(HYPER)GRAPH THEORY AND (HOLOGRAPHIC) ENTANGLEMENT ENTROPY

NING BAO, UC BERKELEY/BROOKHAVEN NATIONAL LAB

BASED ON WORK WITH OOGURI, NEZAMI, STOICA, SULLY, WALTER

AND WORK WITH CHENG, HERNANDEZ-CUENCA, SU

WHAT I DO

- I work at the interface of quantum information theory and high energy physics.
- Interested in how these topics inform each other
- Really mean both ways, not just QI for HEP, but also how lessons from QI for HEP can help guide pure QI

QUANTUM INFORMATION BACKGROUND (PICK 3)

- Entanglement Entropies
- Tensor Networks
- Condensed Matter Physics
- Entanglement of Purification
- Quantum Circuit Models
- Quantum Complexity
- Holevo Bounds
- Quantum Random Walks
- Quantum Algorithms
- Eigenstate Thermalization
- Universality Proofs
- Query Complexity
- Decoherence Effects
- Quantum Error Correction
- Quantum Channels
- Topological Quantum Field Theory

“

HAVE YOU GOTTEN AROUND TO HOLOGRAPHIZING THAT CHAPTER OF NIELSEN AND CHUANG YET?

”

Uncertain, possibly John Preskill, at Preskill group meeting, circa 2016

ENTANGLEMENT

- Entanglement is a precious resource in quantum computing
- What makes quantum computing “quantum”
- Source of many of the speed-ups in quantum algorithms
- Quantifying the amount of this resource is critical

ENTANGLEMENT ENTROPY

- Entanglement entropy is one way of quantifying the amount of entanglement
- For a subsystem of a pure state, defines how many bell pairs can be distilled between the subsystem and its purification to be used as a "quantum resource".
- Zero for pure states, nonzero for mixed states
- Problem: fundamentally difficult to compute for large systems

$$S(A) = -Tr(\rho \log \rho)$$

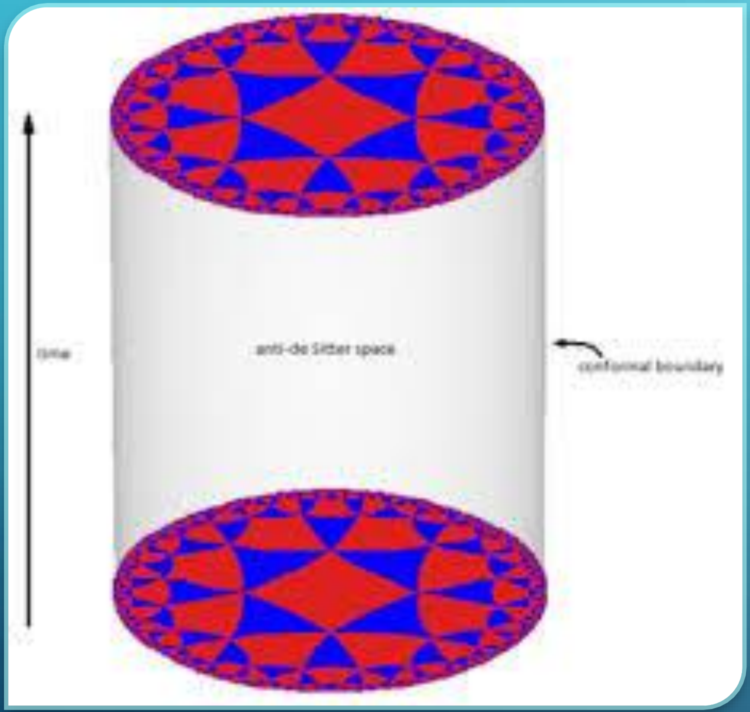
ENTANGLMENT ENTROPY INEQUALITIES

- New approach: constrain what entanglement entropies can be
- All quantum states obey strong subadditivity (and permutations thereof):

