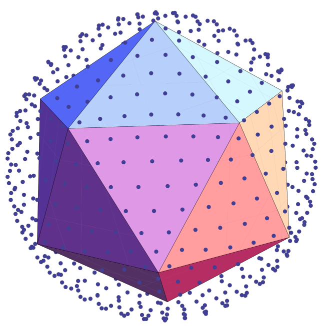
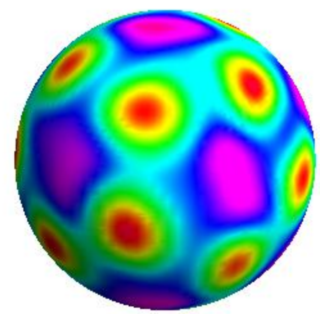
Radial Conformal Lattice Field Theory: Spherical Manifolds



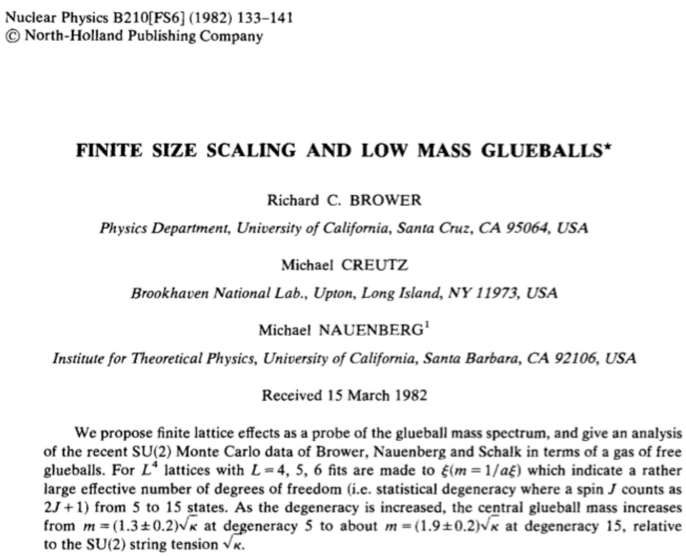
$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$



Richard Brower
Boston University

Creutz Fest 14 BNL Sept 6, 2014

I have (Erdos)/Creutz #1 !



Nuclear Physics B210[FS6] (1982) 133–141

FINITE SIZE SCALING AND LOW MASS GLUEBALLS*

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Received 15 March 1982

We propose finite lattice effects as a probe of the glueball mass spectrum, and give an analysis of the recent SU(2) Monte Carlo data of Brower, Nauenberg and Schalk in terms of a gas of free glueballs. For L^4 lattices with $L=4$, 5, 6 fits are made to $\xi(m=1/a\xi)$ which indicate a rather large effective number of degrees of freedom (i.e. statistical degeneracy where a spin J counts as $2J+1$) from 5 to 15 states. As the degeneracy is increased, the central glueball mass increases from $m=(1.3\pm0.2)\sqrt{\kappa}$ at degeneracy 5 to about $m=(1.9\pm0.2)\sqrt{\kappa}$ at degeneracy 15, relative to the SU(2) string tension $\sqrt{\kappa}$.

- The Erdős number (Hungarian pronunciation: [ˈɛrdøːʃ]) describes the "collaborative distance" between a person and mathematician Paul Erdős, as measured by authorship of mathematical papers.

In 1784, Gustav Filip Creutz was elected a member of the Royal Swedish Academy of Sciences.

My Motivation

- Conformal Field Theories, interesting for
  - BSM composite Higgs
  - AdS/CFT weak-strong duality
  - Model building & Critical Phenomena in general

Potential Huge Advantage for CFT!

- Linear Hypercubic vs Exponential Radial Lattice

$$a < r < aL \to 1 < \log(r) < L$$

Both UV asy freedom and IR conformal on a lattice?

Radial Quantization: *Early History*

- S. Fubini, A. Hanson and R. Jackiw PRD 7, 1732 (1972)

  Abstract: A field theory is quantized covariantly on Lorentz-invariant surfaces. Dilatations replace time translations as dynamical equations of motion. .... The Virasoro algebra of the dual resonance model is derived in a wide class of 2-dimensional Euclidean field theories.

- J. Cardy J. Math. Gen 18 757 (1985).

  Abstract: The relationship between the correlation length and critical exponents in finite width strips in two dimensions is generalised to cylindrical geometries of arbitrary dimensionality d. For $d > 2$ these correspond however, to curved spaces. The result is verified for the spherical model

Radial Quantization

Evolution: $$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

$$ds^2 = dx^\mu dx_\mu = e^{2\tau}[d\tau^2 + d\Omega^2]$$

Can drop Weyl factor!

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

$$\text{"time"} \quad \tau = log(r)\ , \quad \text{"mass"} \quad \Delta = d/2 - 1 + \eta$$

$$D \to x_\mu \partial_\mu = r\partial_r = \frac{\partial}{\partial \tau}$$

Back to the Boostrap! *(CFTs : No local Lagragian)*

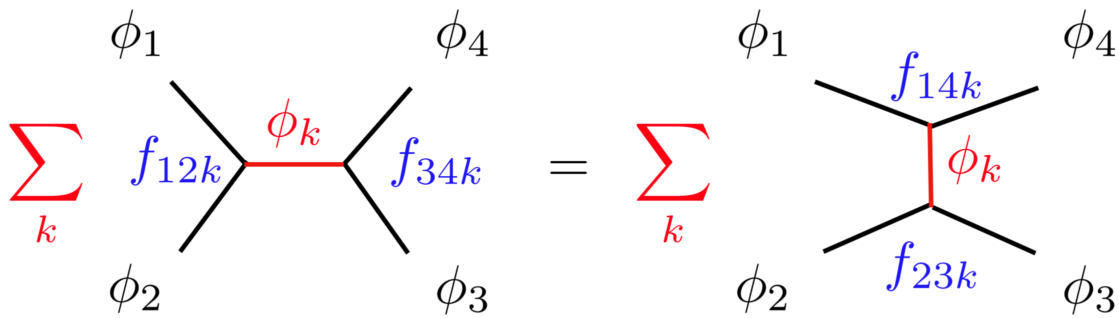
(i.e. Data: spectra + couplings to conformal blocks)

Exact 2 and 3 correlators

$$\langle \phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1 - x_2|^{2\Delta}}$$

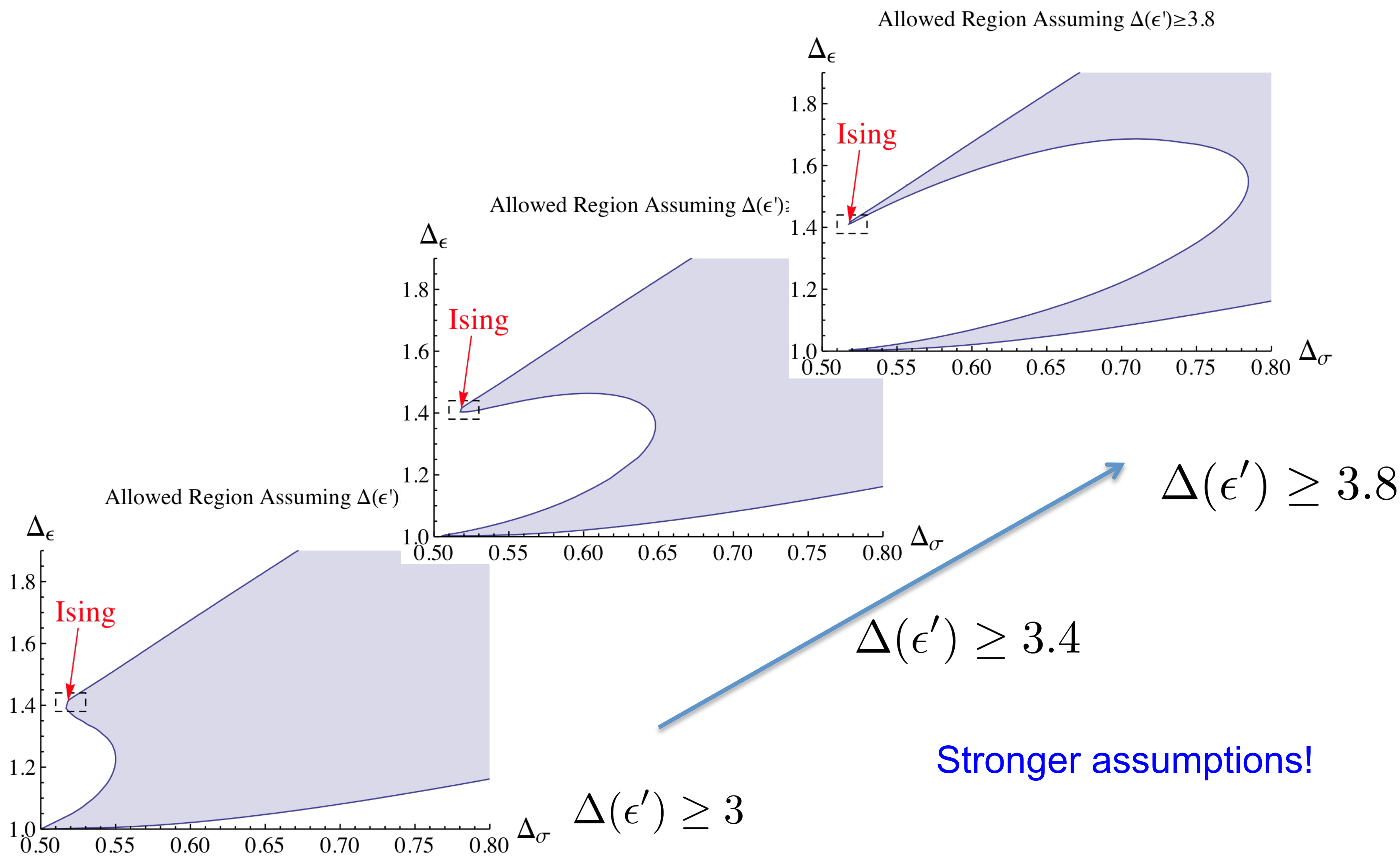
$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1 - x_2|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(0)$$

Only "tree" diagrams!
"partial waves" exp: sum over conformal blocks



CFT Bootstrap: OPE & factorization completely fixed the theory

*Inequalities from Bootstrap**



•“Solving the 3D Ising Model with the Conformal Bootstrap” (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

