Self-similarity, fractality and entropy principle in collisions of hadrons and nuclei at RHIC, Tevatron and LHC

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Motivation & Goals

- Search for and investigation of new symmetries and conservation laws which govern physics at small scales.
- Search for signatures of phase transitions in nuclear matter exploiting scaling properties in suitable representation of data.
- Systematic analysis of inclusive spectra in p+p, p+A and A+A collisions to search for general features of hadron structure, constituent interactions, and fragmentation processes over a wide scale range (RHIC, Tevatron, LHC).
- Development of a unified approach for the description of particle production reflecting the principles of self-similarity, fractality, and locality of hadron interactions at the constituent level.

Principles & Symmetries

"Fundamental symmetry principles dictate the basic laws of physics, control the structure of matter and define the fundamental forces in nature."

Leon M. Lederman

"…for every conservation law there must be a continuous symmetry...."

Emmy Nöether

Discrete (C,P,T,..) and continuous symmetries correspond to fundamental principles (gauge, special, general and scale relativity, …) and conservation laws (charge,….) and vice versa.

- Principles are reflected as regularities in measurable observables and can be usually expressed as scaling in a suitable representation of data.
- z-Scaling of differential cross sections of inclusive particle production in p+p, p+A and A+A is used as a tool to search for and study of principles and symmetries that reflect properties of hadron interactions at constituent level.
- z-Scaling is based on the principles of *self-similarity*, *fractality*, *and locality*.

There exists a symmetry inherent to them:
Symmetry with respect to structural degrees of freedom - structural relativity.

Self-similarity in inclusive reactions

Differential cross section $Ed^3\sigma/dp^3$ for production of an inclusive particle
in the process $M_1+M_2 \to m+X$ depends on
- reaction characteristics (A_1, A_2, P_1, P_2)
- particle characteristics (m, p, θ)
- structural and dynamical characteristics (δ,ε, $dN/d\eta$,…)

- The assumption of self-similarity of hadron interactions at a constituent level transforms to the requirement of universal description of inclusive spectra by a scaling function $\psi(z)$.

- It should be achieved by grouping some characteristics of the inclusive reaction into a suitable variable z.

We search for a solution $\psi(z) \sim Ed^3\sigma/dp^3$ that would depend on adequate, physically meaningful, but still simple self-similarity variable z in a universal way.

Self-similarity types

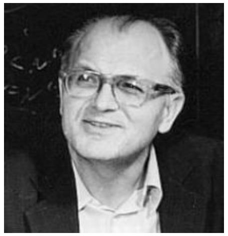
G.I. Barenblatt (1978)

Type I (Dimensional analysis)

There are dimensional quantities F, $\{\alpha_i\}$

Self-similarity variables Π_j are expressed via $\{\alpha_i\}$

Self-similarity functions $\Phi(\Pi_j)$...



V.S. Stavinsky (1972)

Cumulative particle production

$$F \equiv E d^3\sigma / dp^3, \quad \alpha_i \equiv \{P, p, \sqrt{s}\}$$

Self-similarity variables: $\Pi_0 = \sqrt{(x_1 P_1 + x_2 P_2)^2} / m_N, \; \Pi_{1,2} = 1 - x_{1,2}$, cumulative numbers $x_{1,2}$

Self-similarity functions: Φ = exp (-Π/const)

Universality is broken by *power asymptotic* at high p_T !!!

Type II (Intermediate asymptotics)

If $\Phi(\Pi)$ does not converge but has *power asymptotic* for extreme $\{\Pi_0, \Pi_i\}_{i=1}^{N}$

then self-similar solution Φ can be expressed via $\{\Pi_0 / \Pi_i^{\Delta_i}\}$

A.M. Baldin (1998)

Hypothesis of self-similarity in Relativistic Nuclear Physics:

... search for $\Phi(\Pi_0 / \Pi_i^{\Delta_i}, ..)$ …parameters Δ_i should be found from experiment.

Self-similarity of II type & variable z

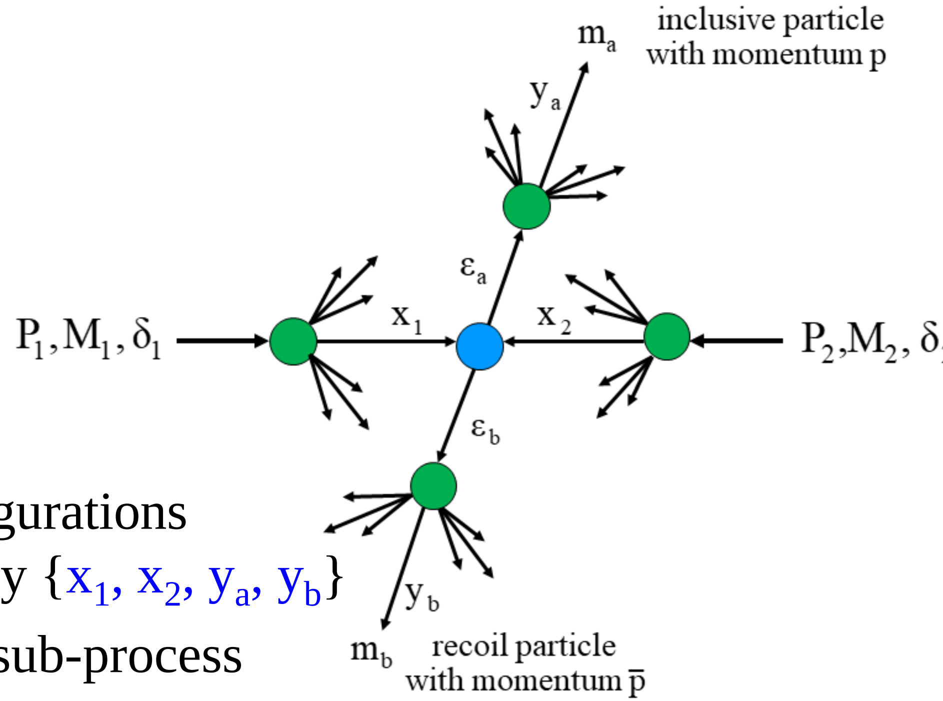
$$\Pi_0 \cong \sqrt{(x_1P_1 + x_2P_2)^2_\perp} / m_N$$

$$\Delta_i$$

$$z \cong \frac{s_\perp^{1/2}}{\Omega}$$

$$\Omega = (1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}$$

Momentum fractions $\{x_1, x_2, y_a, y_b\}$ define a constituent sub-process



z - self-similarity variable of II type
- expressed via momentum fractions
- fractal measure

Model parameters (fractal dimensions):

δ_1, δ_2 - structure of M_1, M_2

ε_a, ε_b - fragmentation processes

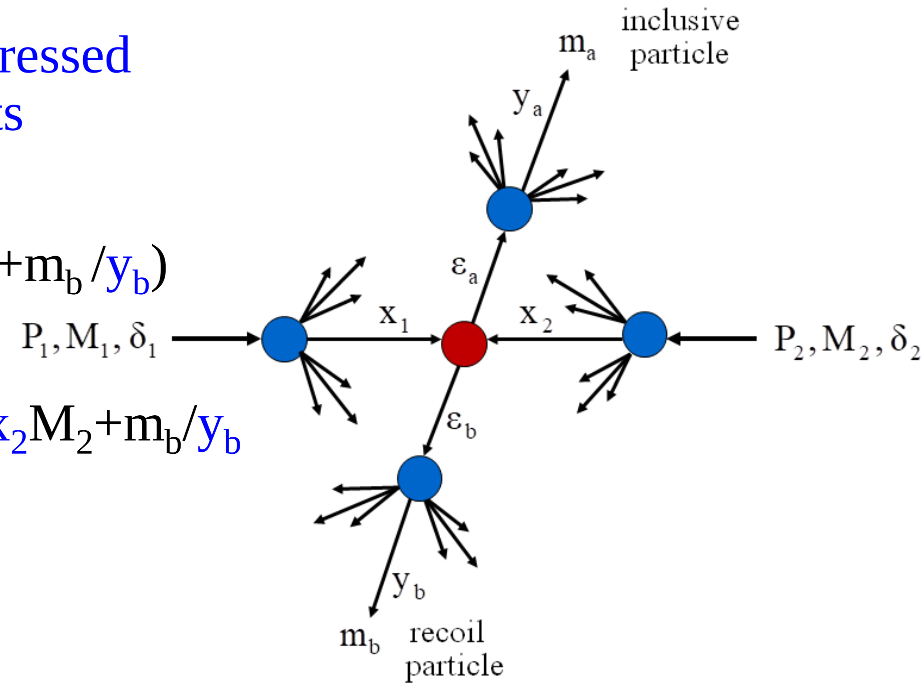
Ω ~ relative number of constituent configurations containing the sub-process defined by $\{x_1, x_2, y_a, y_b\}$