$$S(AB) + S(BC) \geq S(B) + S(ABC)$$

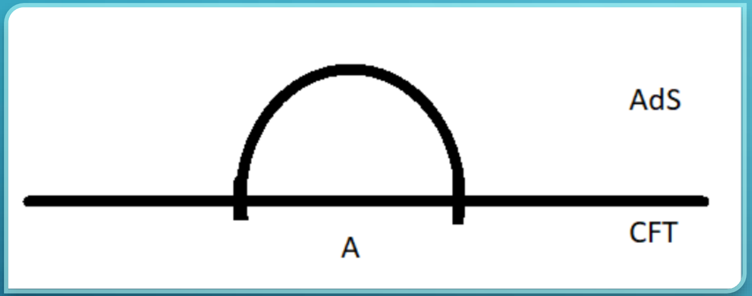
- Original proof of this ~50 pages long, by Lieb and Ruskai (1973)
- Inequalities of 4 or more parties of this type for all quantum states are needed, but not known (almost fifty years now)
- In special quantum states, however, this problem is more tractable

ADS/CFT IN ONE SLIDE



- Duality between a d dimensional conformal field theory (boundary) and d+1 dimensional gravity theory with negative cosmological constant (bulk)
- Strongly coupled boundary theory dual to weakly coupled (semiclassical) bulk theory
- States dual to semiclassical gravity bulks are called “holographic states”
- Takeaway: can learn things at strong coupling about quantum systems by doing classical general relativity

RYU TAKAYANAGI FORMULA



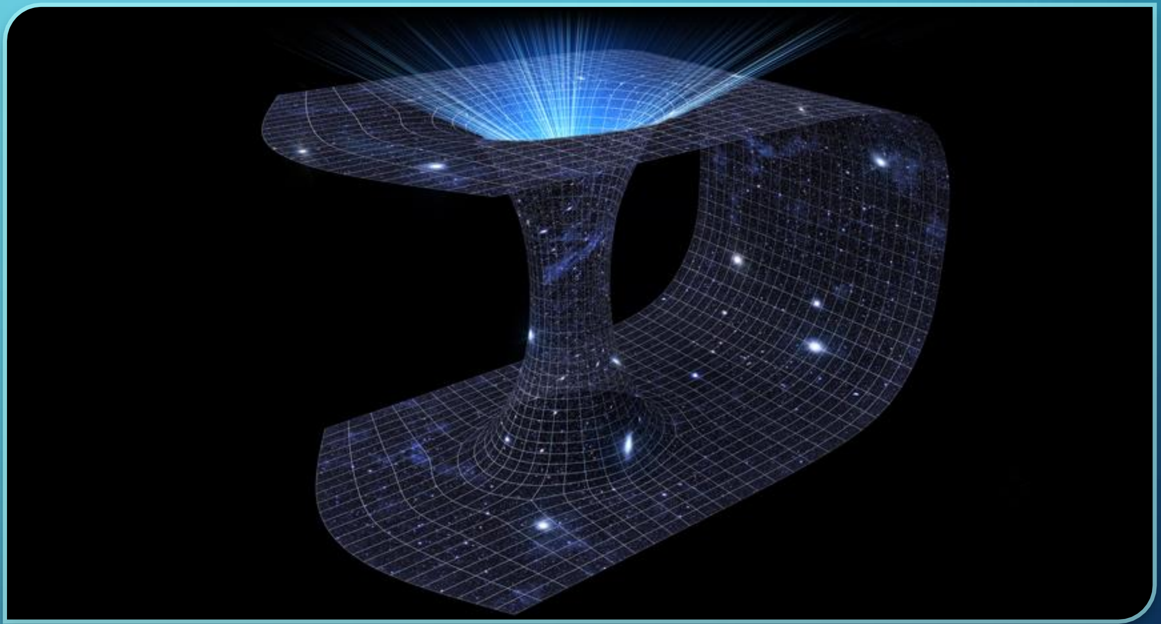
- Holographic formula for entanglement entropy
- Boundary entanglement of a region equal to area of the bulk minimal surface homologous to it when the bulk theory is close to classical gravity:

$$S(A) = \frac{Area(A)}{4G}$$

- Genuine quantum gravitational fact!

