THE GRAVITATIONAL LENS EFFECT*

Sjur Refsdal

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Summary

The so-called gravitational lens effect, previously worked out by Tikhov in 1937, is derived in a simple manner. The effect is caused by the gravitational deflection of light from a star S in the gravitational field of another star B, and occurs when S lies far behind B, but close to the line of sight through B. It turns out that a considerable increase in the apparent luminosity of S is possible. A method is given to determine the mass of a star which acts as a gravitational lens. The possibility of observing the effect is discussed.

1. *Introduction.*—When a star S lies far behind and close enough to the line of sight through another star B, the light from S to the observer O can, due to the gravitational deflection of light, follow two different paths—both in the plane through S, B and O, and on opposite sides of B—corresponding to two "images", S_1 and S_2 of S (Fig. 1). Chwolson (1924) called attention to this phenomenon, but he did not make any calculations. In 1936 Einstein calculated the light intensity of the two "images", assuming the distance to S to be large compared to the distance to B. He found that the intensity of S_1 and S_2 could be much greater than the normal intensity of S, but concluded that the chance of observing the effect was too small to be of practical interest. Tikhov (1937) calculated the intensities in the general case, but his presentation is not easily followed. In the first section of the present paper we try to solve the problem more simply, and a method to determine the mass of a star which acts as a gravitational lens is developed. In the second section the probability of observing the effect is discussed. Due to progress in experimental technique we find, contrary to Einstein, that the effect may be of practical interest.

2. *The gravitational lens effect.*—We assume that the deflection of light is given by Einstein's expression, so that a ray of light passing B at a distance r is deflected towards B by an angle

$$v = 4G\mathscr{M}c^{-2}r^{-1} \equiv Kr^{-1} \tag{1}$$

where $\mathscr{M}$ is the mass of B, and G is the gravitational constant. In Fig. 1, S, B, O and the light rays from S to O, denoted by 1 and 2, are indicated. The distances to B and S are a_B and a_S respectively, and the distance from O to the extension of SB is x. D_1 and D_2 are the points where the light rays are closest

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to B, the distances being r_1 and r_2. The apparent light intensities of the two "images" of S are L_1 and L_2, respectively. Actually the rays are being deflected continuously when passing B, but for our purposes we can safely assume that the deflection occurs only in D_1 and D_2. For practical reasons we choose $r_1 > 0$ and $r_2 < 0$, and $x > 0$ to the right and $x < 0$ to the left of Fig. 1.