Ω^{-1} ~ resolution at which the constituent sub-process can be singled out of the inclusive reaction

Fractal property of z: $z(\Omega) \to \infty$ if $\Omega^{-1} \to \infty$

Locality & Minimal resolution Ω^{-1}

Collisions of hadrons and nuclei are expressed
via interactions of their constituents



Constituent sub-processes

$(x_1M_1) + (x_2M_2) \rightarrow (m_a/y_a) + (x_1M_1+x_2M_2+m_b/y_b)$

Kinematic constraint (*)

$(x_1P_1+x_2P_2-p/y_a)^2 = M_X^2$

Recoil mass

$M_X = x_1M_1+x_2M_2+m_b/y_b$

Fractal measure z

$$z = \frac{s_\perp^{1/2}}{W_{max}}$$

$$W_{max} = (dN_{ch}/d\eta|_0)^c \cdot \Omega_{max}$$

$$W(x_1,x_2,y_a,y_b) = (dN_{ch}/d\eta|_0)^c \cdot \Omega(x_1,x_2,y_a,y_b)$$

c - "specific heat"

$$\Omega = (1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}$$

$\delta_1, \delta_2, \varepsilon_a, \varepsilon_b$ - fractal dimensions

$s_\perp^{1/2}$ – transverse kinetic energy consumed on production of m_a & m_b

W_{max} – max. relative No. of configurations that can lead to production of m_a & m_b

Microscopic features of constituent sub-processes are given in terms of $\{x_1, x_2, y_a, y_b\}$ obtained from minimal resolution Ω^{-1} with the constraint (*).

Scale transformation of z & ψ(z)

Scaling variable

$$z = \frac{s_\perp^{1/2}}{W_{max}}$$

Scaling function

$$\psi(z) = \frac{1}{N\sigma_{inel}} \frac{d\sigma}{dz}$$

ψ(z) in terms of measurable quantities

$$\psi(z) = \frac{\pi}{(dN/d\eta)\ \sigma_{inel}} J^{-1} E \frac{d^3\sigma}{dp^3}$$

J – Jacobian $\{z,\eta\}/\{p_T^2,y\}$

Scale transformation with W_0

$$z' = z / W_0 \qquad \psi'(z') = W_0 \cdot \psi(z)$$

W_0 - absolute No. of configurations of the system (drops out of the z-scaling)

W_0 - depends on type (F) of the inclusive particle

Scale transformation preserves the normalization

$$\int_0^\infty \psi(z)dz = 1$$

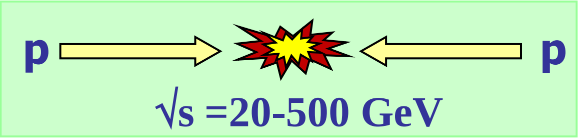
$$\alpha_F \equiv W_0(F)/W_0(\pi)$$

Scale transformation with α_F

$$z \rightarrow \alpha_F \cdot z \qquad \psi \rightarrow \alpha_F^{-1} \cdot \psi$$

Scaling functions for different hadrons collapse to a single curve using transformations with suitable α_F

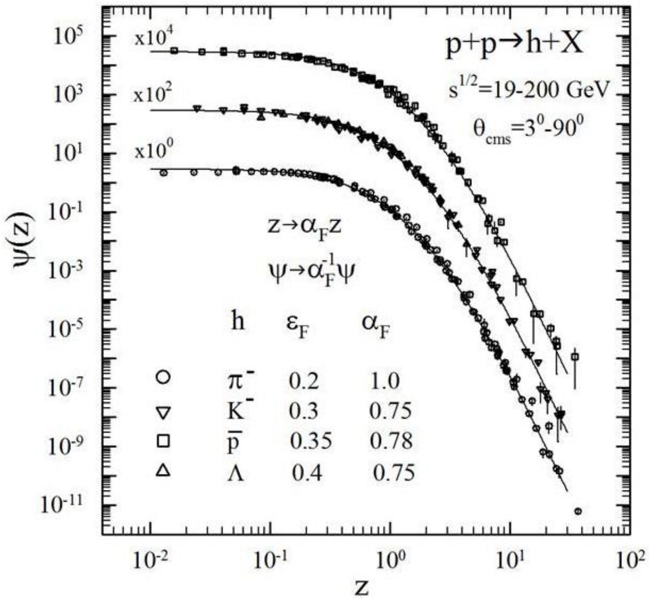
z-Scaling in p+p collisions at RHIC



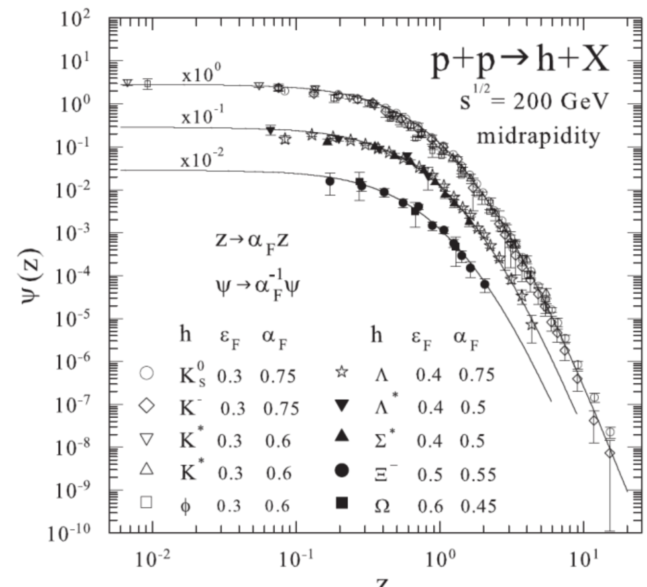
z-Scaling of identified hadrons in p+p at RHIC

"Collapse" of data points onto a single curve

π^-, K^- $\bar{p}$, Λ



K_S^0, K^-, K^*, ϕ, Λ, Ξ, Ω, Σ^*, Λ^*



- Energy & angular independence
- Flavor independence
- Saturation for $z < 0.1$
- Power law $\psi(z) \sim z^{-\beta}$ for high $z > 4$
- Fractal dimensions $\delta = 0.5$, $\varepsilon_F \equiv \varepsilon_a = \varepsilon_b$
- "Specific heat" $c = 0.25$

FNAL & ISR
PRD 19 (1979) 764
NPB 100 (1975) 237
NPB 106 (1976) 1
PLB 64 (1976) 111
NPB 116 (1976) 77
NPB 56 (1973) 333
PRD 40 (1989) 2777

STAR
PRL 97 (2006) 132301
PLB 612 (2005) 181
PRC 71 (2005) 064902
PRC 75 (2007) 064901
PRL 108 (2012) 072302
PLB 616 (2005) 8
PLB 637 (2006) 161

PHENIX
PRD 83 (2011) 052004
PRC 90 (2014) 054905

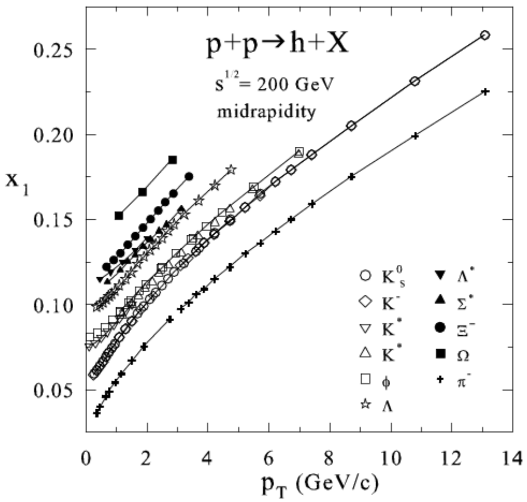
Int. J. Mod. Phys. A 32, 1750029 (2017)