MOTIVATION: SPACETIME EMERGENCE

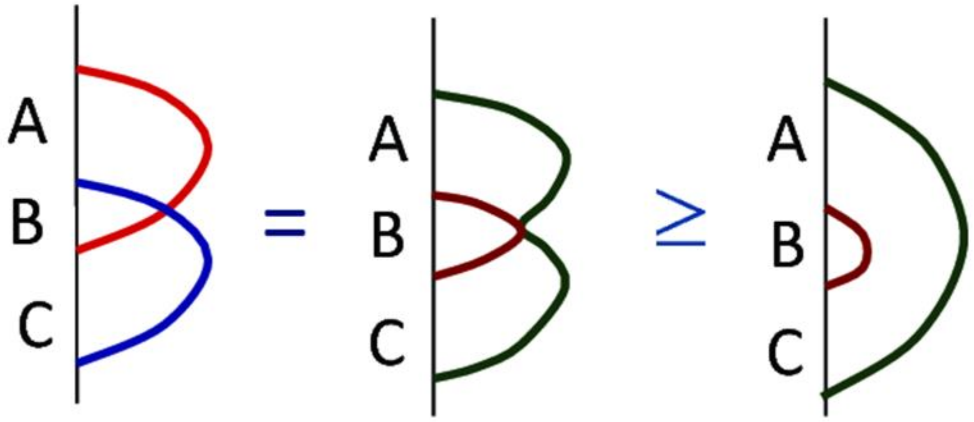
- CFT entanglement $\leftrightarrow$ Geometry
- Is spacetime emerging from entanglement?
- ER=EPR?
- It from Qubit?



MORE ABOUT RYU-TAKAYANAGI

- Dramatically simplifies entanglement entropy calculations
- Converts spectral decomposition into geometry problem
- Known entanglement entropy inequalities very easy to prove in this regime

HOLOGRAPHIC PROOF OF STRONG SUBADDITIVITY



$$\Rightarrow \quad S_{A\cup B} + S_{B\cup C} \geq S_{A\cup B\cup C} + S_B$$

HOLOGRAPHIC ENTANGLEMENT ENTROPY INEQUALITIES

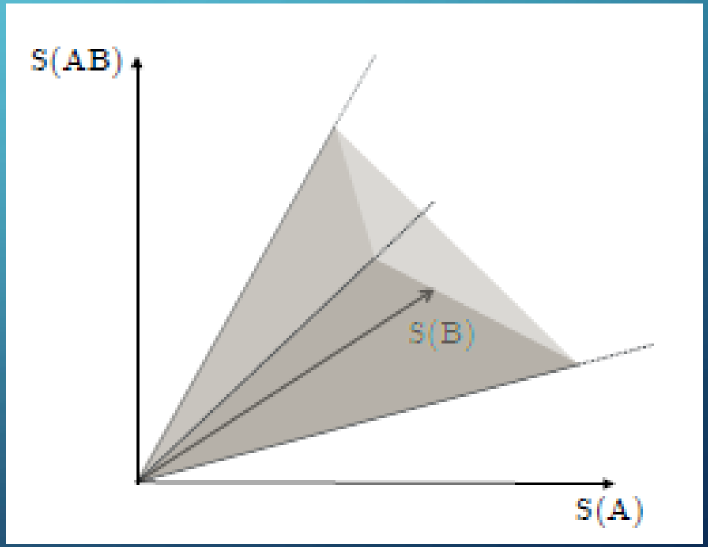
- RT allows us to study holographic entanglement entropy inequalities with far greater proving power
- First such inequality: MMI (HHM)

$$S(AB) + S(BC) + S(AB) \geq S(A) + S(B) + S(C) + S(ABC)$$

- Notably, this inequality is violated by the 4 party GHZ and W states
- Are there more such inequalities?