Exact CFT: Power Law Correlator

Conformal correlator: $\langle \phi(x_1)\phi(x_2)\rangle = C\dfrac{1}{|x_1 - x_2|^{2\Delta}}$

$$
\begin{aligned}
r_1^\Delta r_2^\Delta \langle \phi(\tau_1,\Omega_1)\phi(\tau_2,\Omega_2)\rangle &= C\frac{1}{[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]^\Delta} \\
&\simeq Ce^{-(\log(r_2) - \log(r_1)\Delta} \\
&= Ce^{-\tau\Delta}
\end{aligned}
$$

With $|x_1 - x_2|^2 = r_1 r_2[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]$

as $\tau = \log(r_2) - \log(r_1) \to \infty$

Narrative

1. Review: First attempt & What failed*.
   3D Ising on R x Icosahedron:

2. Finite Elements Methods (FEM) to the rescue ?
   Exact Classical Lagrangian

3. QFT on smooth background manifold

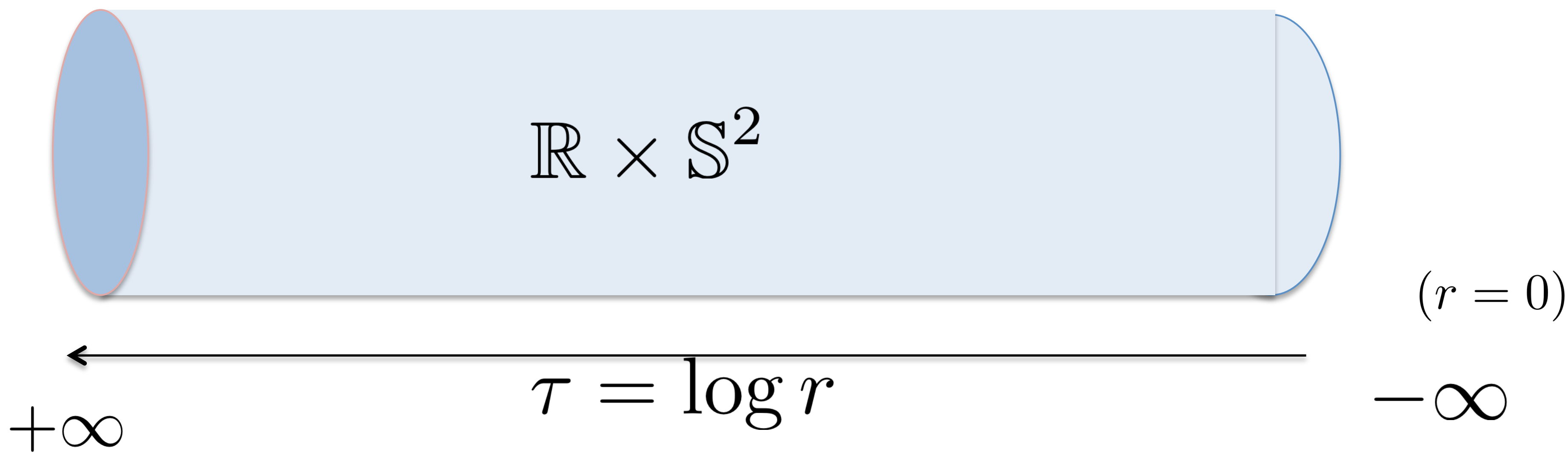
   Regge Calculus (1961). "General relativity without coordinates" (No gravity!)

   Rondom Lattice (1982) Christ, Friedberg, Lee NP (Not random!)

4. Better Lattice & UV counters ?

*R.C.B., G.T. Fleming and H. Neuberger, Phys. Lett. B 721 (2013)

First Attempt: 3-d Ising at Wilson-Fisher FP

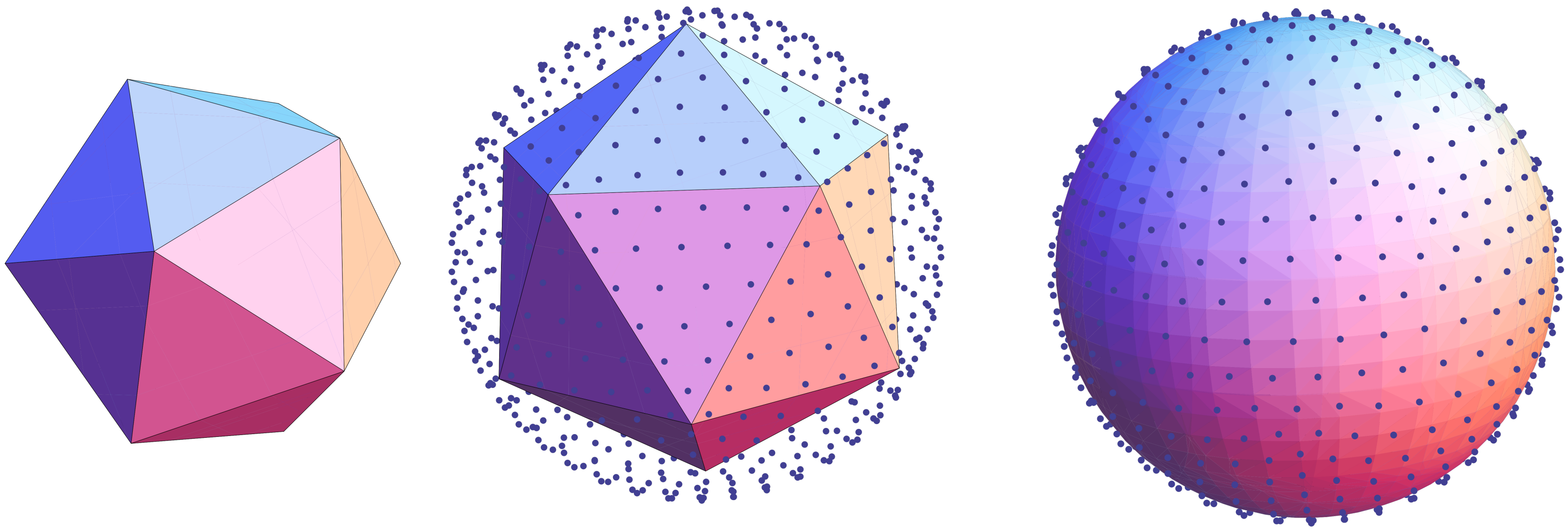


$$Z_{Ising} = \sum_{\sigma(x,t)=\pm 1} e^{\beta \sum_{t,\langle x,y\rangle} \sigma(t,x)\sigma(t,y) + \beta \sum_{t,x} \sigma(t+1,x)\sigma(t,x)}$$

RCB, G. Fleming, H. Neuberger Phys.Lett, B721 (2013)

Order s Refined Triangulated Icosahedron

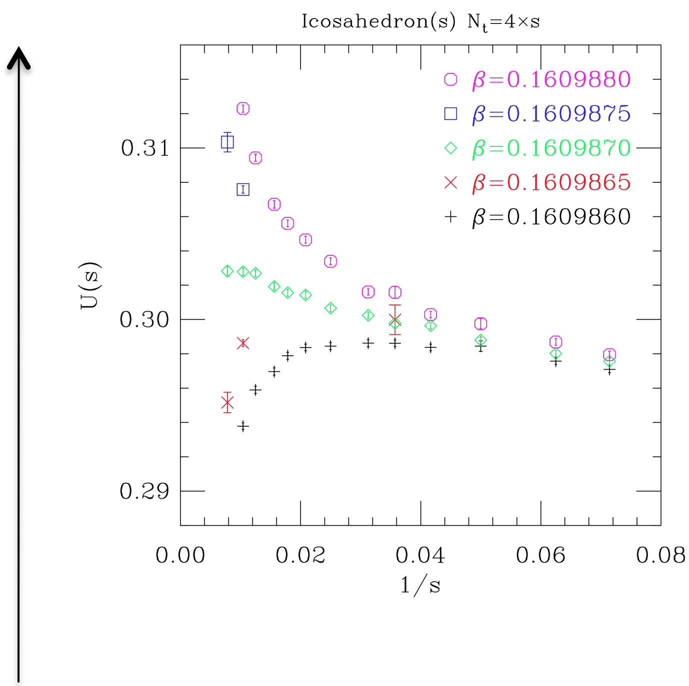
$s = 1$ $s = 8$



l = 0 (A),1 (T1) , 2 (H) are irreducible 120
Iscosahedral subgroup of O(3)

Fitting to Finite scaling

$$U[(\beta - \beta_{cr})L^{1/\nu}, (\lambda - \lambda_{cr})L^{-\omega}, \ldots] \simeq$$
$$U^*(x) + O(L^{-\omega}) \simeq U^*(0) + a_1(\beta - \beta_{cr})L^{1/\nu} + c(\lambda)L^{-\omega} + \cdots$$



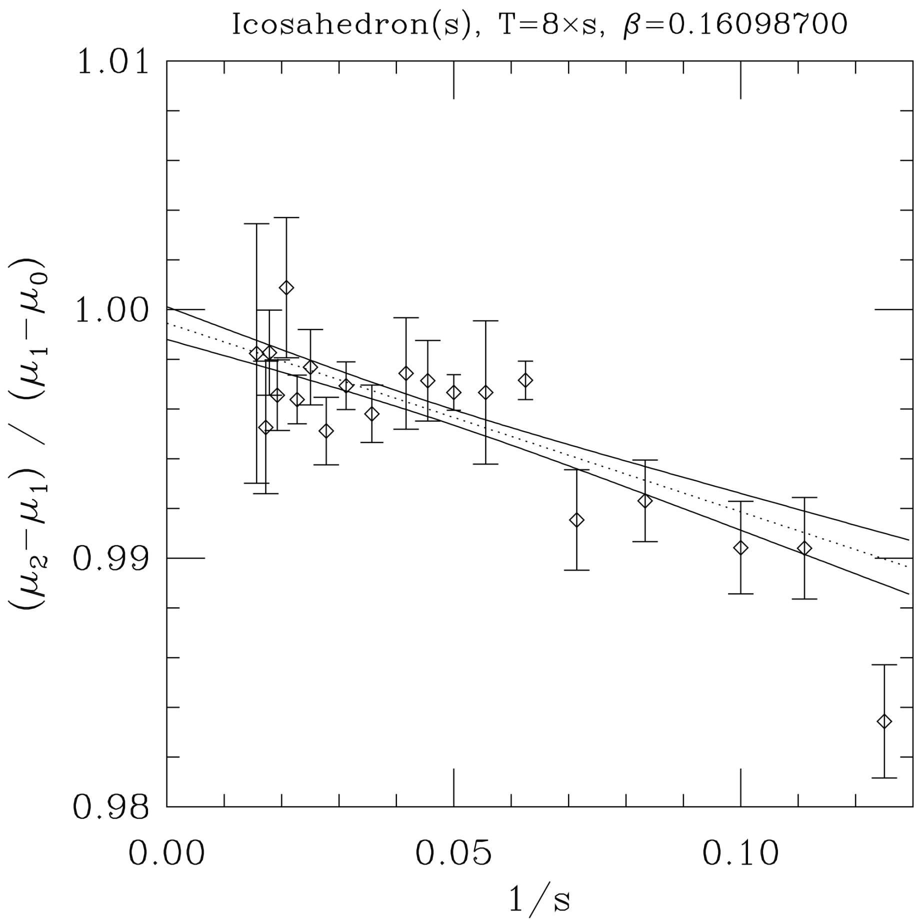
$$U(\beta, s = L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$