Self-similarity of strangeness production in p+p

K_S^0, K^-, K^*, ϕ, Λ, Ξ, Ω, Σ^*, Λ^*

Constituent level of particle production in terms of

Momentum fraction

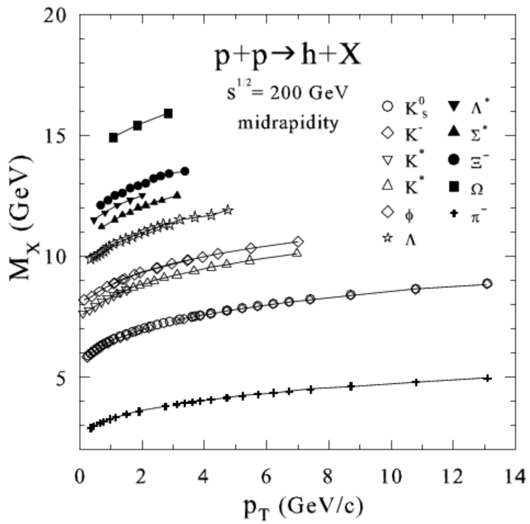


$x_1^{Baryon} > x_1^{Meson}$

The more strangeness, the larger momentum fraction

$x_1^{\Omega} > x_1^{\Xi} > x_1^{\Sigma}$

Recoil mass

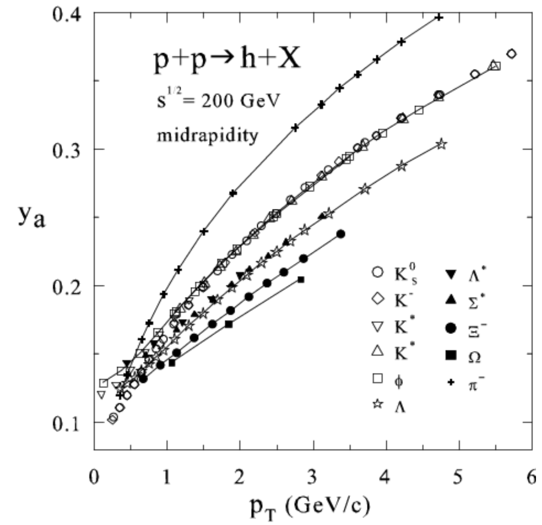


$M_X^{Baryon} > M_X^{Meson}$

The more strangeness, the larger recoil mass

$M_X^{\Omega} > M_X^{\Xi} > M_X^{\Sigma}$

Energy loss $\Delta E/E \sim (1-y_a)$



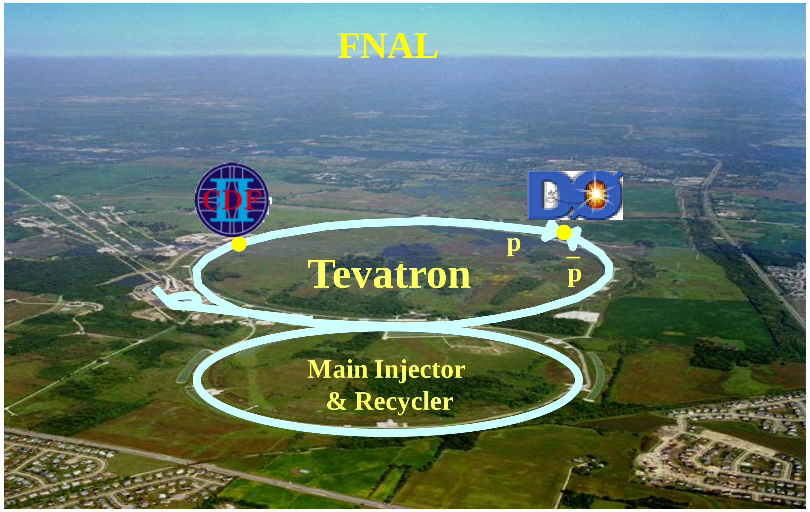
$\varepsilon^{Baryon} > \varepsilon^{Meson}$

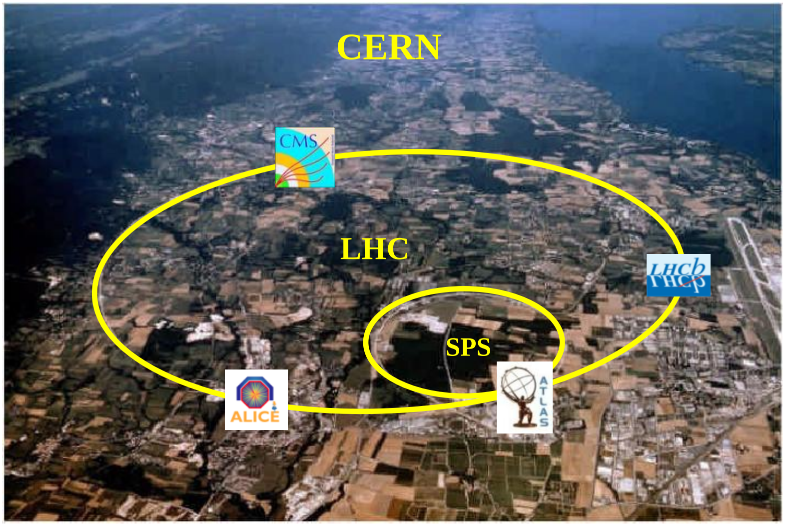
The more strangeness, the larger energy loss

$\varepsilon_{\Omega} > \varepsilon_{\Xi} > \varepsilon_{\Sigma}$

Self-similarity dictates the properties of constituent sub-processes.

z-Scaling at the Tevatron and LHC

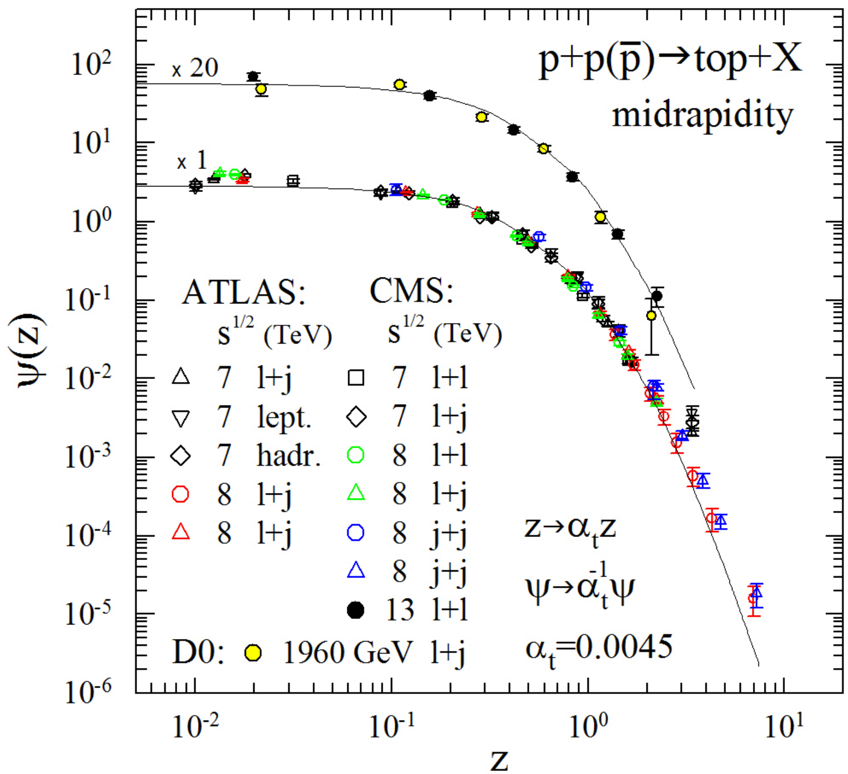




Self-similarity of top quark production at the Tevatron & LHC

$s^{1/2}$ (GeV)		
● (yellow)	1960	PRD90(2014)092006
△	7000	PRD90(2014)072004
▽	7000	JHEP06(2015)100
◇	7000	JHEP06(2015)100
□	7000	EPJC73(2013)2339
◇	7000	EPJC73(2013)2339
○ (red)	8000	PRD93(2016)032009
△ (red)	8000	arXiv:1511.04716
○ (green)	8000	EPJC75(2015)542
△ (green)	8000	EPJC75(2015)542
○ (blue)	8000	EPJC76(2016)128
△ (blue)	8000	arXiv:1605.00116
◆	13000	CMS TOP 16-011

“Collapse” of data points onto a single curve



- Energy independence of $\psi(z)$
- Flavor independence of $\psi(z)$
- Saturation of $\psi(z)$ for or $z < 0.1$
- Fractal dimensions $\delta = 0.5$, $\varepsilon_{top} = 0$
- “Specific heat” $c = 0.25$

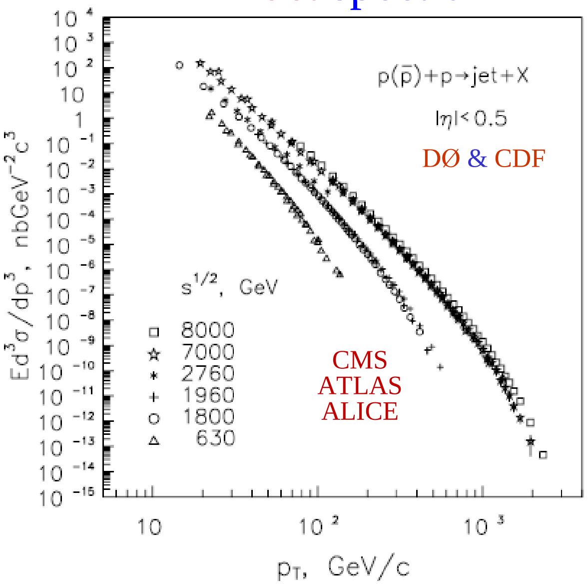
J. Mod. Phys. 32, 815 (2012)
ISMD’16, Jeju Island, South Korea, 2016