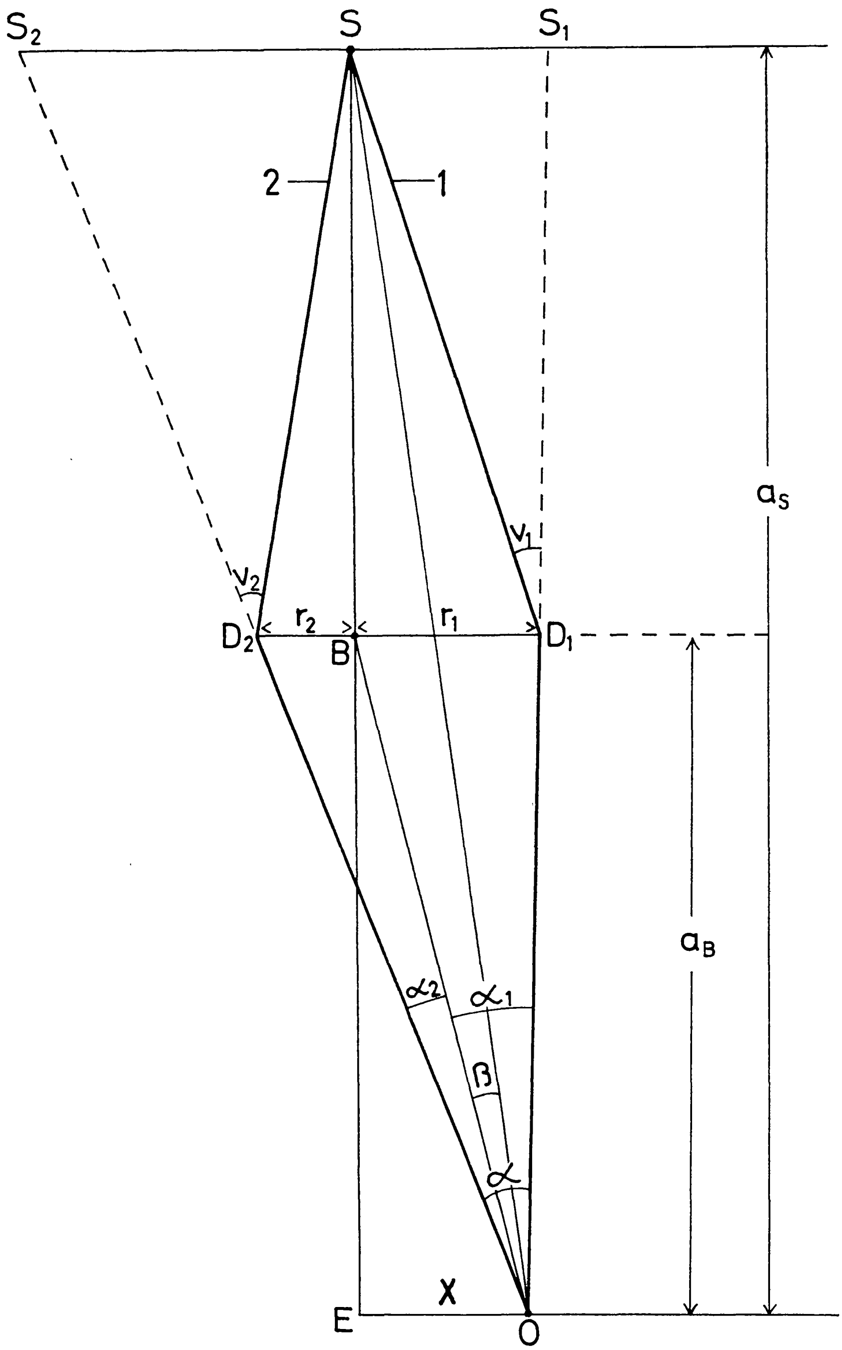

FIG. 1.—*The two light rays from S to O.*

We shall now calculate the intensity L_1 of S_1. In the plane P through B and in the plane P' through O—both planes being normal to SB—we introduce polar coordinates (r, ϕ) and (X, θ), respectively. The origin E in the (X, θ) system lies on the extension of SB and the origin in the (r, θ) system is B. We follow a bundle of light rays from S which delimits an area on the plane P bounded by the lines $r=r_1$, $r=r_1+dr_1$, $\phi=\phi_1$ and $\phi=\phi_1+d\phi_1$ (Fig. 2). The area of intersection is then $dA_P=r_1dr_1d\phi_1$. The same bundle will define an area

$$dA_{P'}=|XdXd\theta|$$

on the plane P'. If the light were not deflected, the area would have been

$$dA_N=n^2dA_P,$$

where $n=a_S/(a_S-a_B)$. SB is an axis of symmetry and consequently $d\theta=d\phi$.