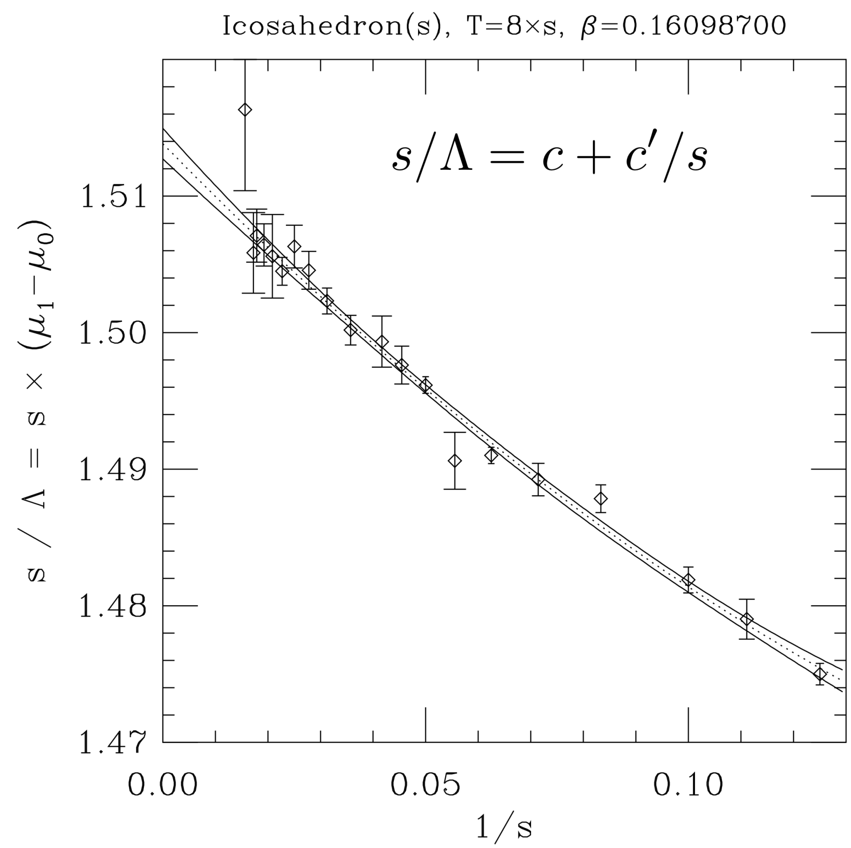
$$\beta_{cross} \simeq \beta_{cr} + c_1 L^{-1/\nu - \omega}$$

$$\beta_{crit} = 0.16098703(3)$$

Double Scaling: $x = (\beta - \beta_{cr})L^{1/\nu}$

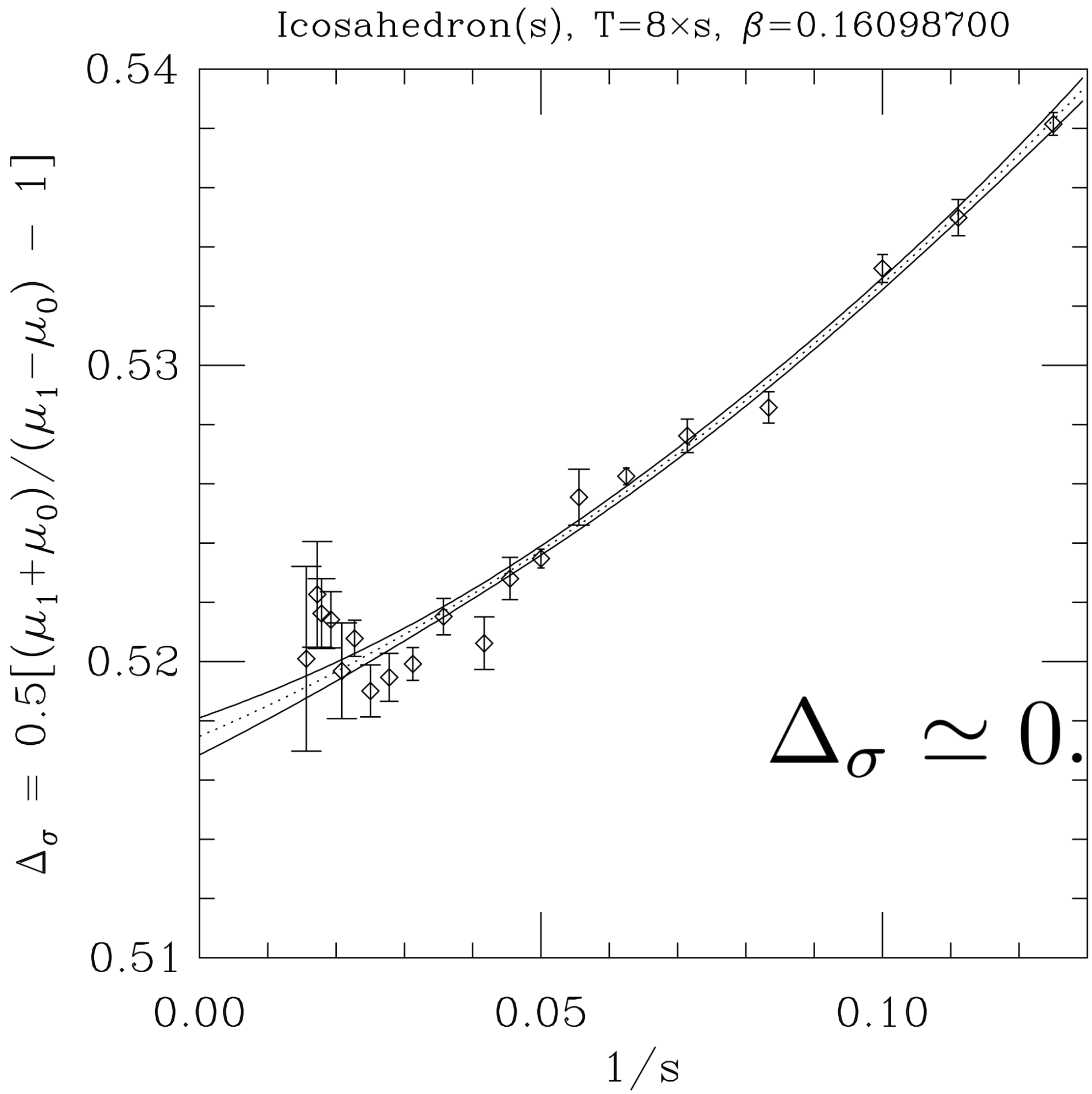
Determine "speed of light" via Descendant Relation & rescale "log(r)"





$$c = 1.5105(7)$$

Current Fit:

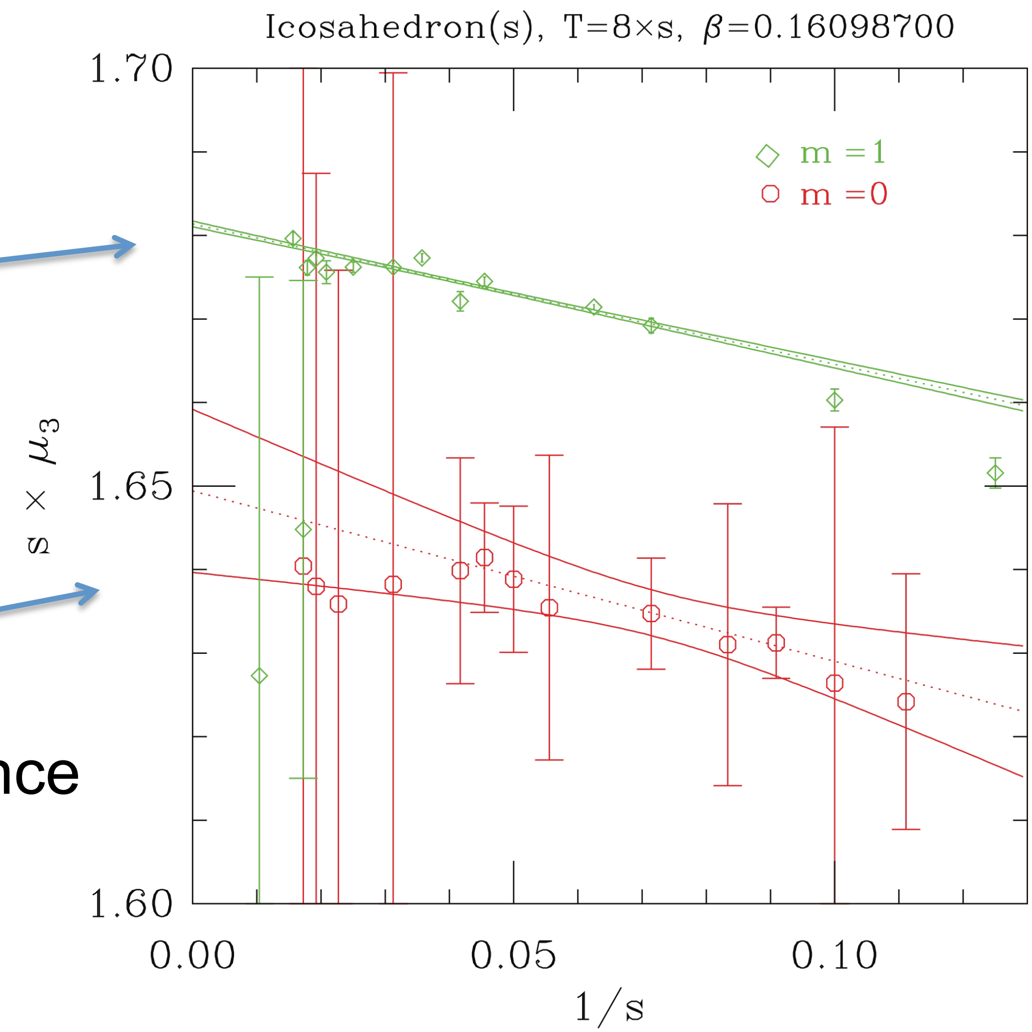


$$\Delta_\sigma \simeq 0.5175$$

1st Failure to recover fullO(4,1) of l = 3?