LHC & Tevatron data
confirm self-similarity
of top quark production in pp & $p\bar{p}$

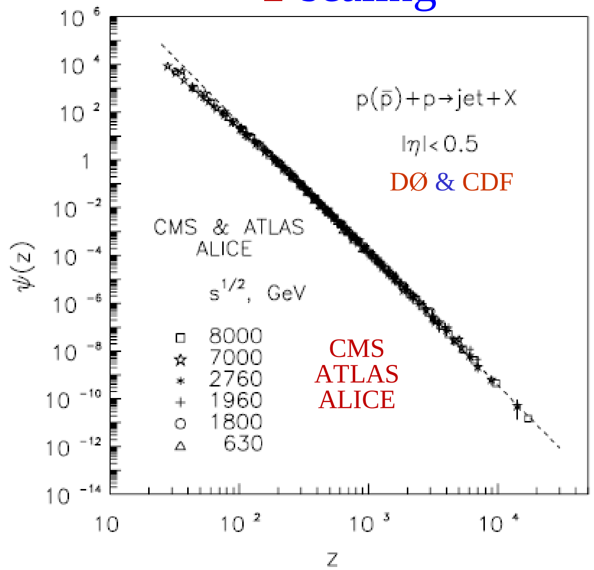
Self-similarity of jet production at the Tevatron and LHC

Energy $\sqrt{s} = 8$ TeV up to the momentum $p_T \approx 2.4$ TeV/c and scale $\sim 8 \cdot 10^{-5}$ fm

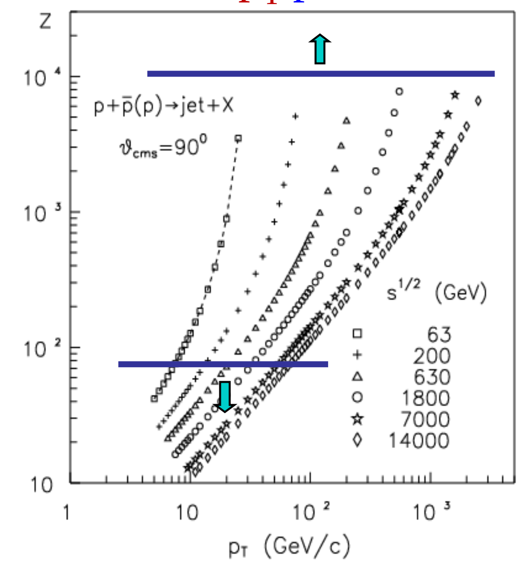
Jet spectra



z-scaling



z-p_T plot



New test of z-scaling at LHC

Structural phenomena ⟺ constituent substructure,…
Self-similarity at small scales ⟺ fractal topology of momentum space,…

M.Tokarev, T.Dedovich, I.Z.
Int.J.Mod.Phys.A15 (2000) 3495
Int.J.Mod.Phys.A27 (2012)1250115

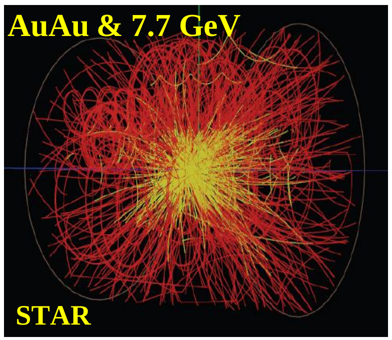
Phys. Part. Nucl. Lett. 51, 141 (2020)

Properties of scaling function ψ (z) in p+p collisions

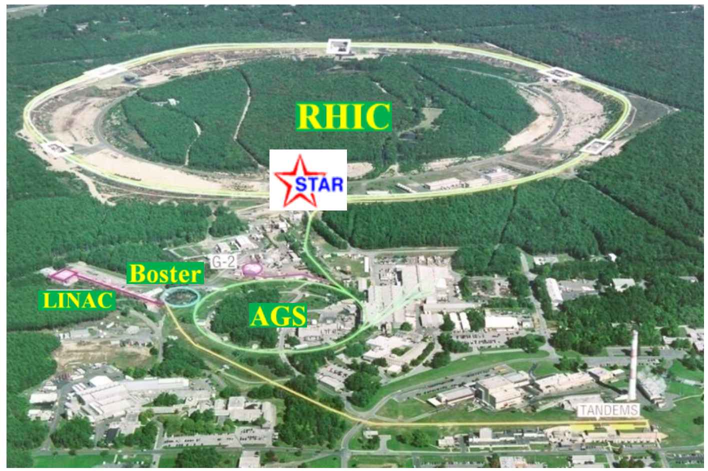
- Energy independence of $\psi(z)$ ($s^{1/2} > 20$ GeV)
- Angular independence of $\psi(z)$ ($\theta_{cms}=3^0$-90^0)
- Multiplicity independence of $\psi(z)$ ($dN_{ch}/d\eta=1.5$-26)
- Saturation of $\psi(z)$ at low z ($z < 0.1$)
- Power law, $\psi(z) \sim z^{-\beta}$, at high z ($z > 4$)
- Flavor independence of $\psi(z)$ (π, K,φ,Λ,..,D,J/ψ,B,ϒ,…, top)

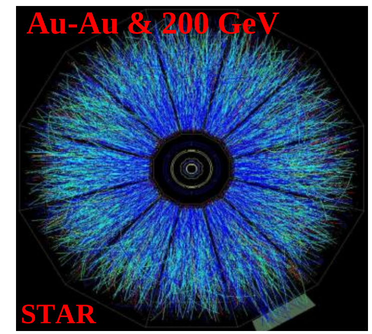
I. Z. and M.V. Tokarev, Phys. Rev. D 75, 094008 (2007)
I.Z. and M.V. Tokarev, Int. J. Mod. Phys. A 24, 1417 (2009)
M.V. Tokarev and I. Z., Int. J. Mod. Phys. A 32, 1750029 (2017)
M.V. Tokarev, A.O. Kechechyan and I. Z., Nucl. Phys. A 993, 121646 (2020)
M.V. Tokarev, I. Z., A.O. Kechechyan and T.G. Dedovich , Phys. Part. Nucl. 51, 141 (2020)

z-Scaling in Au+Au collisions at RHIC



BNL, Upton, Long Island





STAR BES at RHIC

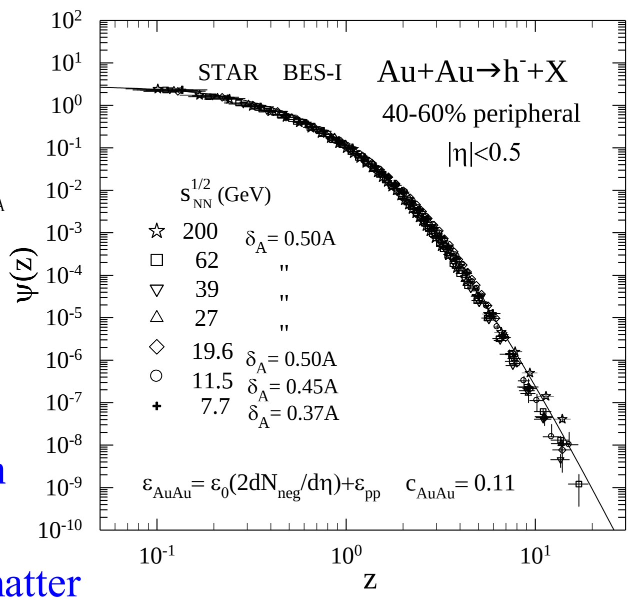
Self-similarity of h^- production in Au+Au collisions

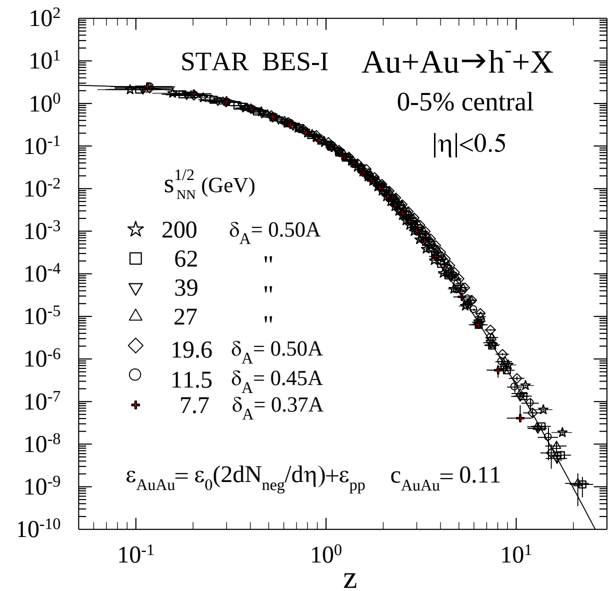
Variable $z = \frac{s_\perp^{1/2}}{W_{max}}$

$$W_{max} = (dN_{ch}/d\eta|_0)^{c_{AA}} \cdot \Omega_{max}$$

$$\Omega = (1-x_1)^{\delta_A}(1-x_2)^{\delta_A}(1-y_a)^{\varepsilon_{AA}}(1-y_b)^{\varepsilon_{AA}}$$