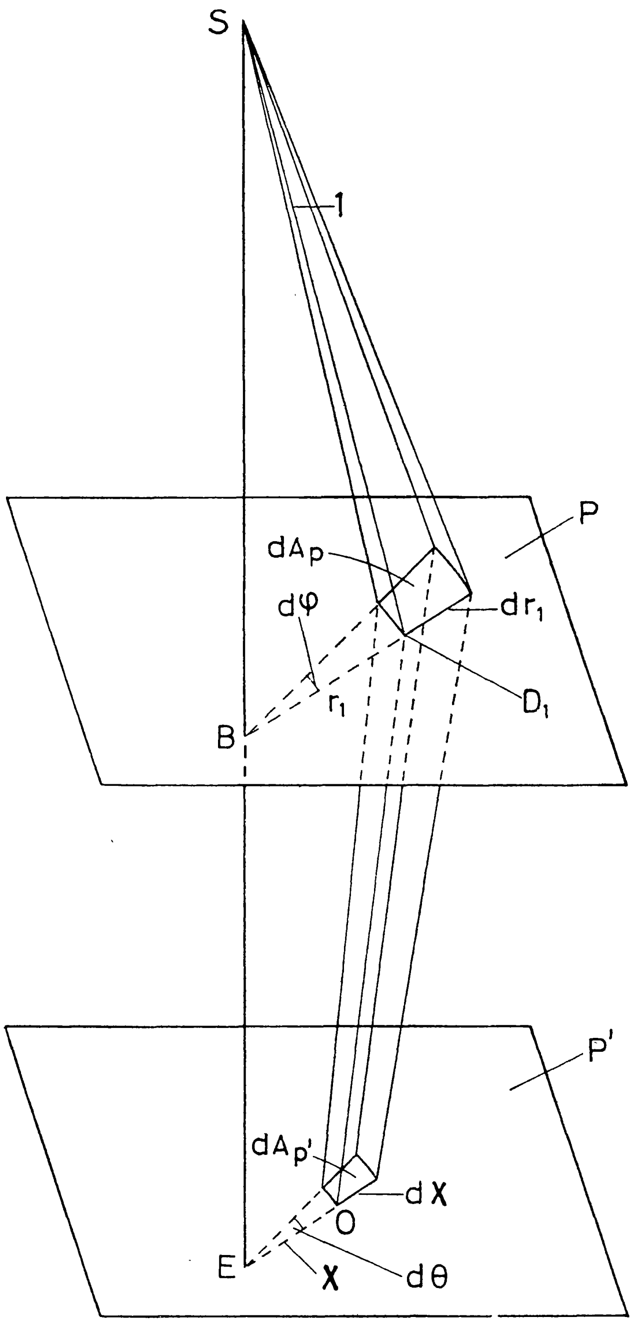

FIG. 2.—*A bundle of rays from S, passing close to O.*

From geometrical optics it is seen that

$$L_1 = \frac{dA_N}{dA_{P1}} L_N = n^2 \frac{r_1 dr_1}{XdX} L_N \tag{2}$$

where L_N is the normal intensity of S. Applying the usual approximations for small angles, we find from Fig. 1 and equation (1)

$$r^2 - Xn^{-1}r - Ka_B n^{-1} = r^2 - Xn^{-1}r - r_0^2 = 0, \tag{3}$$

where $r_0 = \sqrt{Ka_B n^{-1}}$ is the value of r_1 and $|r_2|$ when $X=0$. We then get,

$$r_1 = \frac{1}{2n}(X + \sqrt{X^2 + 4n^2 r_0^2}) \tag{4}$$

$$r_2 = \frac{1}{2n}(X - \sqrt{X^2 + 4n^2 r_0^2}). \tag{4 a}$$

Introducing $\beta = \angle SOB$, $\alpha = \angle D_1OD_2$, $\alpha_1 = \angle D_1OB$ and $\alpha_2 = \angle BOD_2$, we see from Fig. 1,

$$\alpha_1 + \alpha_2 = \alpha \tag{5}$$

$$X = na_B\beta. \tag{6}$$

From (4), (4 *a*) and (5) we find

$$\alpha_1 - \alpha_2 = \frac{r_1 + r_2}{a_B} = \frac{X}{na_B} = \beta \tag{7}$$

where α_2 is chosen positive and therefore $\alpha_2 = -r_2/a_B$. We then obtain

$$\sqrt{X^2 + 4n^2 r_0^2} = \sqrt{n^2 a_B^2 \beta^2 + n^2 a_B^2 \alpha_0^2} = na_B \sqrt{\beta^2 + \alpha_0^2}, \tag{8}$$

where α_0 is the value of α when $X=0$. We have

$$\alpha_0 = \frac{2r_0}{a_B} = 2\sqrt{\frac{K}{na_B}}. \tag{9}$$

From (4) and (4 *a*) it is seen that

$$r_1 - r_2 = n^{-1}\sqrt{X^2 + 4n^2 r_0^2}. \tag{10}$$

We also have, however,

$$r_1 - r_2 = a_B\alpha. \tag{11}$$

We then obtain from (8), (10) and (11)

$$\sqrt{X^2 + 4n^2 r_0^2} = na_B\alpha, \tag{12}$$

$$\alpha = \sqrt{\alpha_0^2 + \beta^2}. \tag{13}$$

Differentiating (4) with respect to X, we obtain

$$\frac{dr_1}{dX} = \frac{1}{2n}\left(1 + \frac{X}{\sqrt{X^2 + 4n^2 r_0^2}}\right). \tag{14}$$

$n(dr_1/dX)$ corresponds to ϕ_{rad} given by equation (7) in Tikhov's paper. From (4), (6), (12) and (14) we get,

$$r_1 = \frac{X}{2n}\left(1+\frac{\alpha}{\beta}\right) \tag{15}$$

and

$$\frac{dr_1}{dX} = \frac{1}{2n}\left(1+\frac{\beta}{\alpha}\right) = \frac{\alpha}{\beta X} r_1. \tag{16}$$

From (2) we then obtain

$$L_1 = \tfrac{1}{4}\left(2+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)L_N. \tag{17}$$

In a similar way we find

$$L_2 = \tfrac{1}{4}\left(-2+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)L_N. \tag{17 a}$$

(17) and (17 *a*) correspond to equations (21) and (22) in Tikhov's paper. The total intensity is given by

$$L_T \equiv L_1 + L_2 = \tfrac{1}{2}\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)L_N, \tag{18}$$

which is given for different values of β/α_0 in Table I. The difference in luminosity is given by

$$L_D \equiv L_1 - L_2 = L_N. \tag{19}$$

For $\beta \ll \alpha_0$ we get

$$L_T \approx \frac{\alpha_0}{2\beta} L_N. \tag{20}$$

From (2), (4) and (16), and the corresponding equations for the second ray, we get

$$\frac{L_1}{L_2} = \frac{r_1^2}{r_2^2} = \frac{\alpha_1^2}{\alpha_2^2}. \tag{21}$$

For $n=1$ this result has previously been derived by Metzner (1963).