G rep

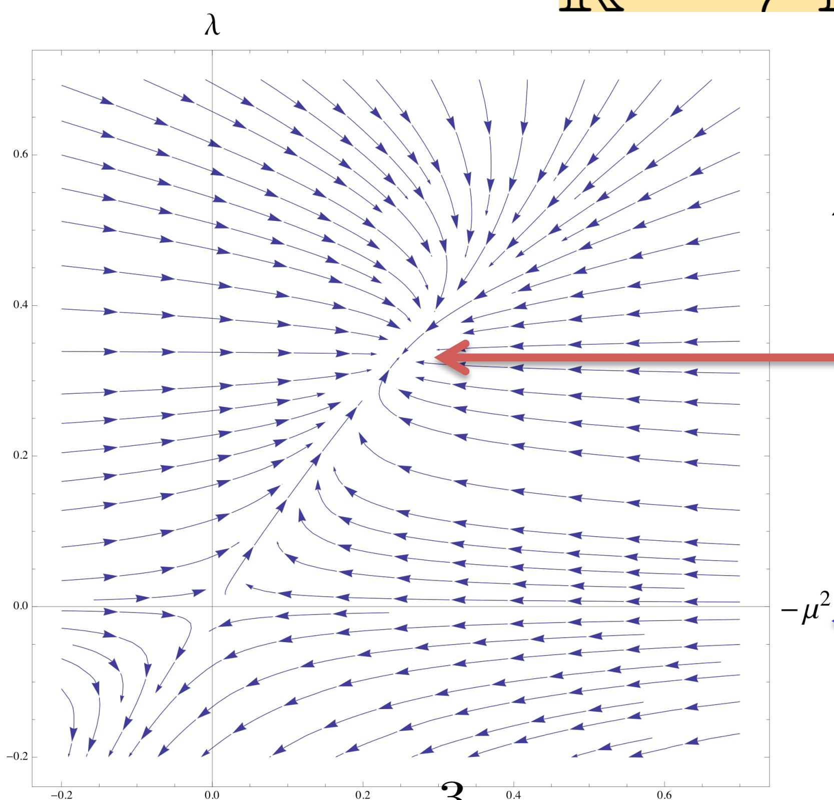
T2 rep



Apparent lack of convergence to a single O(3) irreducible representation for l = 3

Replace Ising Model by phi 4^{th}

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$



$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu/2\lambda)^2$$

Wilson-Fisher FP

Gaussian FP

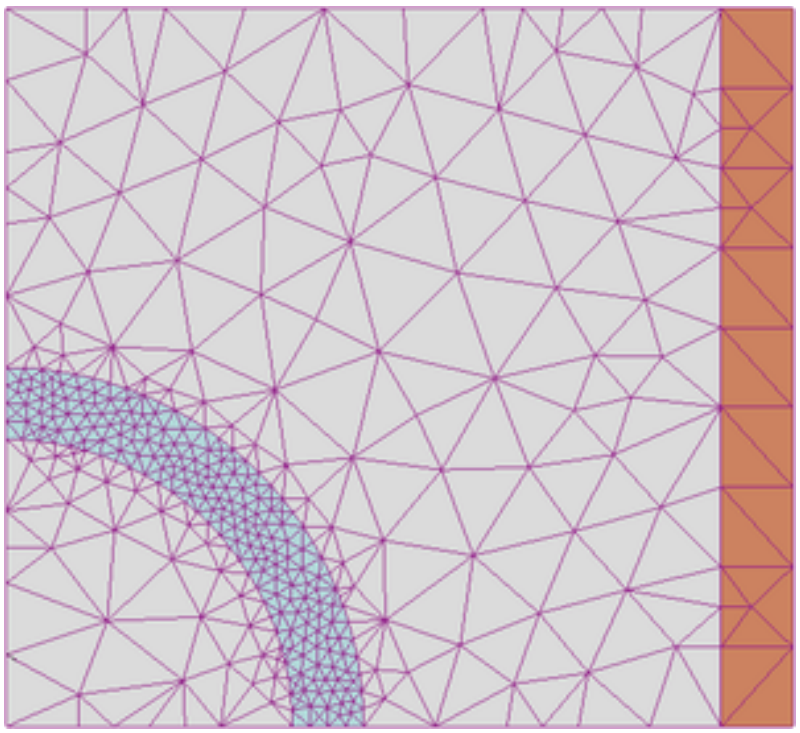
$$\beta_g \quad = \quad \epsilon g - \frac{3}{16\pi^2} g^2 + O(g^3, \epsilon g^2, \mu^4, \mu^2 g;) \qquad \lambda = 4g/4!$$

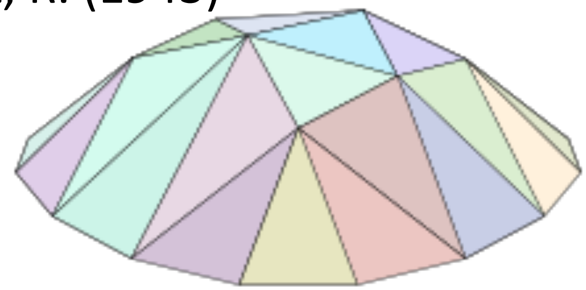
$$\beta_{\mu^2} \quad = \quad 2\mu^2 + ag + \frac{9}{16\pi^2} g\mu^2 + O(\mu^4)$$

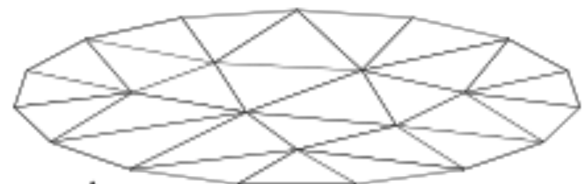
Finite Element Method: What is it?

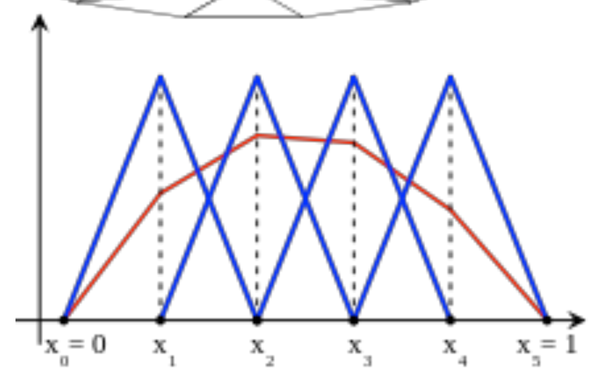
Google: 3,410,000 results

Hrennikoff, Alexander (1941). Courant, R. (1943)





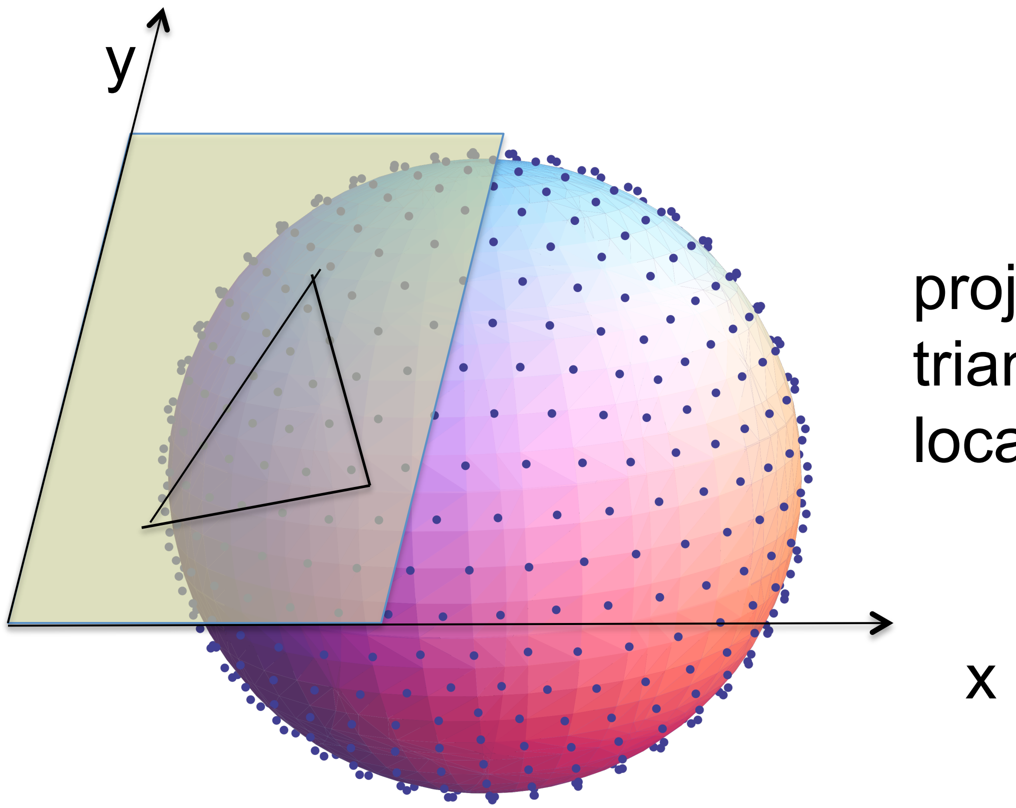




RCB, M. Cheng and G.T. Fleming,
"Improved Lattice Radial Quantization"   PoS LATTICE2013 (2013) 335

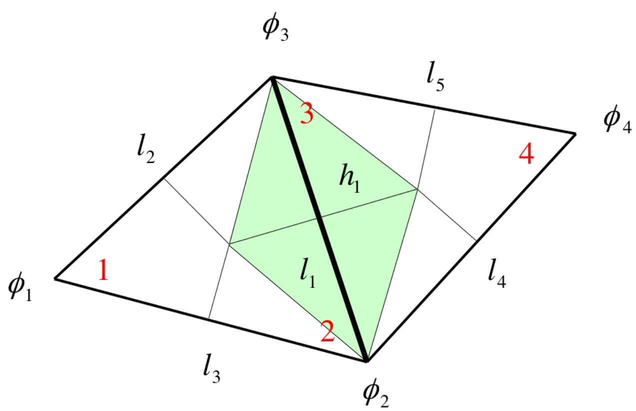
Discretize a Lagrangian on simplicial Manifold?

$$L = \int d^3x[\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \lambda\sqrt{g}(\phi^2 - \mu^2/2\lambda)^2]$$



project spherical triangle onto local tangent plane

Regge Calculus formulation for smooth manifold.