ENTROPY CONES

- An entropy cone is a cone in entropy space whose interior contains all entropy vectors consistent with the type of state considered
- Examples:
  - Quantum (known up to 3 parties)
  - Classical (known up to 3 parties)
  - Stabilizer (known up to 4 parties)



HOLOGRAPHIC ENTROPY CONE

- Need to confirm that the entropy cone is "tight"
- True bounding inequalities mean that the cone is "large enough"
- Ability to reconstruct the extremal entropy vectors with holographic states/geometries means cone is "small enough"
- Two things needed:
  - Inequality proof method
  - Extremal ray construction method

INEQUALITY PROOF METHOD

- Geometric inclusion/exclusion method can work, but becomes cumbersome at higher party number
- This can, however, be converted into an algorithmic process by equating inclusion/exclusion to a contraction map
- Theorem: If the hypercube constructed by inclusion/exclusion of entropies on the LHS of an entropy inequality has the property that, given the single party initial specifications, any two points on this hypercube map into two points on the RHS equivalently defined hypercube that are closer in Hamming distance, then the inequality is true for all holographic states.

	x			$y = f(x)$			
	AB	BC	AC	A	B	C	ABC
O	0	0	0	0	0	0	0
	0	0	1	0	0	0	1
	0	1	0	0	0	0	1
C	0	1	1	0	0	1	1
	1	0	0	0	0	0	1
A	1	0	1	1	0	0	1
B	1	1	0	0	1	0	1
	1	1	1	0	0	0	1

Table 2. Proof by contraction of monogamy of the holographic mutual information.

EXTREMAL RAY CONSTRUCTION METHOD

- Can dessicate bulk geometries to graphs with edges/edge weights given by throats/throat sizes
- Converts matching of entropy vector to matching min cuts on a complete graph with undetermined edge weights
- Deals with divergences neatly
- Naturally multiboundary wormholes
- NOTE: as far as I can tell these graphs are also not equivalent to the graph states of Eisert et. al.

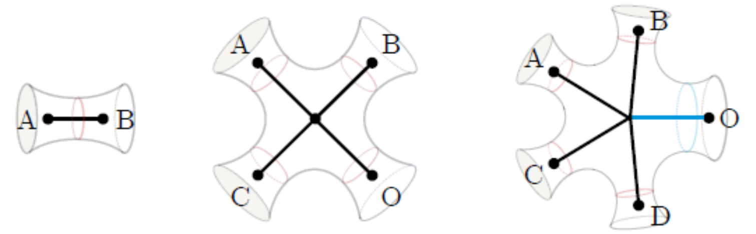


Figure 5. Graph models of the extreme rays of the holographic entropy cone for $n \leq 4$ regions. All black edges have the same weight; the blue edge has twice that weight.

NOW WE'RE IN BUSINESS…

- Define a cone bounded by true inequalities
- Find nonconstructible extreme rays (ray/face duality)
- Search for new inequalities that slice off at least one nonconstructible extreme ray
- Use contraction map proof method to determine inequality correctness
- Once correct inequality found, slice the cone with the new inequality
- Repeat until no nonconstructible extreme rays remain