"Collapse" of data points onto a single curve





A+A collisions:

- δ_A - nucleus fractal dimension
- ε_{AA} - fragmentation dimension
- c_{AA} - "specific heat" of bulk matter
- $dN_{ch}/d\eta|_0$ - multiplicity density in AA

$$\delta_A = A\delta_p$$

additivity of fractal dimensions of nuclei

$$\varepsilon_{AA} = \varepsilon_0(2 \cdot dN_{neg}^{AA}/d\eta) + \varepsilon_{pp}$$

increase of energy loss with multiplicity

$$c_{AuAu} = 0.11 < c_{pp} = 0.25$$

larger temperature fluctuations in AuAu than in pp

- Energy independence of $\psi(z)$
- Centrality independence of $\psi(z)$
- ε_{AA} increases with multiplicity
- Power law in low- and high-z regions

Indication of a decrease of δ for $\sqrt{s_{NN}} < 19.6$ GeV

Nucl. Phys. A993 (2020) 121646

Constituent energy loss in Au+Au collisions

Energy loss: $\Delta E/E \sim (1-y_a)$

p – momentum of the inclusive negative hadron
q – momentum of the scattered constituent

$p = q\, y_a$

p_T =4 GeV in peripheral collisions:

20 % energy loss $q \approx 5$ GeV/c

22 % energy loss $q \approx 5.1$ GeV/c

30 % energy loss $q \approx 5.7$ GeV/c

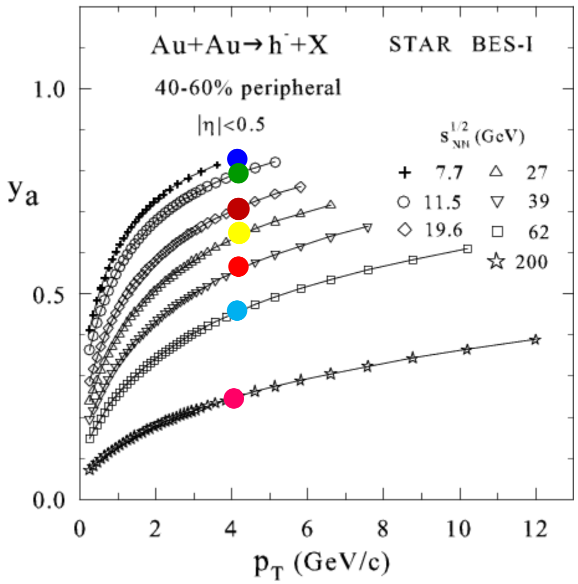
35 % energy loss $q \approx 6.2$ GeV/c

45 % energy loss $q \approx 7.3$ GeV/c

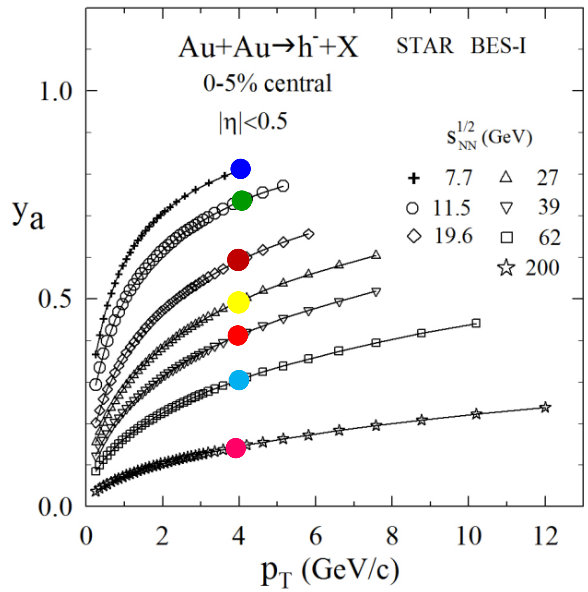
55 % energy loss $q \approx 8.9$ GeV/c

75 % energy loss $q \approx 16$ GeV/c

(40-60)% peripheral



(0-5)% central



p_T =4 GeV in central collisions:

20 % energy loss $q \approx 5$ GeV/c

25 % energy loss $q \approx 5.3$ GeV/c

40 % energy loss $q \approx 6.7$ GeV/c

50 % energy loss $q \approx 8$ GeV/c

60 % energy loss $q \approx 10$ GeV/c

70 % energy loss $q \approx 13.3$ GeV/c

85 % energy loss $q \approx 26.6$ GeV/c

Energy loss as a characteristic of produced medium

- decreases with p_T
- increases with $\sqrt{s_{NN}}$
- increases with centrality

Variable z & Entropy

Variable z is defined in terms of an underlying constituent sub-process

All constituent sub-processes are mutually independent

$$z = \frac{s_\perp^{1/2}}{W_{max}}$$

$$W_{max} = (dN_{ch}/d\eta|_0)^c \cdot \Omega_{max}$$

$$\Omega = (1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}$$

Scale transformation

$$z' = z / W_0 \qquad \psi'(z') = W_0 \cdot \psi(z)$$

Entropy - thermodynamical

$$S = c_V \ln T + R \cdot \ln V + \text{const.}$$

statistical

$$S = k \cdot \ln W$$

Entropy of the remaining part of the system

$$S = c \cdot \ln(dN/d\eta|_0) + \ln\Omega + \ln W_0$$

$$S_{max} = \ln W_{max} + \ln W_0$$

- $dN/d\eta|_0$ characterizes "temperature" of the produced system
- local equilibrium ⇨ $dN/d\eta|_0 \sim T^3$ (for high T and small μ)
- c - "specific heat" of the produced medium
- $\delta_1, \delta_2, \varepsilon_a, \varepsilon_b$ – fractal dimensions in space of momentum fractions $\{x_1, x_2, y_a, y_b\}$
- entropy S increases with $dN/d\eta|_0$ and decreases with increasing resolution Ω^{-1}

Max. entropy $S(x_1,x_2,y_a,y_b)$ ⟺ Max. number of configurations $W(x_1,x_2,y_a,y_b)$

under condition: $(x_1P_1+x_2P_2-p/y_a)^2 = (x_1M_1+x_2M_2+m_b/y_b)^2$ ⇨ Ω_{max} ⇨ z

Maximum entropy principle & Momentum fractions

Entropy

$$S_\Omega(x_1, x_2, y_a, y_b) = \ln\Omega(x_1, x_2, y_a, y_b) + \ln\Omega_0$$

$$\Omega = (1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}$$

Kinematic constraint

$$(x_1P_1 + x_2P_2 - p/y_a)^2 = (x_1M_1 + x_2M_2 + m_b/y_b)^2$$

$$\varphi = x_1x_2 - x_1\lambda_2 - x_2\lambda_1 - \lambda_0$$

$$\varphi(x_1, x_2, y_a, y_b) = 0$$

Lorentz invariants

$$\lambda_{1,2} = \frac{\kappa_{1,2}}{y_a} + \frac{\nu_{1,2}}{y_b}$$

$$\lambda_0 = \frac{\nu_b}{y_b^2} - \frac{\nu_a}{y_a^2}$$

$$\kappa_{1,2} = \frac{(P_{2,1}p)}{(P_1P_2) - M_1M_2}$$

$$\nu_{1,2} = \frac{M_{2,1}m_b}{(P_1P_2) - M_1M_2}$$

$$\nu_{a,b} = \frac{0.5m_{a,b}^2}{(P_1P_2) - M_1M_2}$$

Maximization of the functional Φ with a Lagrange multiplicator β

$$\Phi(x_1, x_2, y_a, y_b) = \Omega(x_1, x_2, y_a, y_b) + \beta\cdot\varphi(x_1, x_2, y_a, y_b)$$

for determination of the momentum fractions $\{x_1, x_2, y_a, y_b\}$.