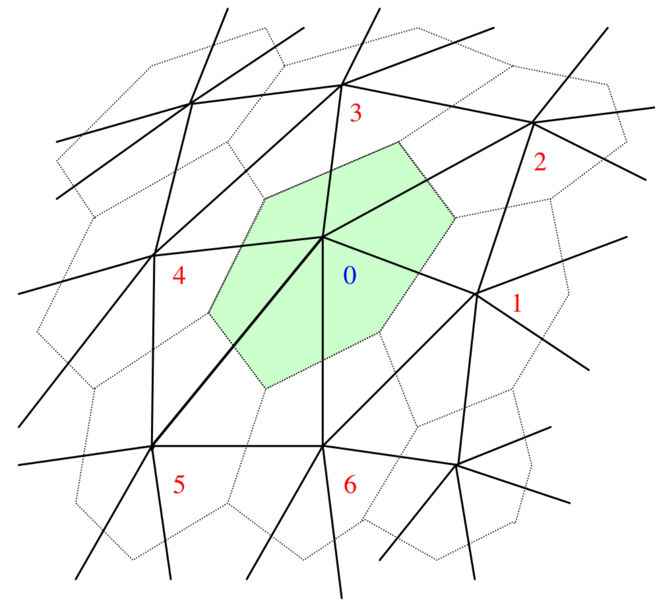
$$FEM : A_d \frac{(\phi_2 - \phi_3)^2}{l_1^2}$$

Delaunay Link Area: $A_d = h_1 l_1$

$$\sum_{\triangle_{kij}} \sqrt{g(k)} g^{ij}(k) \frac{(\phi_k - \phi_i)(\phi_k - \phi_j)}{l_{ki} l_{kj}}$$

H. Hamber, S. Liu, *Feynman rules for simplicial gravity,* NP B475 (1996)

Einstein Regge Curvature



$$\delta_v = 2\pi - \sum_{i \in V} \theta_i$$

$$\sum_v \delta_v = 2\pi\chi = 2\pi(F - E + V)$$

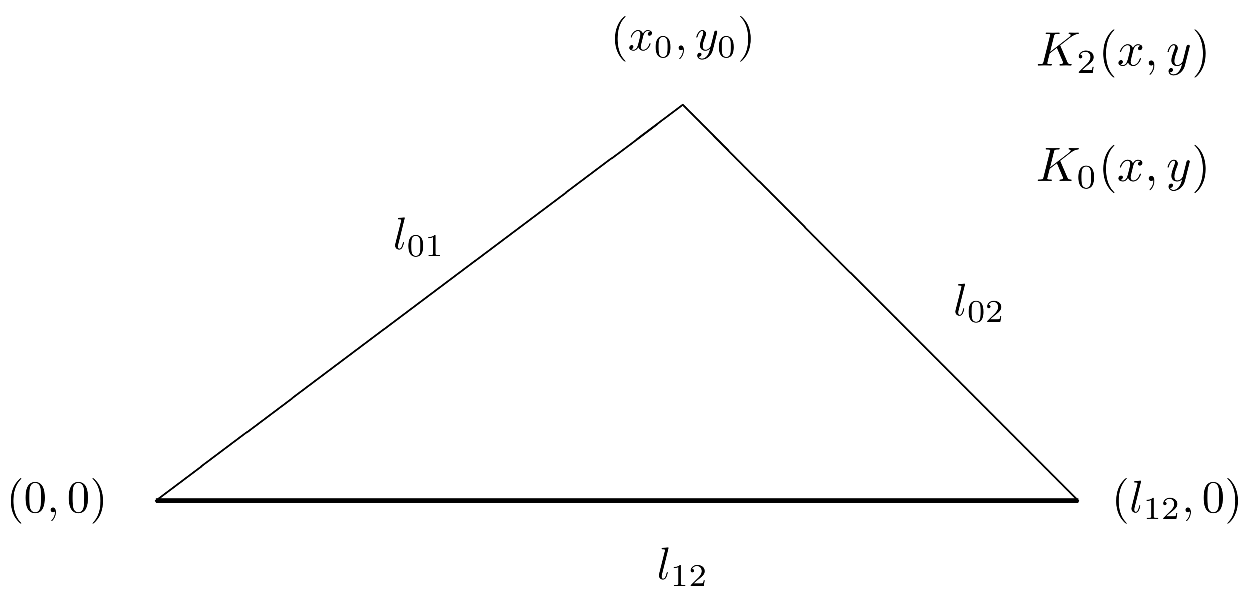
Could Optimize adaptive Delaney triangles on unit sphere

$$\int d^2x\sqrt{g}[\lambda - kR^2 + aR^2] \implies \sum_v A_v[\lambda - 2kR_v + aR_v^2]$$

$$R_v = 2\delta_v/A_v$$

flat triangles: $\delta_v = 4\pi/A_v$

Kinetic term for Linear Element



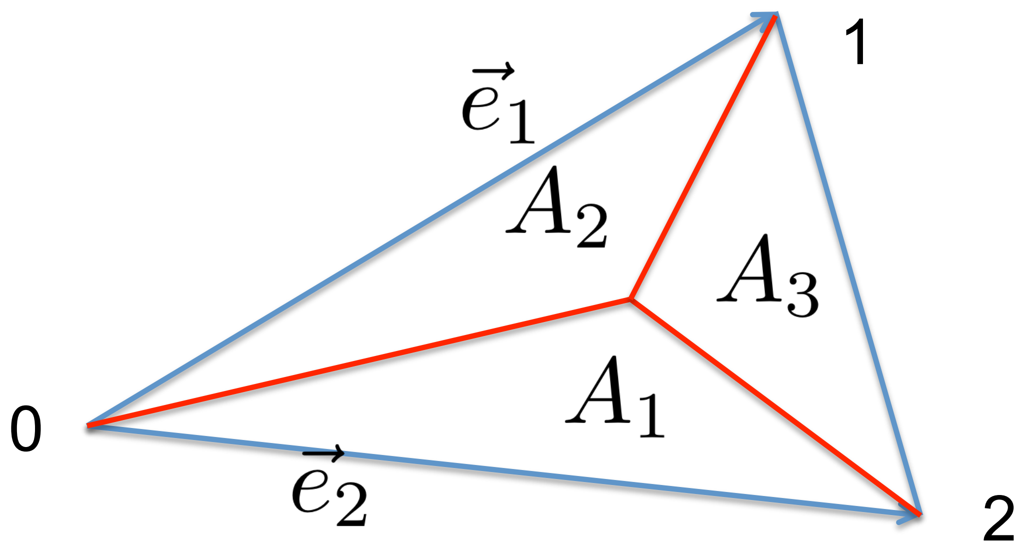
$$
\begin{aligned}
K_1(x,y) &= [l_{12} - x - \frac{(l_{12} - x_0)y}{y_0}]/l_{12} \\
K_2(x,y) &= [x - \frac{x_0 y}{y_0}]/l_{12} \\
K_0(x,y) &= \frac{y}{y_0}
\end{aligned}
$$

On each triangle expand: $\phi(x,y) = \sum_i K_i(x,y)\phi_i$ an integrate

$$
\int_{A_{012}} dxdy \partial_\mu \phi(x,y) \partial_\mu \phi(x,y) = \frac{1}{2A_{012}}[(l_{01}^2 + l_{20}^2 - l_{12}^2)(\phi_1 - \phi_2)^2 + \text{cyclic}]
$$

Linear Finite Element Method for triangulate Manifold

$$g_{ij}(0) = \vec{e}_i \cdot \vec{e}_j$$



$$K_i^{\triangle}(r) = \frac{A_i}{A_{123}}$$

$$W_i(r) = \sum_{\triangle} K_i^{\triangle}(r)$$

$$\phi(r) = \sum_i W_i(r)\phi_i$$

Piecewise linear
subspace of Hilbert space

$$d\vec{x} = \vec{e}_i dx^i \qquad ds^2 = d\vec{x} \cdot d\vec{x} = g_{ij} dx^i dx^j$$

FEM have "spectral fidelity"

- Taylor expansion on hypercubic lattice:

$$a^{-1} \sum_{\pm\mu} (\phi(x) - \phi(x + a\mu))^2 \simeq (\nabla \phi)^2 + O(a^2)$$

- Taylor series for FEM does not work!

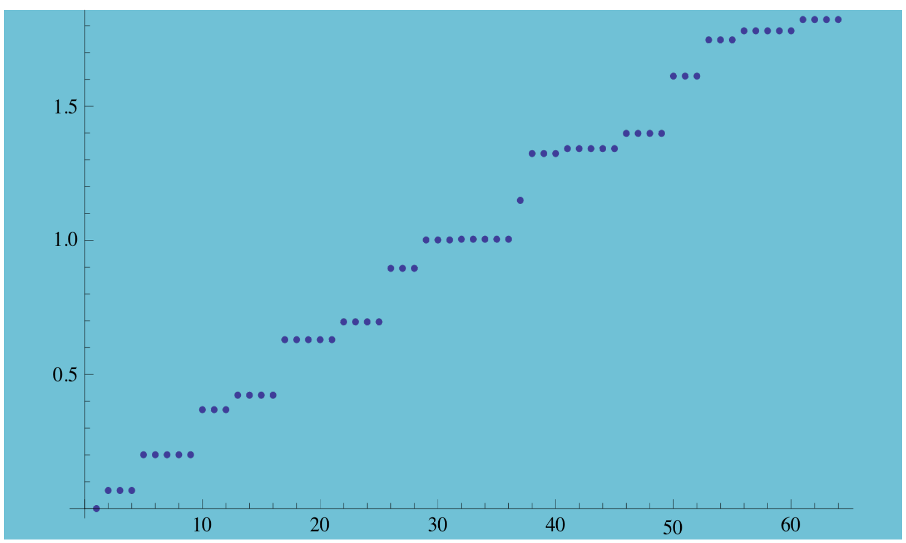
$$a^2 \sum_{y} K(x,y)(\phi(x) - \phi(y))^2 \simeq c_{\mu\nu} \partial_\mu \phi \partial_\nu \phi + O(a^2)$$

- FEM "meta-theorem": spectrum < cut-off to O(a^2) if "triangles are regular enough"