HOLOGRAPHIC ENTANGLEMENT ENTROPY INEQUALITIES

- Recall MMI:

$$S(AB) + S(BC) + S(AB) \geq S(A) + S(B) + S(C) + S(ABC)$$

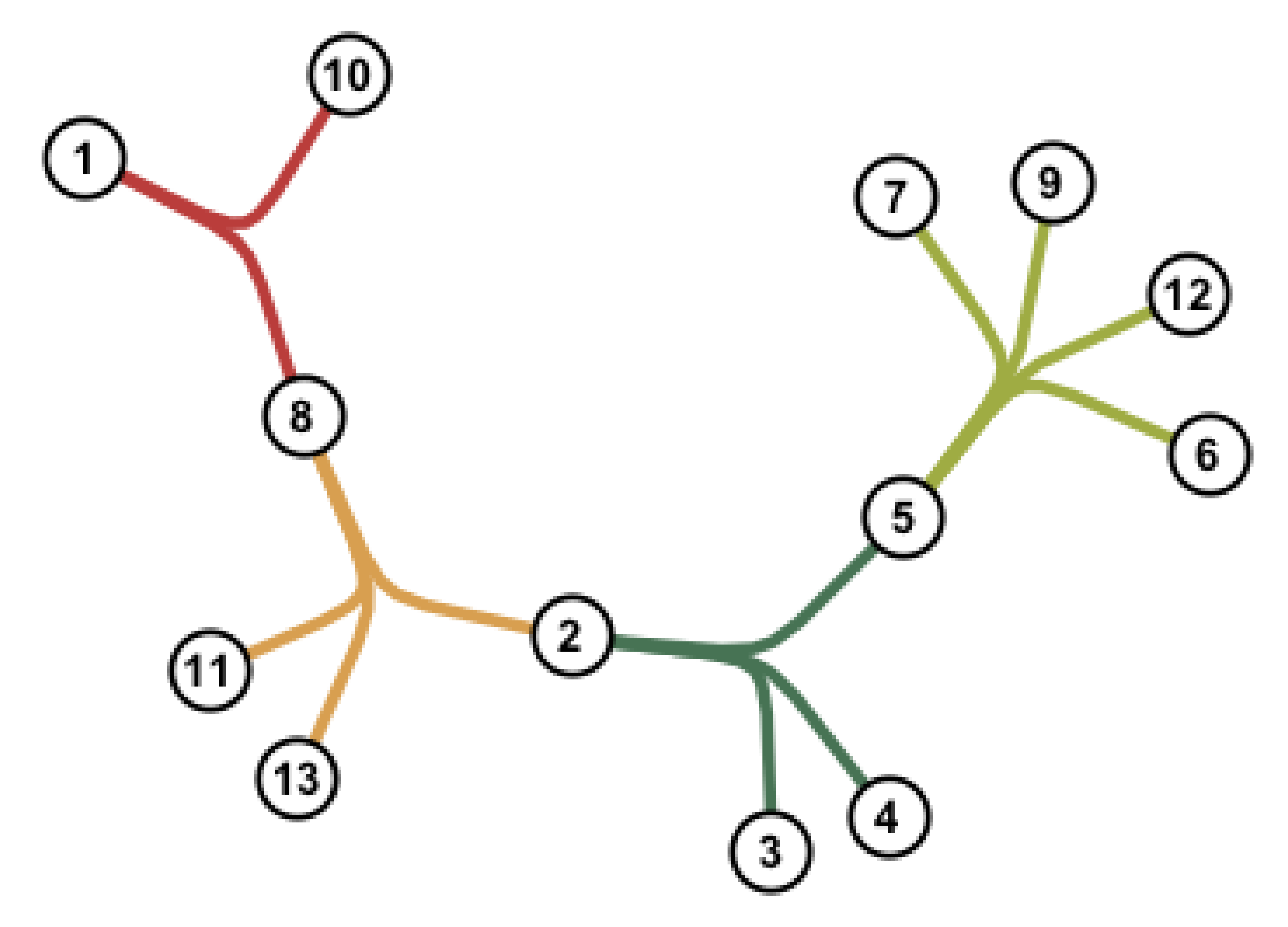
- An infinite generalization of this inequality to all odd party number exists, along with exactly 4 others at 5 parties (BNOSSW, Hernandez-Cuenca)
- Therefore, the holographic entropy cone is known up to five parties.

CAN WE DO BETTER?