We have so far regarded S and B as points. Denoting the radius of B by r_B we must require $r_1 > r_B$ and $|r_2| > r_B$, otherwise at least one of the rays from S will be absorbed or scattered. If the mass and radius of B equal those of the Sun, and $\beta < \alpha_0$, it is sufficient that a_B and $(a_S - a_B)$ are both $> 0{\cdot}01$ pc. Except for real double stars, this condition will usually be satisfied. Assuming B to be spherically symmetric, no more changes due to the extent of B are necessary. Correction terms for the extent of S must be introduced when β is less than, or of the same order of magnitude as the angular radius of S, $u = r_S/a_S$, r_S being the radius of S. Usually α_0 will be larger than u by several orders of magnitude. The total intensity is of the most interest for us, and we shall therefore only give the correction terms for L_T. The angle β no longer has a precise meaning, but it is natural to define it by $\beta = \angle C_S O C_B$ where C_S and C_B are the respective centres of S and B. Regarding S as a circular disk with constant surface brightness, we find by integration over the surface and power expansion,

$$\beta < u, \quad L_T = \frac{\alpha_0}{u}\left(1 - \tfrac{1}{4}\frac{\beta^2}{u^2} - \frac{3}{64}\frac{\beta^4}{u^4} - \ldots\right) L_N, \tag{22}$$

$$u < \beta < 5u, \quad L_T = \frac{\alpha_0}{2\beta}\left(1 + \tfrac{1}{8}\frac{u^2}{\beta^2} + \frac{3}{64}\frac{u^4}{\beta^4} + \ldots\right) L_N, \tag{23}$$

$$5u < \beta < \tfrac{1}{10}\alpha_0, \quad L_T = \frac{\alpha_0}{2\beta}\left(1 + \tfrac{1}{8}\frac{u^2}{\beta^2} + \tfrac{3}{2}\frac{\beta^2}{\alpha_0^2} + \ldots\right) L_N. \tag{24}$$

From (22) we see that L_T has a maximum when $\beta = 0$,

$$L_T(\max) \approx \frac{\alpha_0}{u} L_N. \tag{25}$$

We shall give an example:

Let $a_S = 100$ pc, $a_B = 10$ pc, $\mathcal{M} = \mathcal{M}_\odot$, $r_S = r_\odot$. We then obtain $\alpha_0 = 5{\cdot}5 \times 10^{-2}{}''$ and $u = 5{\cdot}7 \times 10^{-5}{}''$, hence

$$L_T(\max) \approx 1100 L_N. \tag{26}$$

We see from consideration of symmetry that when $\beta = 0$, the "image" of S will be a circular ring with centre C_B and an angular diameter α_0. It can be shown that the angular thickness of the ring is u. For $u > \beta > 0$ the "image" of S will be a ring, similar to that for $\beta = 0$, but with variable thickness. For $\beta > u$ two separate "images" appear, corresponding to S_1 and S_2 (Fig. 3). The surface brightness is of course constant, and equal to the normal surface brightness of S. The increase in luminosity is caused by the increase in solid angle covered by the "images".

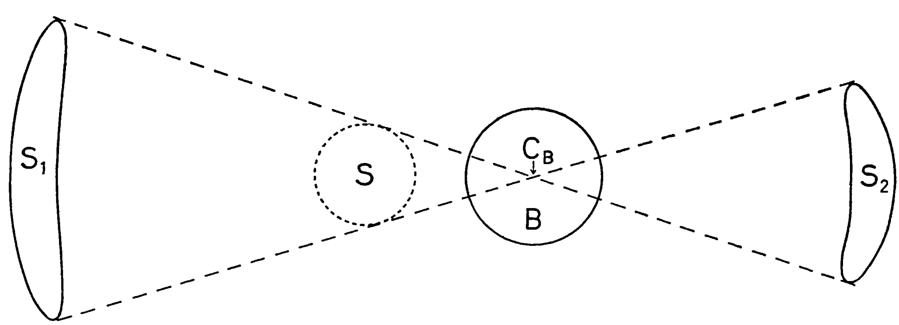

FIG. 3.—*The shape and position of S_1 and S_2.*

Assuming a_B and n to be known, we see from (9) that K, and consequently $\mathcal{M}$, can be determined if α_0 is known. α_0 can be determined in two different ways. The first one is based on measurement of α and L_T/L_N. Neglecting the correction terms in L_T, we see from (18) that L_T/L_N determines α/β, which together with α determines α_0. From Table I we see that if L_2 shall not be too small to be observable, we must have $\beta \gtrsim \alpha_0$. However, α_0 will in the most favourable cases be of the order of $0{\cdot}1''$, and therefore a precise measurement