FEM fixes the huge Spectral defects

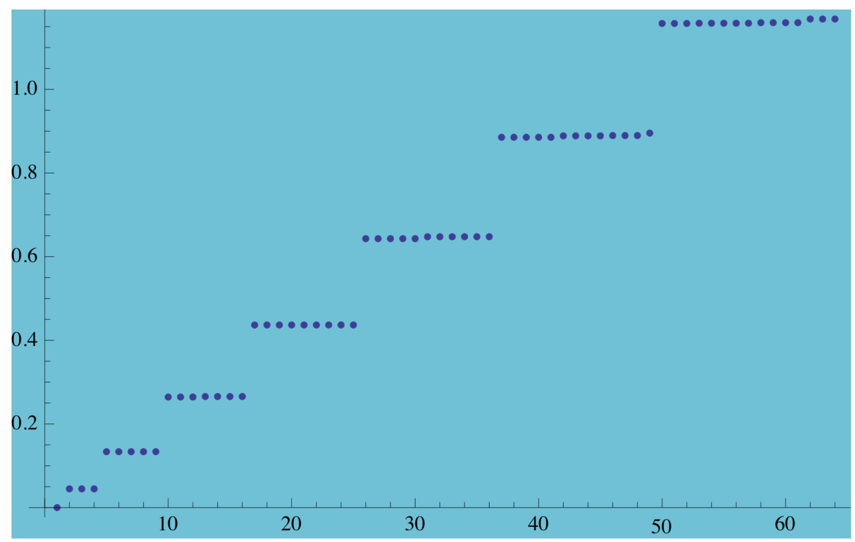
For s = 8 first (l+1)*(l +1) = 64 ev

BEFORE (K = 1)



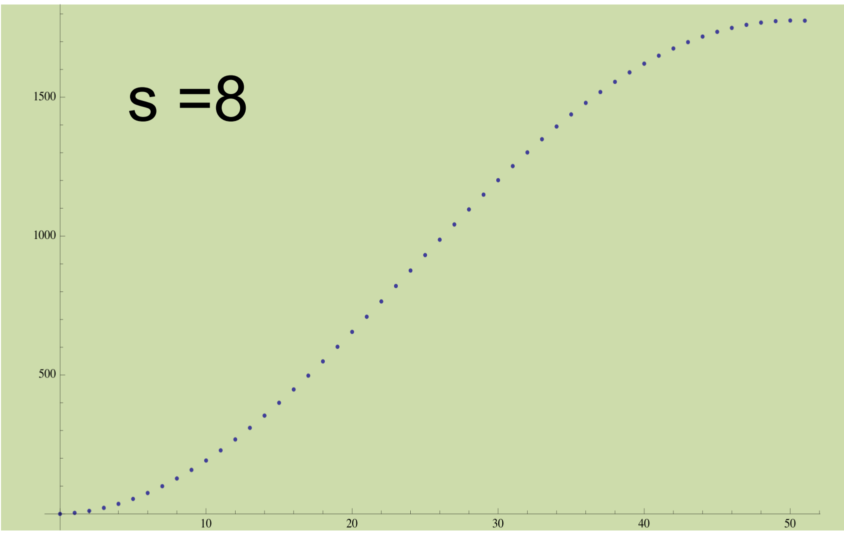
l, m

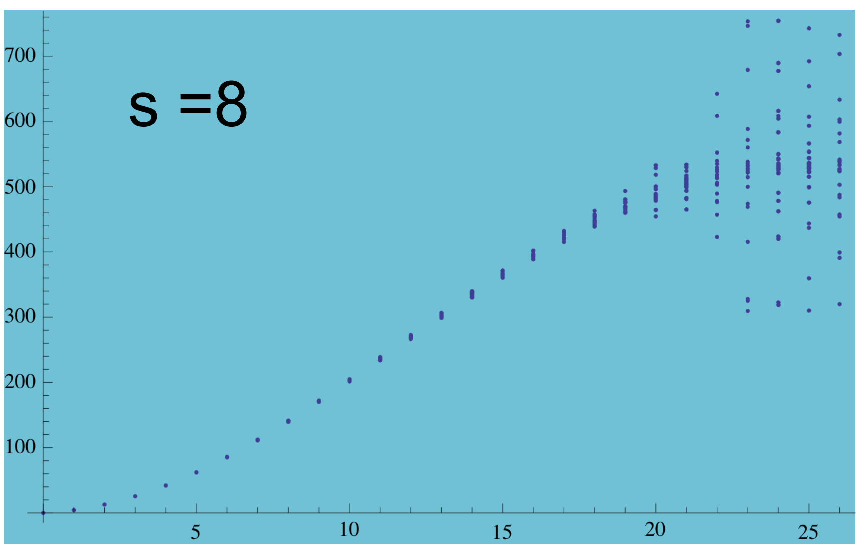
AFTER (FEM K's)

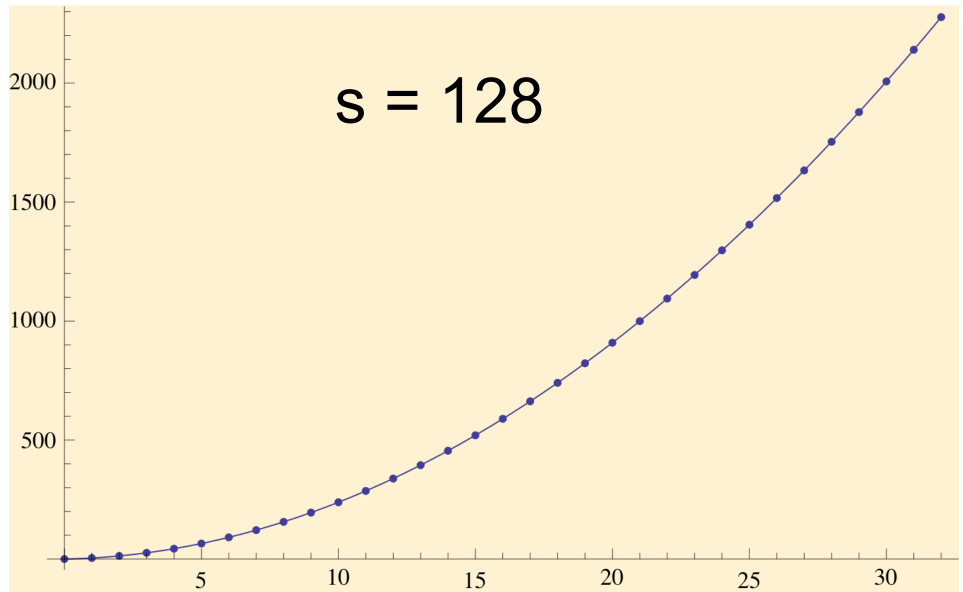


l, m

Spectrum of FE Laplacian on a sphere





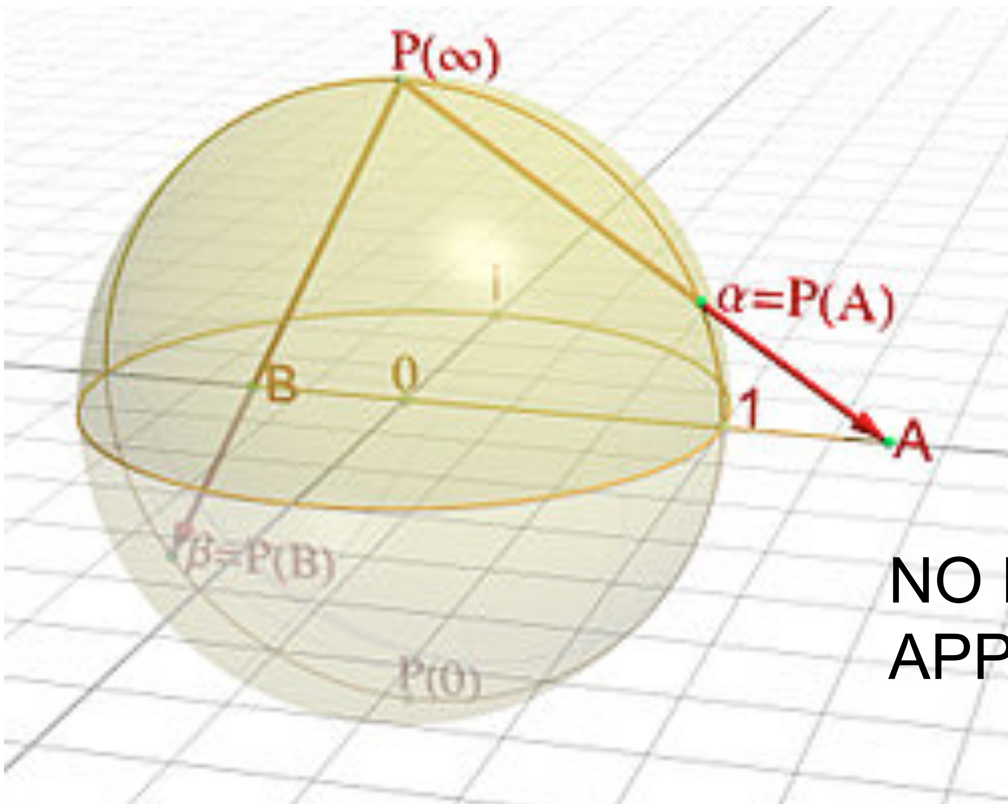


Fit

$$l + 1.00012\, l^2 - 13.428110^{-6}\, l^3 - 5.5724410^{-6}\, l^4$$

Test as Phi 4th on the Rieman sphere!

projection $$\xi = \tan(\theta/2)e^{-i\phi} = \frac{x+iy}{1+z}$$



NO FINITE VOLUME
APPROXIMATION!