- Two things always bothered me about this method
- First, can we give something back to QI?
- Second, why can't we make GHZ or W states?
- With some generalization, the answer to both of these is yes (BCHS)

ENTER HYPERGRAPHS

- A natural way of generalizing graph representations of entropy vectors is to generalize to hypergraphs
- Hypergraphs: edges become hyperedges, which can connect more than two vertices (k-edges)
- Entropies still given by the min cut necessary to separate some subsystem from the remainder of the system
- Max flow/min cut still works


10
1
8
7
9
12
6
5
2
11
13
3
4

MAKING 4 PARTY GHZ

- It is now trivial to write down a GHZ state in this framework
- It is simply a 4-edge connecting 4 vertices (bag with 4 jellybeans)
- Easy to see that all of the entropies of subsystems are the same, consistent with GHZ
- Also straightforward to extend to arbitrary party GHZ

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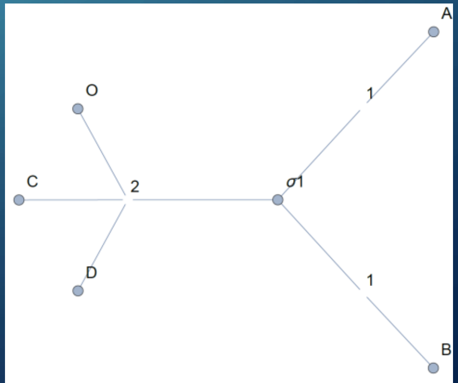
WHAT ELSE CAN WE MAKE?

- With simple hypergraphs one can make all of the extremal rays of the 4 party stabilizer cone, generated by SSA, Araki-Lieb, and the Ingleton inequality
- Further, of the classical linear rank cone (a cone that contains the 5 party stabilizer cone) all of the extremal rays can be realized with 2 or fewer interior vertices. As of this moment, 5 more can be constructed with 3 internal vertices, and the remainder require 4.



PROVING INEQUALITIES

- Constructing extremal rays was only half the battle
- Inequalities must also be proven
- The extension of the contraction map theorem to hypergraphs goes through
- Instead of being a single contraction condition, it becomes a sequence of contraction conditions
- If each contraction condition is met, then the inequality is correct for hypergraphs.

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- The contraction map can be generalized to work for hypergraphs, as the extension of the contraction map theorem to hypergraphs goes through
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GENERALIZED CONTRACTION MAP II

- Insight: can consider each k-uniform subgraph separately
- Theorem: Given a k-uniform subgraph, if for any k points on the LHS hypercube hypergraph, the minimum number of k-edges needed to connect those k points must be greater than or equal to the minimum number of k-edges needed to connect the points these k points map to on the RHS hypercube hypergraph, then the entropy inequality that assigns this contraction mapping is correct for all k-uniform hypergraph states.