TABLE I

β/α_0	L_1/L_N	L_2/L_N	L_T/L_N
10	1·0000063	0·0000063	1·000013
5	1·0001	0·0001	1·0002
3	1·0014	0·0014	1·0028
1·5	1·0084	0·0084	1·017
1	1·030	0·030	1·06
0·6	1·116	0·116	1·23
0·4	1·27	0·27	1·54
0·3	1·44	0·44	1·88
0·2	1·83	0·83	2·66
0·15	2·23	1·23	3·46
0·1	3·04	2·04	5·08
0·05	5·52	4·52	10·0
0·01	25·5	24·5	50·0

of α seems difficult at present. The other method is based on a determination of the time dependence of L_T during a passage. Assuming S, B and O to have constant velocities, β will depend on the time in the following manner,

$$\beta = \beta(t) = \sqrt{\beta^2(O) + \mu^2 t^2} \tag{27}$$

where t is put equal to zero when β has its minimum, and μ is the relative angular velocity of B relative to S, as seen from O. We see from (13) and (18) that L_T/L_N determines β/α_0. During the passage we can only expect to be able to measure the total incoming light from S and B, $L_T + L_B$, where L_B is the luminosity of B. μ, L_B and L_N can be determined when S and B can again be optically separated some time after the passage, or they may have been determined by observations before the passage. From measurements during the passage of

$$(L_T(t) + L_B)$$

we can then calculate $L_T(t)/L_N$, and also

$$\beta(t)/\alpha_0 = \frac{1}{\alpha_0}\sqrt{\beta^2(O) + \mu^2 t^2}.$$

Taking the ratio of this quantity at $t = 0$ and $t = t$ we obtain

$$g(t) = \frac{\beta(O)}{\sqrt{\beta^2(O) + \mu^2 t^2}}. \tag{28}$$

The value of $g(t)$ for one value of t different from zero is sufficient to determine $\beta(O)$. In practice, however, more than one value of t will be used and $\beta(O)$ will be chosen as some average of the values thus obtained. From $\beta(O)$ and $\beta(O)/\alpha_0$ we then find α_0 which, as seen before, determines $\mathscr{M}$ when a_B and n are known.

We have assumed that the deflection of light is given by (1). We can assume more generally that the deflection is given by

$$v = \frac{4kG\mathscr{M}}{c^2 r^\omega} \tag{29}$$

where k and ω are numbers. For theoretical reasons ω is usually put equal to unity. If $\mathscr{M}$ is known, we can then find k and compare it to the Einstein value, $k = 1$.

One should note, however, that a wide range of values of ω is possible from the present experimental data, mainly because the range of possible values of r has been too small, from $2r_{\odot}$ and up to about $8r_{\odot}$. But in our case, r will be of the order of $100\,r_{\odot}$ or more, and a much better determination of ω should be possible.

From theory it is reasonable to believe that the deflection of light is independent of frequency. However, this independence has not been tested by experiment with any accuracy. This should now be possible, because L_T/L_N will depend on the frequency if there is such a dependence.

We will now briefly discuss some objections that may be raised against the use of geometrical optics in the present problem. The following objections seem to be the most important ones:

1. The two light rays may interfere.
2. Equation (25) is evidently false when $u \to 0$.

To the first objection we can make the following remarks. Let dS be a surface element on S and $\gamma = \angle dSOC_B$. For $\gamma = 0$ it is seen from considerations of symmetry that the time of light travel from dS to O will be the same for all possible paths. We then see that, for $\gamma \neq 0$, the difference in travel time Δt, for the two possible paths, must be

$$\left.\begin{aligned} \Delta t &= c^{-1}\int_0^X \alpha\, dX = na_B c^{-1}\int_0^\gamma \alpha\, d\gamma \\ &\approx na_B\alpha\gamma c^{-1}\left(1 - \tfrac{1}{3}\frac{\gamma^2}{\alpha^2}\right) \approx na_B\alpha_0\gamma c^{-1}\left(1 + \tfrac{1}{6}\frac{\gamma^2}{\alpha_0^2}\right) \approx na_B\alpha_0\gamma c^{-1} \end{aligned}\right\} \tag{30}$$

where (6) and (13) have been used, and β replaced by γ. Choosing as before

$$a_S = 100\,\text{pc}, \quad a_B = 10\,\text{pc}, \quad \mathscr{M} = \mathscr{M}_{\odot}$$