Exact Solution to CFT

Exact Two point function

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \to \frac{1}{|1 - cos\theta_{12}|^{\Delta}}$$

$$\Delta = \eta/2 = 1/8$$

$$x^2 + y^2 + z^2 = 1$$

4 pt function

$$(x_1, x_2, x_3, x_4) = (0, \xi, 1, \infty)$$

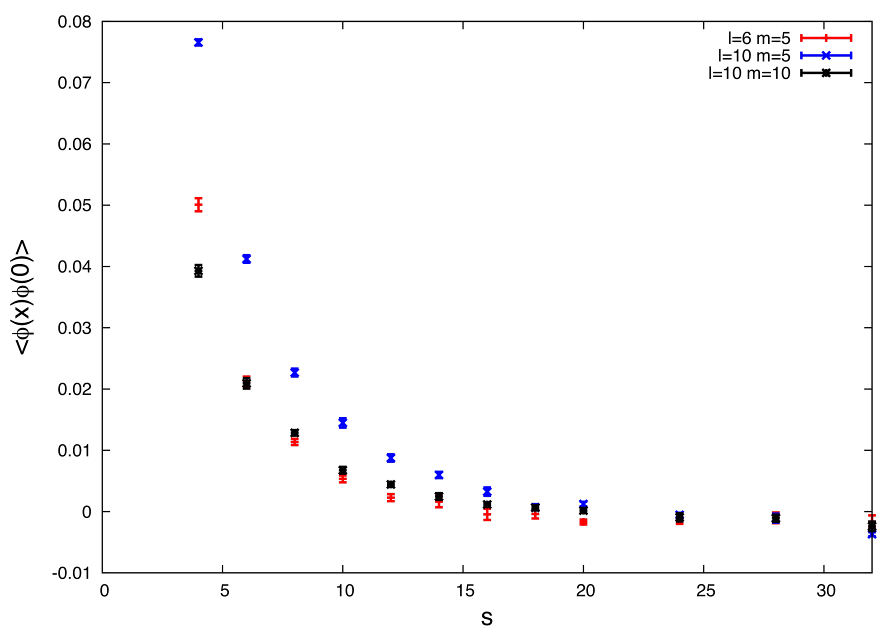
$$g(0, \xi, 1, \infty) = \frac{1}{2|\xi|^{1/4}|1 - \xi|^{1/4}}[1 + \sqrt{1 - \xi}| + |1 - \sqrt{1 - \xi}|]$$

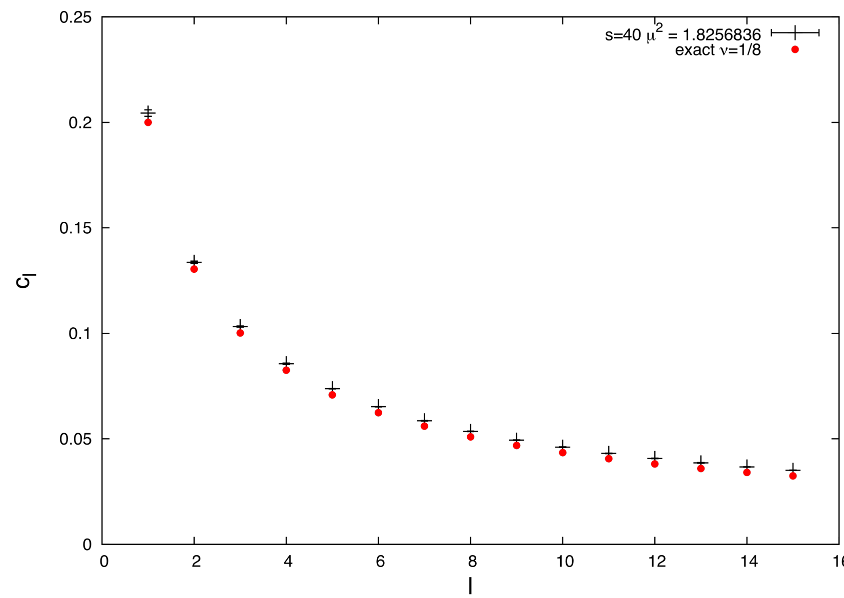
Critical Binder Commulant

$$U_B^* = 1 - \frac{\langle M^4\rangle}{3\langle M^2\rangle\langle M^2\rangle} = 0.567336$$

Test of rotational symmetry?

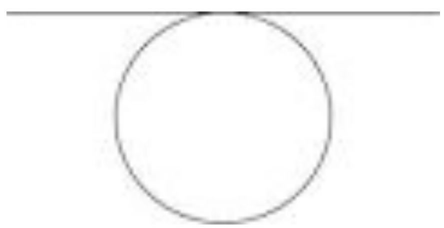
Ylm projection.



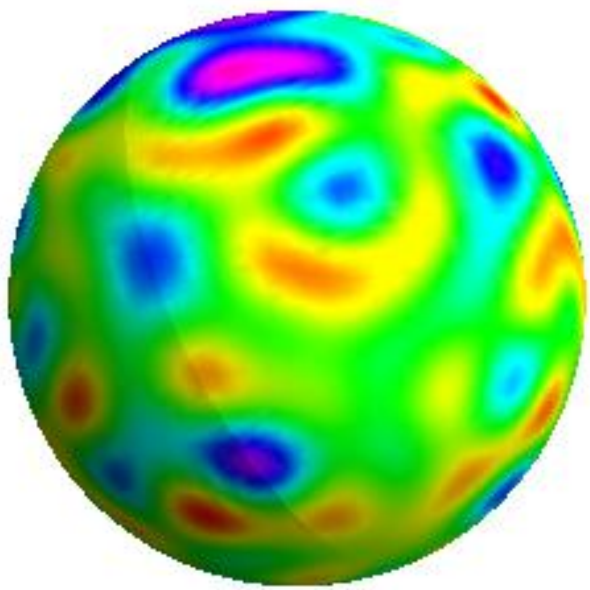


$$\Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

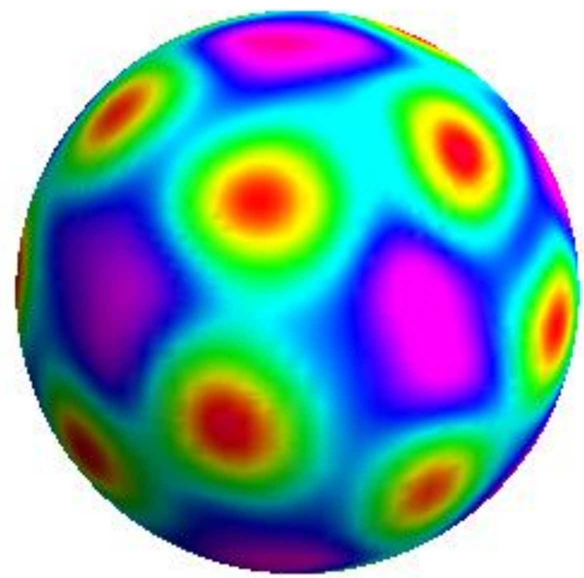
2nd Failure: UV divergence breaks Rotations



$$\delta m^2 = \lambda \langle \phi(x) \phi(x) \rangle \to \frac{1}{K_{xx}}$$



one configuration



average of config.

3 possible solutions?

(i) Pauli-Villars* 1949 (like gradient flow?)

