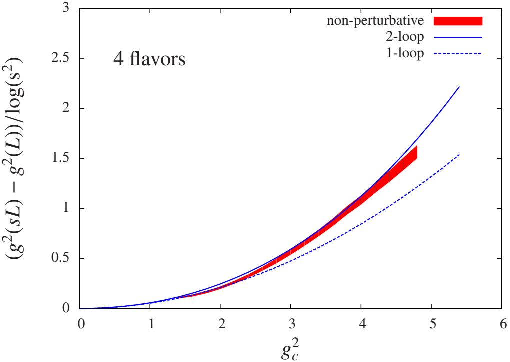


Cut-off corrections with our action are small

All τ_0 shifts predict the same continuum value $\rightarrow$ consistency check!

$N_f = 4$

Discrete β function

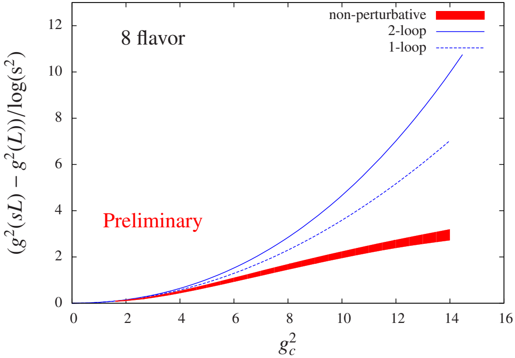


Close agreement with 2-loop perturbative value

$\tau_0 = -0.02 - 0.0$ in the investigated g^2_{GF} range

$N_f = 8$

Expected to be chirally broken but very strongly coupled



Very different from 2-loop perturbative

$\tau_0 = 0.0 - 0.04$ with 1x nHYP

$\tau_0 = 0.12 - 0.20$ with 2x nHYP

t-shift optimization is essential