and $r_S = r_{\odot}$, we obtain

$$c\Delta t = 10^{11}\gamma, \; m = 2\cdot 10^{17}\gamma\lambda \tag{31}$$

where we have put the wavelength of the light, λ, equal to $5\cdot 10^{-7}$ m. If it had been possible to observe only the light from dS, an interference effect would have been observed for monochromatic light. Constructive interference would have occurred for $c\Delta t = j\lambda$, and destructive interference for $c\Delta t = (j+\frac{1}{2})\lambda$, j being an integer. We note, however, that even in this ideal case the interference will gradually diminish as Δt increases, and for $c\Delta t \approx 1$ m, the interference effect will disappear, because the length of the optical wave trains is about 1 m. From the fact that the change in $c\Delta t$ is about $10^8\lambda = 50$ m when γ changes by $2u \approx 5\cdot 10^{-10}$, and that the light from S is far from being monochromatic, it is evident that all interference effects must be erased.

To counter the second objection we can make the following remarks. By using physical optics and the same numerical values as before, one can show that for a point source and $\beta = 0$

$$L_T \approx 10^{12} L_N. \tag{32}$$

A comparison with (20) shows that this corresponds to $u \approx 3\cdot 10^{-19}$. For $u \gg 10^{-18}$ we can then safely use (20).

3. *Possibility of observing the effect.*—We will now calculate the expected number of passages per year, with regard, of course, to the type of passage. We have noted earlier that the easiest quantity to measure during a passage is $L_T(t)+L_B$. Hence, an important quantity is the growth number F defined by

$$L_T(0)+L_B=F(L_N+L_B). \tag{33}$$

In order that a passage shall be an F passage, $\beta(O)$ must be equal to $h(\Delta m, F)\alpha_0$, where Δm is the difference between the natural apparent magnitude of S and that of B, $\Delta m = m_S - m_B$. In Table II h is tabulated for $F=1\cdot5$, 5 and 20, and for $|\Delta m| \leqslant 2$. By a passage stronger than F we mean a passage for which the growth number is larger than F. A passage will be stronger than F if

$$\beta(O) < h(\Delta m, F)\alpha_0.$$

TABLE II

Values of $h(\Delta m, F)$.

Δm	$F=1\cdot5$	$F=5$	$F=20$
−2	0·38	0·18	0·043
−1	0·35	0·15	0·036
0	0·28	0·11	0·025
1	0·19	0·066	0·015
2	0·11	0·033	0·007

The average number of stars per unit volume in our neighbourhood with absolute visual magnitude from $M-1/2$ to $M+1/2$ we denote by $\Phi(M)$, and this is given by Allen (1963), who also gives the average number of stars per square degree with apparent visual magnitude from $m-1/2$ to $m+1/2$, which we denote by A_m. We will assign the absolute magnitude M to all stars with absolute magnitude from $M-1/2$ to $M+1/2$, and the apparent magnitude m to all stars with apparent magnitude from $m-1/2$ to $m+1/2$. The expected mass, $\bar{\mathscr{M}}$, of a star with an absolute bolometric luminosity $\mathscr{L}$, we find from the mass luminosity law as given by Allen,

$$\log\frac{\mathscr{L}}{\mathscr{L}_\odot} = 3\cdot3 \log\frac{\bar{\mathscr{M}}}{\mathscr{M}_\odot}. \tag{34}$$

For simplicity we assume that (34) is also valid if $\mathscr{L}$ is the absolute visual luminosity.

For the majority of passages in which we are interested, $a_S \gg a_B$, and the angular velocity of S can to a good approximation be put equal to zero. The mean secular parallax for stars is 4·2 times the annual parallax (Allen 1963). Taking account of the contribution from the motion of the Earth around the Sun, the mean value of μ will approximately be $\bar{\mu}(a_B)=10/a_B$, where a_B is given in parsecs and $\bar{\mu}$ in sec of arc per year.

We divide space into distance intervals: 7·4 pc–11·8 pc, mean distance 10 pc, 11·8 pc–18·6 pc, mean distance 16 pc, and so forth. The mean distance divides the volume of each interval into two equal parts. The ratio between the mean distances for two neighbouring intervals is $\sqrt[5]{10} \approx 1\cdot6$, corresponding to a change of one unit in the apparent magnitude of a star. The interval with mean distance a we denote the a interval. We assign the distance a to all stars in the a interval.