Maximum entropy principle & New conservation law

Principle of maximal entropy

The momentum fractions x_1, x_2, y_a, y_b are determined in a way to maximize the entropy S_Ω with a kinematic constraint

Maximum of the functional Φ

$$\Phi = \Omega + \beta \cdot \varphi$$

$$\begin{cases} \partial\Phi/\partial x_1 = 0 & \partial\Phi/\partial y_a = 0 \\ \partial\Phi/\partial x_2 = 0 & \partial\Phi/\partial y_b = 0 \end{cases}$$

Constraint on momentum fractions

$$(x_1 P_1 + x_2 P_2 - p/y_a)^2 = M_X^2$$

Mass of the recoil system

$$M_X = x_1 M_1 + x_2 M_2 + m_2/y_b$$

Resolution w.r.t. sub-processes

$$\Omega^{-1} = (1-x_1)^{-\delta_1}(1-x_2)^{-\delta_2}(1-y_a)^{-\varepsilon_a}(1-y_b)^{-\varepsilon_b}$$

Conservation law

$$\delta_1 \frac{x_1}{1-x_1} + \delta_2 \frac{x_2}{1-x_2} = \varepsilon_a \frac{y_a}{1-y_a} + \varepsilon_b \frac{y_b}{1-y_b}$$

for arbitrary $P_1, P_2, p, \delta_1, \delta_2, \varepsilon_a, \varepsilon_b$!!!

Conserved quantity

$$C(D,\zeta) = D \cdot g(\zeta) \qquad g(\zeta) = \frac{\zeta}{1-\zeta}$$

D - fractal dimension

ζ - momentum fraction

Conservation of $C(D,\zeta)$

$$\sum_i^{in} C(D_i, \zeta_i) = \sum_j^{out} C(D_j, \zeta_j)$$

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Properties of the quantity $C(D,\zeta)$

$$C(D,\zeta) = D \cdot \frac{\zeta}{1-\zeta}$$

$D = (\delta_1, \delta_2, \varepsilon_a, \varepsilon_b)$ – fractal dimension
$\zeta = (x_1, x_2, y_a, y_b)$ – momentum fraction

$C(D,\zeta)$ characterizes:

- property of a fractal-like object (or fractal-like process) with fractal dimension D to form a "structural aggregate" with certain degree of local compactness which carries the momentum fraction ζ
- ability of the fractal systems to create structural constituents
- cumulative property of internal structure of hadrons and nuclei with local compactness due to the Heisenberg uncertainty principle
- aggregation property of fractal-like fragmentation processes

$C(D,\zeta)$ is proportional to the fractal dimension D of a respective fractal system. The larger momentum fraction ζ carries a structural constituent (or an aggregated part) of the fractal-like system, the larger value of $C(D,\zeta)$ it has.

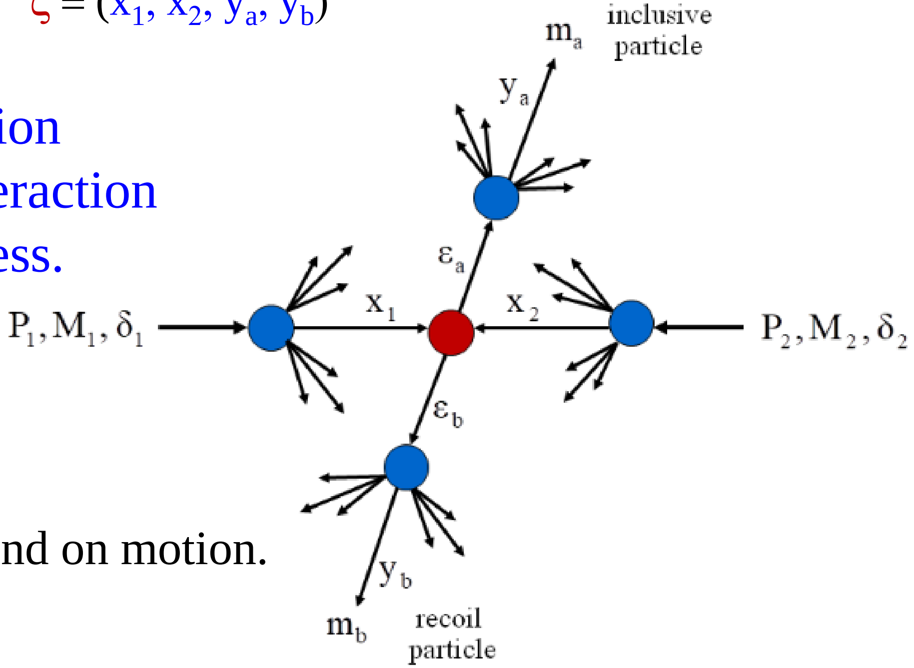
$C(D,\zeta)$ is named "fractal cumulativity" of a fractal-like structure with the dimension D carried by its constituent with the momentum fraction ζ

Conservation of fractal cumulativity C(D,ζ)

$$C(D,\zeta) = D \cdot \frac{\zeta}{1-\zeta}$$

$D = (\delta_1, \delta_2, \varepsilon_a, \varepsilon_b)$

$\zeta = (x_1, x_2, y_a, y_b)$



Fractal cumulativity before interaction
is equal to fractal cumulativity after interaction
for any binary constituent sub-process.

$$\sum_{i}^{in} C(D_i, \zeta_i) = \sum_{j}^{out} C(D_j, \zeta_j)$$

Conservation law for C(D,ζ) does not depend on motion.
It depends only on resolution.

We assume that

- hadron constituents possess fractal structure
- fragmentation processes have fractal-like character
- corresponding structures are mutually similar
- compactness of the fractal systems is governed by the Heisenberg uncertainty principle

Composition rule for C(D,ζ)

$$C(D,\zeta)=D\cdot g(\zeta)$$
$$g(\zeta)=\frac{\zeta}{1-\zeta}$$

$$\boxed{C(\zeta'')=C(\zeta)+C(\zeta')+D^{-1}C(\zeta)\cdot C(\zeta')}$$

$$\boxed{g''=g+g'+gg'}$$

$$\boxed{\zeta''=\zeta'+\zeta-\zeta'\cdot\zeta}$$

ζ – momentum fraction

Composition rule for C(D,ζ) leads to q-exponential type of the distributions with non-extensivity parameter q-1~1/D

Properties of C(D,ζ) for a fractal with dimension D

- different ζ correspond to different levels of resolution
- additivity for small C(D,ζ)
- non-additivity for large C(D,ζ)

Cumulativity $C(D,\zeta)$ & energy $E(M_0,\beta)$

"Fractal-like" object | Analogy | "Point-like" object

$$C(D,\zeta) = D \cdot g(\zeta) \quad g(\zeta) = \frac{\zeta}{1-\zeta}$$

conservation law

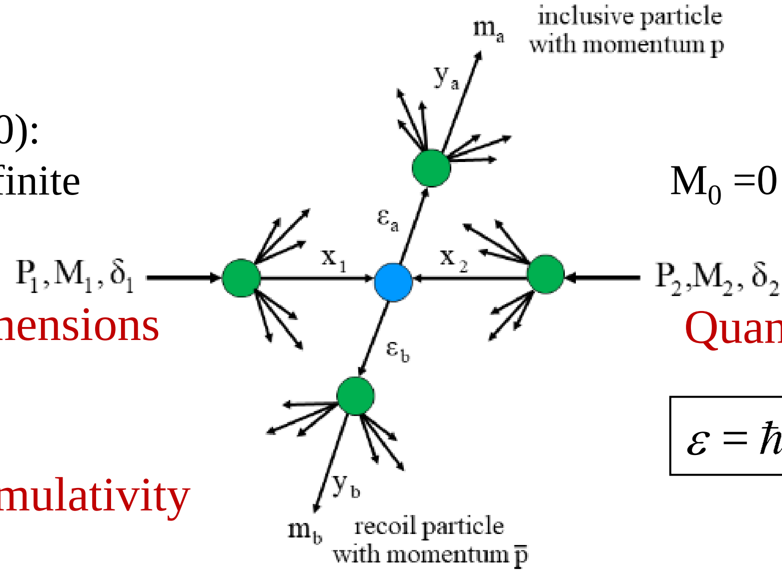
$$E(M_0,\beta) = M_0 \cdot \gamma(\beta) \quad \gamma(\beta) = \frac{1}{\sqrt{1-\beta^2}}$$

$C(D,\zeta)$ is function of the momentum fraction ζ in a resolution dependent ref. system $\{P_1,P_2,p\}$ (state of resolution)

$E(M_0,\beta)$ is function of the velocity fraction β in a motion dependent ref. system $\{V_1,V_2,V_3\}$ (state of motion)

Non-structural objects (D=0):
$D=0 \rightarrow \zeta=1$, but $C(0,1)$ is finite

Photon ($M_0=0$):
$M_0=0 \rightarrow \beta=1$, but $E(0,1)$ is finite



Quantization of fractal dimensions

$$D = n_D \cdot d$$

Quantization of fractal cumulativity

$$C = n_C \cdot d$$

Quantization of energy

$$\varepsilon = \hbar\omega \quad \vec{p} = \hbar\vec{k} \quad p = \varepsilon / c$$

$$E = n_E \cdot \varepsilon$$

Differences

$C(D,\zeta)$ is Lorentz invariant.
There is no absolute scale for ζ.
Dimension D can depend on other characteristics.