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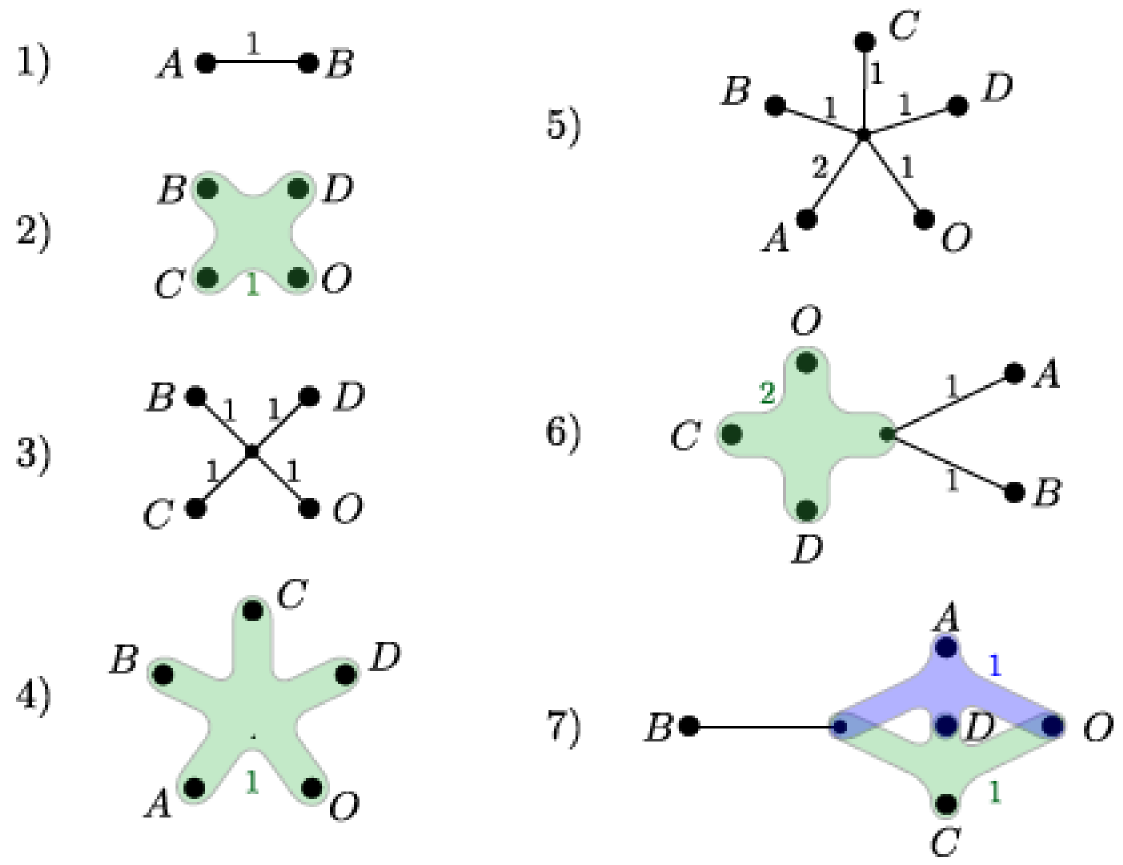

Figure 4: Hypergraphs corresponding to each symmetry orbit of extreme rays of the 4-party QLR cone, which notably coincides with the stabilizer cone. We have organized them by their labels as enumerated in [37] for the stabilizer cone. Similar to the holographic case, note that very few bulk vertices are needed – in fact, at most one. These are denoted by smaller, unlabeled points. As can be seen, some of the families of extreme rays admit realizations with no hyperedges, while others (such as the GHZ state) genuinely require hyperedges for their construction.

HYPERGRAPHS AND STABILIZER STATES

- Are hypergraph entropy vectors all realizable by stabilizer states?
- YES (Walter, Witteveen)
- Are all stabilizer entropy vectors realizable by hypergraph min cuts?
- In progress, with Cheng, Hernandez-Cuenca, and Su
- If the second is affirmative, the hypergraph cone is equal to the stabilizer cone
- Can potentially give a efficient method of determining all entanglement entropies of stabilizer states if given the circuit

FURTHER EXTENSIONS

- Proof method is combinatorically expensive, but there are some streamlining possibilities
- Can we prove higher party hypergraph inequalities?
- Can we further generalize to realize all quantum states?
- Directed (hyper)graphs more efficient?

OTHER THINGS I'M THINKING ABOUT

- Proving the eigenstate thermalization hypothesis by using quantum channels and the Grover search query complexity lower bound
- Understanding Hayden-Preskill (decoupling theorems) in the context of noise
- Designing wide but shallow circuits to perform state preparations of large scale invariant spin chain ground states via MERA bootstrapping
- Magic state distillation of scale invariant spin chains using a mixture of tensor network techniques and Bravyi-Kitaev

THANKS FOR LISTENING!

I welcome questions both on the main subject, and also on any ancillary topics you want me to say more about.