TABLE III

Mean distance $=a$ (pc)	Distance interval (pc)	Volume (10^4 pc^3)	$\bar{\mu}$ (sec of arc per year)	m_v	11	12	13	14	15	16	17	18	19	20
	7·4			$N(10, m)$	47	58	65	70	65	50	45	39	35	30
10		0·51	1	$1000\bar{\alpha}_0(10, m)$	23	20	17	15	13	11	10	8·7	7·7	6·5
	11·8			$H(10, m)$	1·1	1·2	1·2	1·1	0·85	0·55	0·45	0·33	0·25	0·2
				$V(10, m)$	1·1	1·2	1·2	1·1	0·85	0·55	0·45	0·33	0·25	0·2
	11·8			$N(16, m)$	160	190	230	260	280	260	200	180	160	140
16		2·0	0·63	$1000\bar{\alpha}_0(16, m)$	21	18	16	14	12	10	9·1	7·9	6·8	5·9
	18·6			$H(16, m)$	2	2·2	2·3	2·3	2·1	1·7	1·1	0·9	0·65	0·5
				$V(16, m)$	3·1	3·4	3·5	3·4	3·0	2·3	1·6	1·2	0·9	0·7
	18·6			$10^{-1}\times N(25, m)$	44	62	74	92	100	110	100	80	72	62
25		8·0	0·4	$1000\bar{\alpha}_0(25, m)$	19	16	14	12	11	9·4	8·3	7·2	6·2	5·4
	29·4			$H(25, m)$	3·5	4	4·3	4·7	4·5	4·2	3·5	2·3	1·8	1·3
				$V(25, m)$	6·6	7·4	7·8	8·1	7·5	6·5	5·1	3·5	2·7	2·0
	29·4			$10^{-1}\times N(40, m)$	140	170	250	300	370	410	450	410	320	290
40		32	0·25	$1000\bar{\alpha}_0(40, m)$	17	15	13	11	9·8	8·6	7·6	6·6	5·6	4·9
	46·7			$H(40, m)$	6·5	7	8	8·5	9·5	9·0	8·5	7·0	4·5	3·5
				$V(40, m)$	13	14	16	17	17	16	14	11	7·2	5·5
	46·7			$10^{-2}\times N(63, m)$	50	57	69	98	120	150	160	180	160	130
63		128	0·16	$1000\bar{\alpha}_0(63, m)$	16	14	12	10	9·0	7·9	6·9	6·0	5·1	4·5
	74			$H(63, m)$	12	13	14	16	17	19	18	17	14	9·0
				$V(63, m)$	25	27	30	33	34	35	32	28	21	14
	74			$10^{-3}\times N(100, m)$	18	20	23	28	39	47	58	65	70	65
100		510	0·1	$1000\bar{\alpha}_0(1000, m)$	14	12	11	10	8·2	7·2	6·3	5·5	4·7	4·1
	118			$H(100, m)$	25	24	25	27	32	34	36	36	33	27
				$V(100, m)$	50	51	55	60	66	69	69	64	54	41
	118			$10^{-3}\times N(160, m)$	60	72	80	90	110	160	190	230	260	280
160		2000	0·063	$1000\bar{\alpha}_0(160, m)$	13	11	10	8·7	7·5	6·6	5·7	5·0	4·3	3·7
	186			$H(160, m)$	52	50	48	50	54	64	68	74	72	66
				$V(160, m)$	100	100	100	110	120	130	140	140	130	110
	186			$10^{-4}\times N(250, m)$	17	24	29	32	36	44	62	74	92	100
250		8000	0·04	$1000\bar{\alpha}_0(250, m)$	12	10	9·1	7·9	6·8	6·0	5·2	4·6	3·9	3·4
	274			$H(250, m)$	84	100	100	96	100	110	130	140	150	140
				$V(250, m)$	180	200	200	210	220	240	270	280	280	250

Values of quantities used in the calculation of $P(a, m, F)$.

The number of stars in the a interval with apparent magnitude m is denoted $N(a, m)$, and is easily found from $\phi(M)$. Assuming $a_S \gg a_B$, it is seen from (9) and (34) that the expected value of α_0 depends only on m and the interval a in which B is situated; we denote it $\bar{\alpha}_0(a, m)$. In Table III $N(a, m)$ and $\bar{\alpha}_0(a, m)$ are given for different values of a and m. On an average we have $a_S \approx 10a_B$, giving $n = 1 \cdot 1$, which has been used in the tabulation of $\bar{\alpha}_0$.

The expected number of passages per year stronger than F, the nearest star having an apparent magnitude, m, and lying in the a interval, and the distant star having a natural apparent magnitude $m + \Delta m$, is then

$$\left.\begin{aligned} p(a, m, \Delta m, F) &= N(a, m) \times \bar{\alpha}_0(a, m) \times \bar{\mu}(a) \\ &\qquad \times 2h(\Delta m, F) \times A_{m+\Delta m} \times 60^{-4} \\ &= H(a, m) \times 2h(\Delta m, F) \times A_{m+\Delta m} \times 60^{-4}, \end{aligned}\right\} \tag{35}$$

α_0 is given in sec of arc, and

$$H(a, m) \equiv N(a, m) \times \bar{\alpha}_0(a, m) \times \bar{\mu}(a).$$

We introduce
$$V(a, m) = \sum_{a'=10}^{a'=a} H(a', m).$$

In Table III H and V are given for different values of a and m.