$E(M_0,\beta)$ is not a Lorentz invariant.
Velocity β has absolute scale.
Mass M_0 is a relativistic invariant.

Max. entropy & constraints for momentum fractions

Max. entropy ⟹ max. $\Phi = \Omega + \beta \cdot \varphi$ ⟹ $\frac{\partial\Phi}{\partial x_1} = 0, \frac{\partial\Phi}{\partial x_2} = 0, \frac{\partial\Phi}{\partial y_a} = 0, \frac{\partial\Phi}{\partial y_b} = 0$

1 $$F_1(x_1, x_2, y_a, y_b, \kappa_1, \kappa_2) \equiv x_1 x_2 - \kappa_2 \frac{x_1}{y_a} - \kappa_1 \frac{x_2}{y_a} - \nu_2 \frac{x_1}{y_b} - \nu_1 \frac{x_2}{y_b} - \nu_b \frac{1}{y_b^2} + \nu_a \frac{1}{y_a^2} = 0$$

2 $$F_2(x_1, x_2, y_a, y_b, \kappa_1, \kappa_2) \equiv \left[x_1 x_2 - \kappa_2 \frac{x_1}{y_a} - \nu_2 \frac{x_1}{y_b} \right] \frac{(1-x_1)}{\delta_1 x_1} - \left[x_1 x_2 - \kappa_1 \frac{x_2}{y_a} - \nu_1 \frac{x_2}{y_b} \right] \frac{(1-x_2)}{\delta_2 x_2} = 0$$

3 $$G_1(x_1, x_2, y_a, y_b) \equiv \left[\frac{\varepsilon_a y_a}{1-y_a} + \frac{\varepsilon_b y_b}{1-y_b} \right]^{-1} - \left[\frac{\delta_1 x_1}{1-x_1} + \frac{\delta_2 x_2}{1-x_2} \right]^{-1} = 0$$

4 $$G_2(x_1, x_2, y_a, y_b) \equiv \left[\nu_2 \frac{x_1}{y_b} + \nu_1 \frac{x_2}{y_b} + \frac{2\nu_b}{y_b^2} \right] \frac{(1-y_b)}{\varepsilon_b y_b} - \left[x_1 x_2 - \nu_2 \frac{x_1}{y_b} - \nu_1 \frac{x_2}{y_b} - \frac{\nu_a}{y_a^2} - \frac{\nu_b}{y_b^2} \right] \frac{(1-y_a)}{\varepsilon_a y_a} = 0$$

Momentum fractions are functions of κ_1, κ_2 :

$$x_1 = x_1(\kappa_1, \kappa_2), \; x_2 = x_2(\kappa_1, \kappa_2), \; y_a = y_a(\kappa_1, \kappa_2), \; y_b = y_b(\kappa_1, \kappa_2)$$

$$\kappa_{1,2} = (P_{2,1} p) / [(P_1 P_2) - M_1 M_2] \qquad \nu_{1,2} = \frac{M_{2,1} m_b}{(P_1 P_2) - M_1 M_2} \qquad \nu_{a,b} = \frac{0.5 m_{a,b}^2}{(P_1 P_2) - M_1 M_2}$$

Momentum fractions near fractal limit $\Omega^{-1} \to \infty$

Fractal limit (L): $x_1=x_2=y_a=y_b=1$

Constraints for momentum fractions in the region $x_1, x_2, y_a, y_b \to 1$:

$$\left.\frac{\partial F_i}{\partial x_1}\right|_L (1-x_1) + \left.\frac{\partial F_i}{\partial x_2}\right|_L (1-x_2) + \left.\frac{\partial F_i}{\partial y_a}\right|_L (1-y_a) + \left.\frac{\partial F_i}{\partial y_b}\right|_L (1-y_b) + \left.\frac{\partial F_i}{\partial \kappa_1}\right|_L (\overline{\kappa_1}-\kappa_1) + \left.\frac{\partial F_i}{\partial \kappa_2}\right|_L (\overline{\kappa_2}-\kappa_2) = 0$$

$$\left.\frac{\partial G_i}{\partial x_1}\right|_L (1-x_1) + \left.\frac{\partial G_i}{\partial x_2}\right|_L (1-x_2) + \left.\frac{\partial G_i}{\partial y_a}\right|_L (1-y_a) + \left.\frac{\partial G_i}{\partial y_b}\right|_L (1-y_b) = 0$$

$$(1-e_1-e_2) = \left(\overline{\lambda_1}+\overline{\lambda_0}\right)(1-x_1) + \left(\overline{\lambda_2}+\overline{\lambda_0}\right)(1-x_2) + (1-\nu)(1-y_a) + \left(\nu+\overline{\lambda_0}\right)(1-y_b)$$

$$0 = \delta_1^{-1}\left(\overline{\lambda_1}+\overline{\lambda_0}\right)^2(1-x_1) + \delta_2^{-1}\left(\overline{\lambda_2}+\overline{\lambda_0}\right)^2(1-x_2) - \varepsilon_a^{-1}(1-\nu)^2(1-y_a) - \varepsilon_b^{-1}\left(\nu+\overline{\lambda_0}\right)^2(1-y_b)$$

$$0 = -\delta_1^{-1}\left(\overline{\lambda_1}+\overline{\lambda_0}\right)(1-x_1) + \delta_2^{-1}\left(\overline{\lambda_2}+\overline{\lambda_0}\right)(1-x_2)$$

$$0 = \varepsilon_a^{-1}(1-\nu)(1-y_a) - \varepsilon_b^{-1}\left(\nu+\overline{\lambda_0}\right)(1-y_b)$$

Over-lined symbols are calculated at fractal limit (L); Expressions in red depend on p_T

$$e_1+e_2 \equiv \kappa_1+\kappa_2+\nu_1+\nu_2+\nu_b-\nu_a \qquad \nu \equiv \nu_1+\nu_2+\nu_b+\nu_a$$

Momentum fractions near fractal limit $\Omega^{-1} \to \infty$

Fractal limit (L): $x_1 = x_2 = y_a = y_b = 1$

Momentum fractions in the region $x_1, x_2, y_a, y_b \to 1$:

$$1 - x_1 = \frac{(1 - e_1 - e_2)}{\left(\overline{\lambda_1} + \overline{\lambda_0}\right)} \frac{\delta_1}{(\delta_1 + \delta_2 + \varepsilon_a + \varepsilon_b)}, \quad 1 - y_a = \frac{(1 - e_1 - e_2)}{(1 - \nu)} \frac{\varepsilon_a}{(\delta_1 + \delta_2 + \varepsilon_a + \varepsilon_b)}$$

$$1 - x_2 = \frac{(1 - e_1 - e_2)}{\left(\overline{\lambda_2} + \overline{\lambda_0}\right)} \frac{\delta_2}{(\delta_1 + \delta_2 + \varepsilon_a + \varepsilon_b)}, \quad 1 - y_b = \frac{(1 - e_1 - e_2)}{\left(\nu + \overline{\lambda_0}\right)} \frac{\varepsilon_b}{(\delta_1 + \delta_2 + \varepsilon_a + \varepsilon_b)}$$

$\overline{\lambda_0}, \overline{\lambda_1}, \overline{\lambda_2}, \nu$ are known functions of $P_1, P_2, M_1, M_2, m_a, m_b$

$$\Omega = (1 - x_1)^{\delta_1} (1 - x_2)^{\delta_2} (1 - y_a)^{\varepsilon_a} (1 - y_b)^{\varepsilon_b}$$

Maximal entropy near the fractal limit

$$S_\Omega = \ln \Omega + \ln \Omega_0$$

Entropy S_Ω near fractal limit $\Omega^{-1} \to \infty$

Entropy decomposition

$$S_\Omega = S_\Upsilon - S_\Gamma + S_0$$

Entropy S_Υ depends on momenta and masses of the colliding and inclusive particles

Entropy S_0 is a constant

Entropy S_Γ depends *solely* on fractal dimensions

$$S_\Gamma = (\delta_1+\delta_2+\varepsilon_a+\varepsilon_b)\ln(\delta_1+\delta_2+\varepsilon_a+\varepsilon_b) - \delta_1\ln\delta_1 - \delta_2\ln\delta_2 - \varepsilon_a\ln\varepsilon_a - \varepsilon_b\ln\varepsilon_b$$