$$\frac{1}{p^2} - \frac{1}{p^2 + M_{PV}^2} \equiv \frac{1}{p^2 + p^4/M_{PV}^2}$$

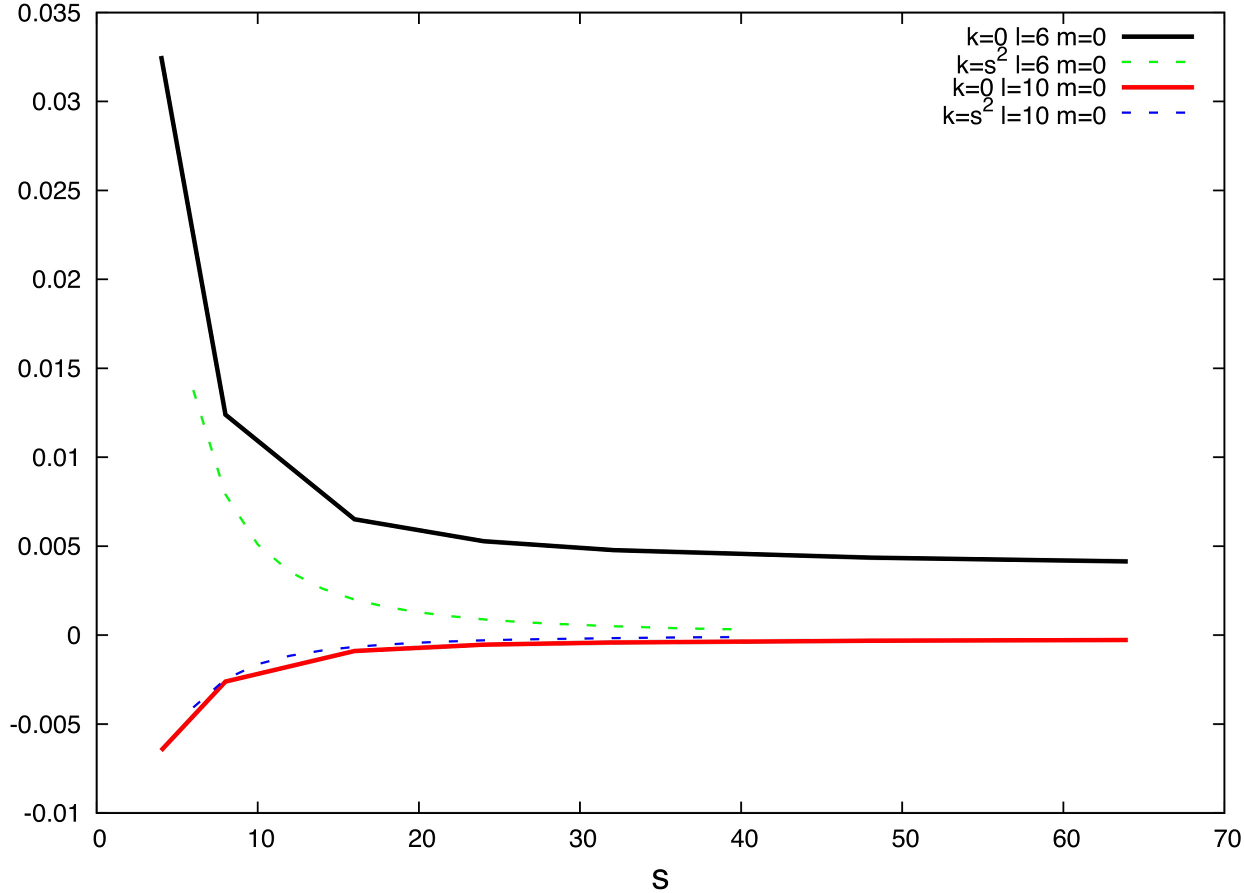
$$1/\xi \ll M_{PV} \ll \pi/a$$

(ii) Subtract x-dependent mass Counter term

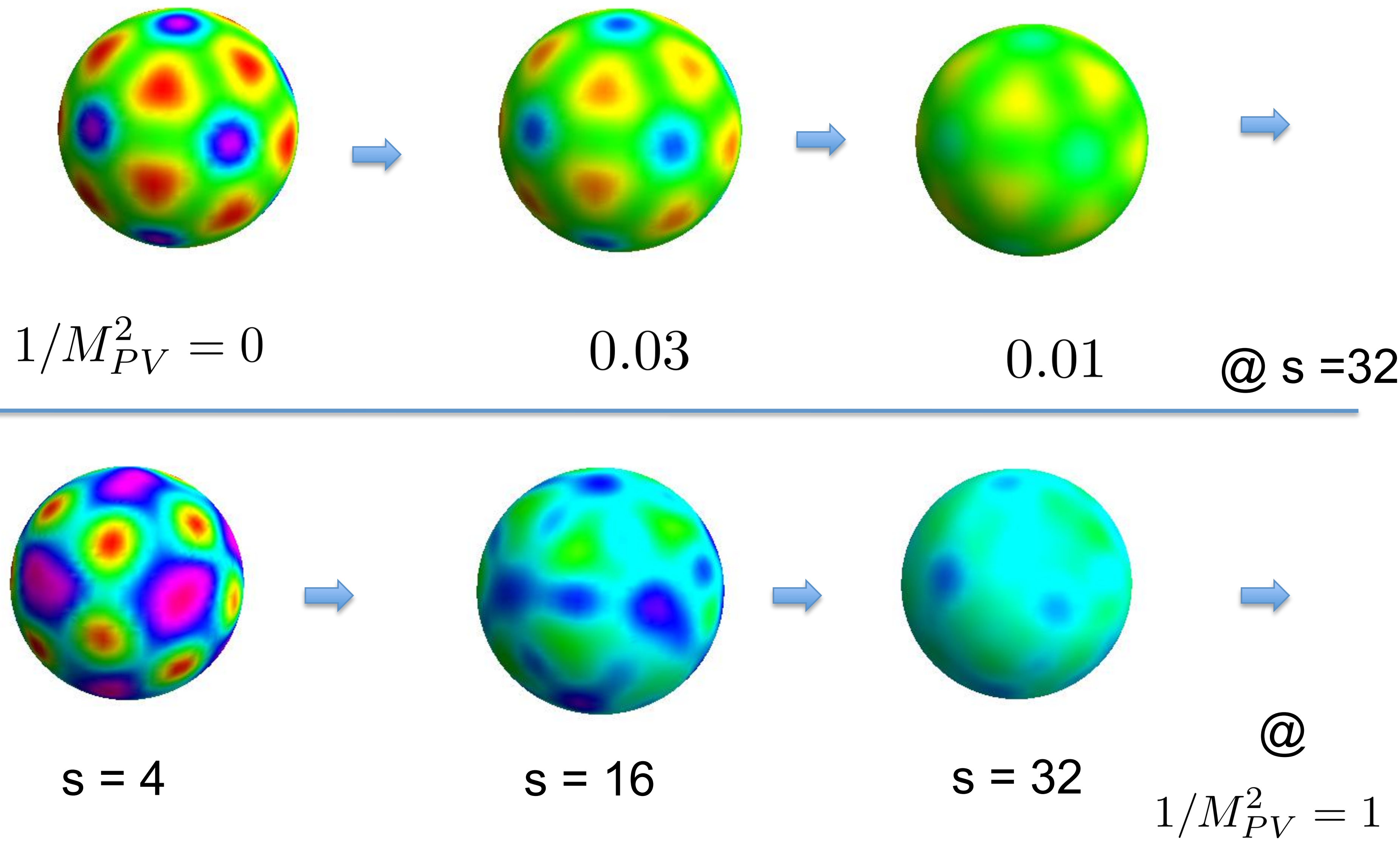
(ii) Better simplex distribution (Exact const curvature density)

*(*Richard Feynman, Ernst Stueckelberg)*

PV does improve symmetry



Impact of PV term on UV Divergent CT



Fermions and Gauge fields
"FEM" or discrete Exterior Calculus

Fermions on smooth manifold: $$"\bar{\psi} e_\mu (\partial_\mu - \omega_\mu)\psi"$$

$$e_\mu(x) = e_\mu^a \gamma_a$$

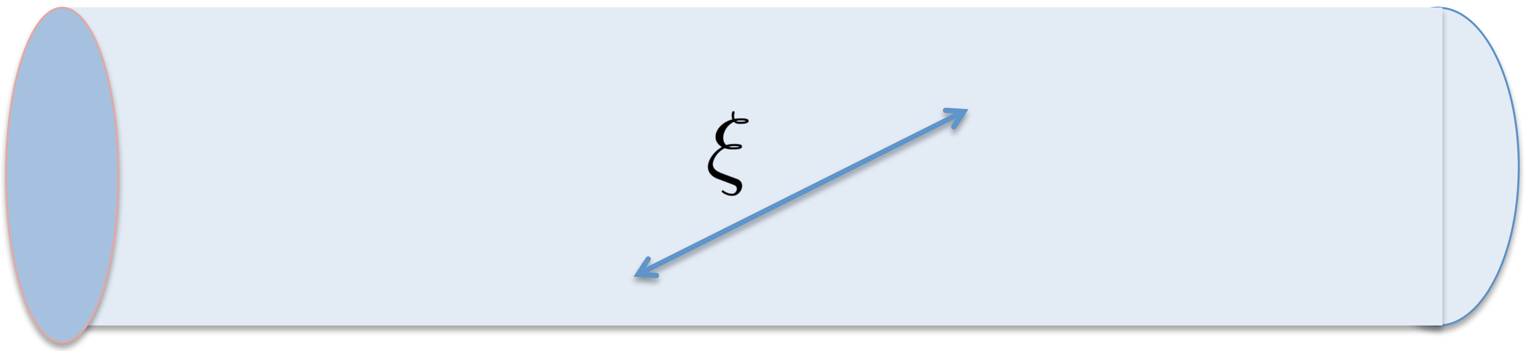
$$\omega_\mu(x) = \omega_\mu^{ab}\gamma_a\gamma_b \qquad g_{\mu\nu}(x) = e_\mu^a(x) e_\nu^a(x)$$

(1) Lattice Fermions are on simplicial curved manifolds must be done with great care: Spin connection does not always exist!

(2) Compact gauge links can be using Nedelic/Whitney "edge" elements the spirit of Christ, Friedberg, Lee

What about Radial Quantization of Yang Mills? (even QCD)?

$\mathbb{R} \times \mathbb{T}^3$ vs $\mathbb{R} \times \mathbb{S}^3$



1. Short distances: Asymptotically free OPE
2. In confining phase: Hamiltonian evolution!
3. In Conformal Window: Dilatation evolution!
4. Relevant mass deformation

(Spectral flow: Anamalous Dimension ➔ Energies!)

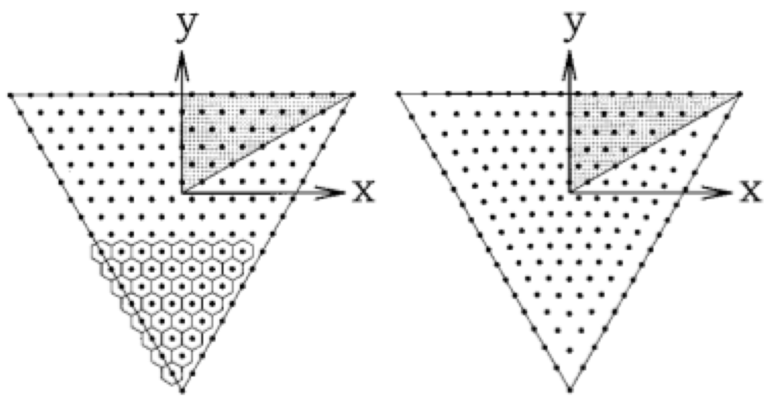
Michael we need your help!

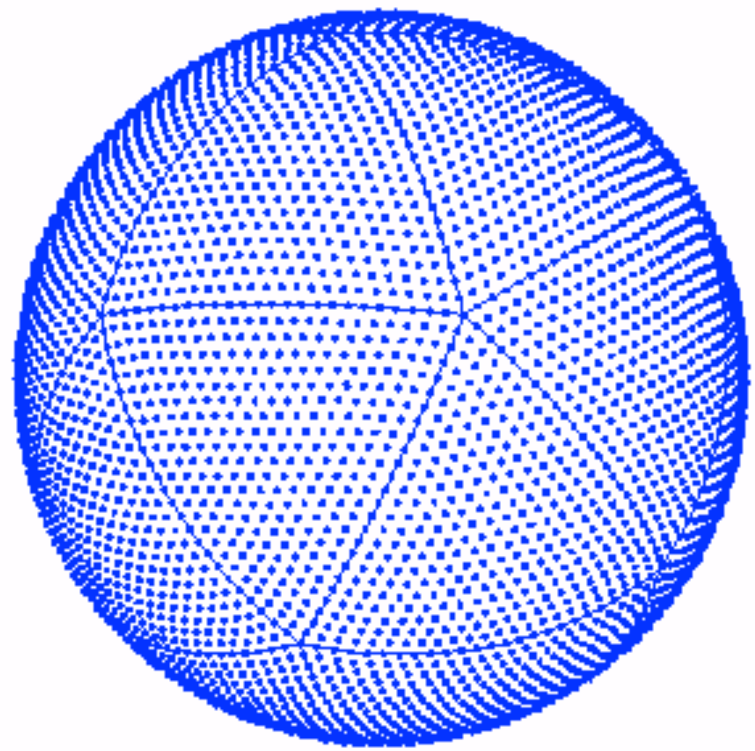
- This can also be viewed as generalizing graphene lattices!

Extra

Accurate Results require
Better Triangulation, Counter Terms etc

Equal AREA simplex
for WMAP Ski!





Graduate student help: Andrew Gasbarro & Timothy Raben\

An icosahedron-based method for pixelizing the celestial sphere
MAX TEGMARK THE ASTROPHYSICAL JOURNAL, 470:L81–L84, 1996 October 20

FUN IDENTITY

$$g(1,2,3,4) =$$

$$\frac{1}{2}\sqrt{\left[\frac{g(1,3)g(2,4)g(1,4)g(2,3)}{g(1,2)g(3,4)}\right]^2 + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)}$$

s-t t-u u-s

$$\sqrt{2 + 2\sqrt{(1-z)(1-\bar{z})} + 2\sqrt{z\bar{z}}} = |1 + \sqrt{1-z}| + |1 - \sqrt{1-z}|$$

Simplicial FEM Essentially equivalent to

- Regge Calculus T. Regge, Nuovo Cimento 19 (1961) 558.

- Random Lattices: N. H. Christ, R. Friedberg, and T. D. Lee, Nucl. Phys. B 202, 89 (1982).

- FEM: Discrete Exterior Calculus (de Rahm Complex, Whitney, etc, etc.), even ‘t Hooft