We now consider passages that satisfy the following conditions:

1. The passage is stronger than F.
2. The nearest star lies in the a interval or nearer.
3. The apparent magnitude of the nearest star is m or smaller.
4. $|\Delta m| \leqslant 2$.

The expected number of passages per year is then

$$P(a, m, F) = \sum_{i=i_0}^{i=m} \sum_{\Delta m=-2}^{\Delta m=2} V(a, i) \times 2h(\Delta m, F) \times A_{i+\Delta m} \times 60^{-4}. \tag{36}$$

For $m \geqslant 14$, we can safely choose $i_0 = 11$, because the contribution to P from lower values of i is very small. In Table IV P is given for different values of a, m, and F. The contribution from stars nearer than the 10 interval has been neglected, because they are too few to be treated statistically. It is easily shown, however,

TABLE IV

Values of $P(a, m, F)$, the expected number of passages per year.

	$a \backslash m$	14	15	16	17	18	19
	40	0·0028	0·0052	0·011	0·02	0·033	0·05
$F = 1·5$	100	0·068	0·018	0·043	0·088	0·17	0·28
	250	0·027	0·066	0·15	0·33	0·66	1·2
	40	0·00074	0·0018	0·0038	0·0072	0·012	0·018
$F = 5$	100	0·0018	0·006	0·015	0·032	0·06	0·10
	250	0·0094	0·026	0·054	0·12	0·24	0·46
	40	0·00017	0·00042	0·00088	0·0016	0·0028	0·004
$F = 20$	100	0·00062	0·0015	0·0036	0·0074	0·014	0·024
	250	0·0022	0·0052	0·012	0·028	0·056	0·10

that the chance for a passage to occur among these stars is very small. For $F > 10$, P will be approximately proportional to F^{-1}, as far as S can be approximated by a point so that equation (20) can be used for the calculation of $h(\Delta m, F)$.

Of special interest to us are the white dwarfs, because they have larger masses than indicated in equation (34). Very few white dwarf masses are known, and passages where the nearest star is a white dwarf will thus be of great importance. In the future observations from places outside the Earth will be possible to perform. As an example we will estimate the expected number of passages per year within a distance from the Sun equal to five times our own distance to the Sun. The procedure will be as before, but in equation (35) $2h(\Delta m, F) \times \bar{\alpha}_0$ has to be replaced by $10/a$, giving

$$p'(a, m, \Delta m) = N(a, m) \times \bar{\mu}(a) \times 10a^{-1} \times A_{m+\Delta m} \times 60^{-4}. \tag{37}$$

The expected number of passages will increase by a factor between 10 and 100 as compared to the number of passages observable from the Earth, and we note that p' is independent of F. The accuracy in angular measurements required to "find" an F passage is $h(\Delta m, F) \times \alpha_0$.

To get an idea of the duration of a passage, we calculate the expected time interval T for which $\beta < \sqrt{2} \times \beta(O)$. We easily get

$$T = 2\beta(O) \times \mu^{-1} = 2\alpha_0 \times h \times \mu^{-1}. \tag{38}$$

Choosing $\Delta m = 0$, $F = 1{\cdot}5$ and $m = 14$, and taking the expected values of α_0 and μ, we obtain $T = 0{\cdot}005\sqrt{a}$ years. For $a = 100$ pc, we get $T = 20$ days.

We have so far assumed S and B to be single stars. Actually about one-third of all stars are double or multiple systems, so that for about 50 per cent of all passages at least one of the stars will be double or multiple, and the description of the phenomenon will be more complicated.

4. *Conclusion.*—It seems safe to conclude that passages observable from the Earth occur rather frequently. The problem is to find where and when the passages take place. By comparing photographs of the sky taken at different times, the angular velocity of a great number of stars can be determined, and passages may be predicted.

After this paper was submitted for publication, a paper by S. Liebes appeared in which similar problems are discussed (Liebes 1964).

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Institute of Theoretical Physics,
University of Oslo, Blindern,
Norway:

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References

C. W. Allen, 1963, *Astrophysical Quantities*, The Athlone Press, London.
O. Chwolson, 1924, *Astr. Nachr.*, **221**, 329–330.
A. Einstein, 1936, *Science*, **84**, 506.
S. Liebes, 1964, *Phys. Rev.*, **133**, 835.
A. W. K. Metzner, 1963, *J. Math. Phys.*, **4**, 1194–1205.
G. A. Tikhov, 1937, *Dokl. Akad. Nauk. S.S.S.R.*, **16**, 199–204.