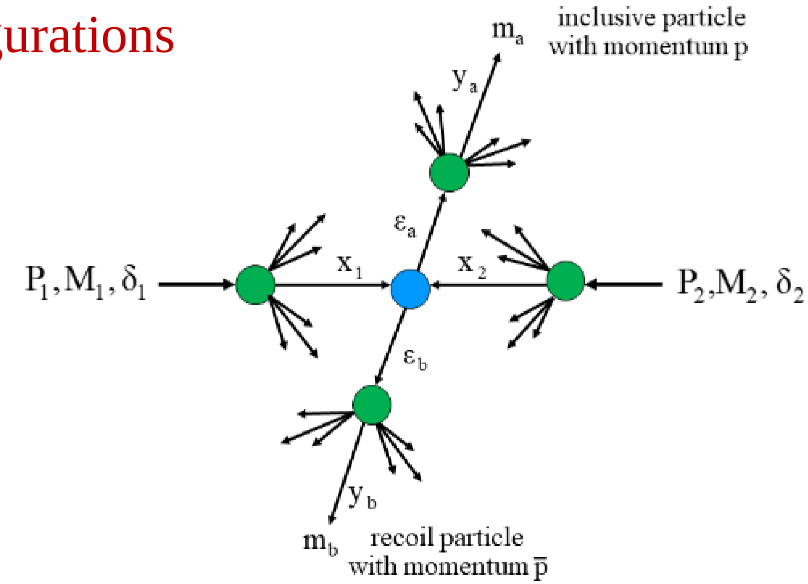
Symmetry property of S_Γ

S_Γ is invariant under the transformation

$$D_i \leftrightarrow D_j, \quad D = (\delta_1, \delta_2, \varepsilon_a, \varepsilon_b)$$

Entropy S_Γ of a statistical ensemble

Statistical ensemble of interacting fractal configurations



Large collection of the interacting fractals

- with random configurations $\{x_1, x_2, y_a, y_b, \ldots\}$
- with the same fractal dimensions $\{\delta_1, \delta_2, \varepsilon_a, \varepsilon_b\}$

Number of configurations

n_{δ_1} - internal structure of M_1

n_{δ_2} - internal structure of M_2

n_{ε_a} - fragmentation process to m_a

n_{ε_b} - fragmentation process to m_b

Entropy S_Γ of a single "average" fractal configuration of the system

$$S_I\left(\frac{\varepsilon}{\delta}\right) + S_I\left(\frac{\delta_2}{\delta_1}\right) + S_I\left(\frac{\varepsilon_b}{\varepsilon_a}\right)$$

$$\varepsilon \equiv \varepsilon_a + \varepsilon_b$$

$$\delta \equiv \delta_1 + \delta_2$$

$$S_I(r) = d\left[(1+r)\ln(1+r) - r\ln r\right]$$

Entropy S_Γ of the whole statistical ensemble is additive

$$S_\Gamma = n_\delta S_I\left(\frac{\varepsilon}{\delta}\right) + n_{\delta_1} S_I\left(\frac{\delta_2}{\delta_1}\right) + n_{\varepsilon_a} S_I\left(\frac{\varepsilon_b}{\varepsilon_a}\right)$$

$$n_\delta \equiv n_{\delta_1} + n_{\delta_2}$$

Quantization of fractal dimensions

The entropy of the statistical ensemble of interacting fractal configurations

$$S_\Gamma = n_\delta S_I\left(\frac{\varepsilon}{\delta}\right) + n_{\delta_1} S_I\left(\frac{\delta_2}{\delta_1}\right) + n_{\varepsilon_a} S_I\left(\frac{\varepsilon_b}{\varepsilon_a}\right)$$

$$\varepsilon \equiv \varepsilon_a + \varepsilon_b$$
$$\delta \equiv \delta_1 + \delta_2$$
$$n_\delta \equiv n_{\delta_1} + n_{\delta_2}$$

$$S_I(r) = d\left[\left(1+r\right)\ln\left(1+r\right) - r\ln r\right]$$

The entropy S_Γ can be represented by all possible arrangements, in which the fractal dimensions of the interacting fractal structures are composed of identical dimensional quanta, each of size d, provided the fractal dimensions have the quantum form:

$$\delta_1 = d \cdot n_{\delta_1}\ ,\ \ \delta_2 = d \cdot n_{\delta_2}\ ,\ \ \varepsilon_a = d \cdot n_{\varepsilon_a}\ ,\ \ \varepsilon_b = d \cdot n_{\varepsilon_b}$$

Statistical interpretation of entropy S_Γ

The entropy S_Γ can be expressed as the logarithm of number of different ways in which identical dimensional quanta can be distributed among the fractal dimensions of interacting fractal structures.

$$S_\Gamma = d \cdot \ln\left(\Gamma_{\delta_1,\delta_2,\varepsilon_a,\varepsilon_b}\right)$$

$$\Gamma_{\delta_1,\delta_2,\varepsilon_a,\varepsilon_b} \equiv \frac{\left(n_{\delta_1}+n_{\delta_2}+n_{\varepsilon_a}+n_{\varepsilon_b}\right)!}{n_{\delta_1}!n_{\delta_2}!n_{\varepsilon_a}!n_{\varepsilon_b}!} = \Gamma_{\delta,\varepsilon}\cdot\Gamma_{\delta_1,\delta_2}\cdot\Gamma_{\varepsilon_a,\varepsilon_b}$$

$$n_\delta \equiv n_{\delta_1} + n_{\delta_2}$$
$$n_\varepsilon \equiv n_{\varepsilon_a} + n_{\varepsilon_b}$$

$$\Gamma_{\delta,\varepsilon} = \frac{\left(n_\delta + n_\varepsilon\right)!}{n_\delta!\ n_\varepsilon!} \qquad \Gamma_{\delta_1,\delta_2} = \frac{\left(n_{\delta_1} + n_{\delta_2}\right)!}{n_{\delta_1}!\ n_{\delta_2}!} \qquad \Gamma_{\varepsilon_a,\varepsilon_b} = \frac{\left(n_{\varepsilon_a} + n_{\varepsilon_b}\right)!}{n_{\varepsilon_a}!\ n_{\varepsilon_b}!}$$

The statistical interpretation of entropy S_Γ is only possible by the quantization of fractal dimensions

$$\delta_1 = d\cdot n_{\delta_1}\ ,\ \ \delta_2 = d\cdot n_{\delta_2}\ ,\ \ \varepsilon_a = d\cdot n_{\varepsilon_a}\ ,\ \ \varepsilon_b = d\cdot n_{\varepsilon_b}$$

Conservation of Number of Quanta of Cumulativity NQC

Quantization of fractal dimensions $D = d \cdot n$ is connected with quantum character of fractal cumulativity $C(D,\zeta)$

$$C(D,\zeta) = D\frac{\zeta}{1-\zeta}$$

Conservation law for Number of Quanta of Cumulativity

$$NQC(n,\zeta) = n \cdot \frac{\zeta}{1-\zeta}$$

Number of Cumulativity Quanta before and after a binary sub-process is the same.

$$\sum_{i}^{in} NQC(n_i,\zeta_i) = \sum_{j}^{out} NQC(n_j,\zeta_j)$$

The number of quanta of fractal cumulativity is conserved at any resolution given by arbitrary momenta P_1, P_2, and p of the colliding and inclusive particles.

The quantization of D and $C(D,\zeta)$ is based on the assumptions of

- fractal self-similarity of the internal hadron structure
- fractal nature of fragmentation processes
- locality of hadron interactions at a constituent level up to the kinematic limit

Summary I

- z-Scaling is a specific feature of high-p_T particle production established in p-(anti)p collisions at the U70, ISR, S$\bar{p}$pS, Tevatron, RHIC and LHC.
- It reflects self-similarity, locality, and fractality of hadron interactions at a constituent level.
- The scaling behavior was confirmed for inclusive production of different hadrons, jets, heavy quarkonia and top quark.
- Hypothesis of self-similarity and fractality was tested in Au+Au collisions at RHIC using z-presentation of spectra of negative hadrons.
- Analysis of STAR BES-I data indicates approximate energy and multiplicity independence of the scaling function $\psi(z)$.
- The variable z depends on multiplicity density, “heat capacity”, and entropy of constituent configurations of the interacting system.
- Constituent energy loss as a function of energy and centrality of collision and momentum of inclusive particle was estimated.

Summary II

- We have shown that z-scaling containing the principle of maximum entropy includes conservation of the quantity C(D,ζ) ("fractal cumulativity").
- Fractal cumulativity reflects ability of fractal systems to create structural constituents.
- The cumulativity C(D,ζ) of a fractal object or a fractal-like process is proportional to its fractal dimension D and represents a simple function of the momentum fraction ζ carried by the corresponding constituent.
- A composition rule for C(D,ζ) connects the fractal cumulativity at different scales.
- Fractal dimension D is interpreted as a quantity which has quantum nature.
- The quantization of fractal dimension D results in preservation of the number of quanta of cumulativity NQC(n,ζ) in binary sub-processes at any resolution.

Thank you for your attention !