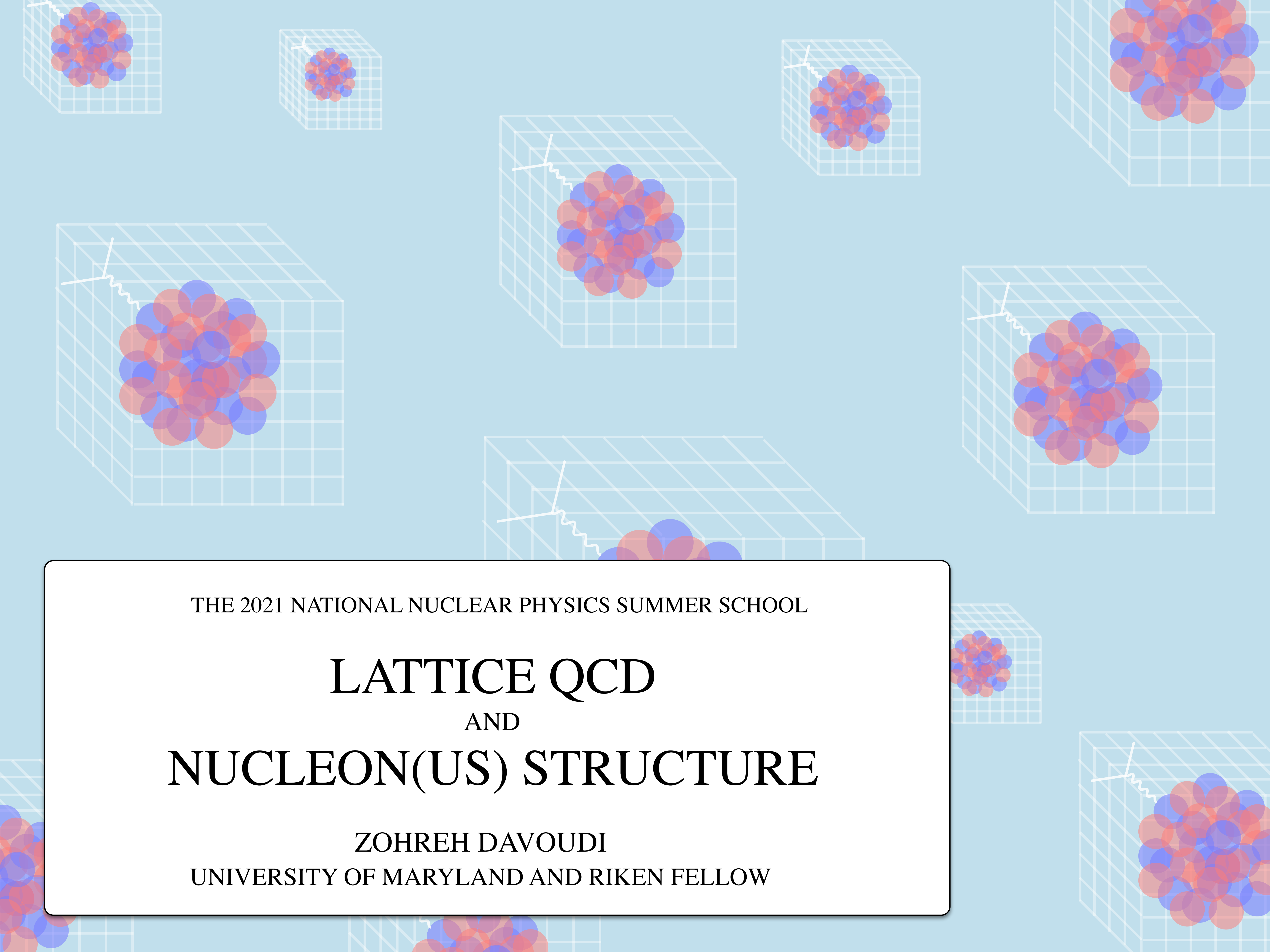


THE 2021 NATIONAL NUCLEAR PHYSICS SUMMER SCHOOL

LATTICE QCD
AND
NUCLEON(US) STRUCTURE

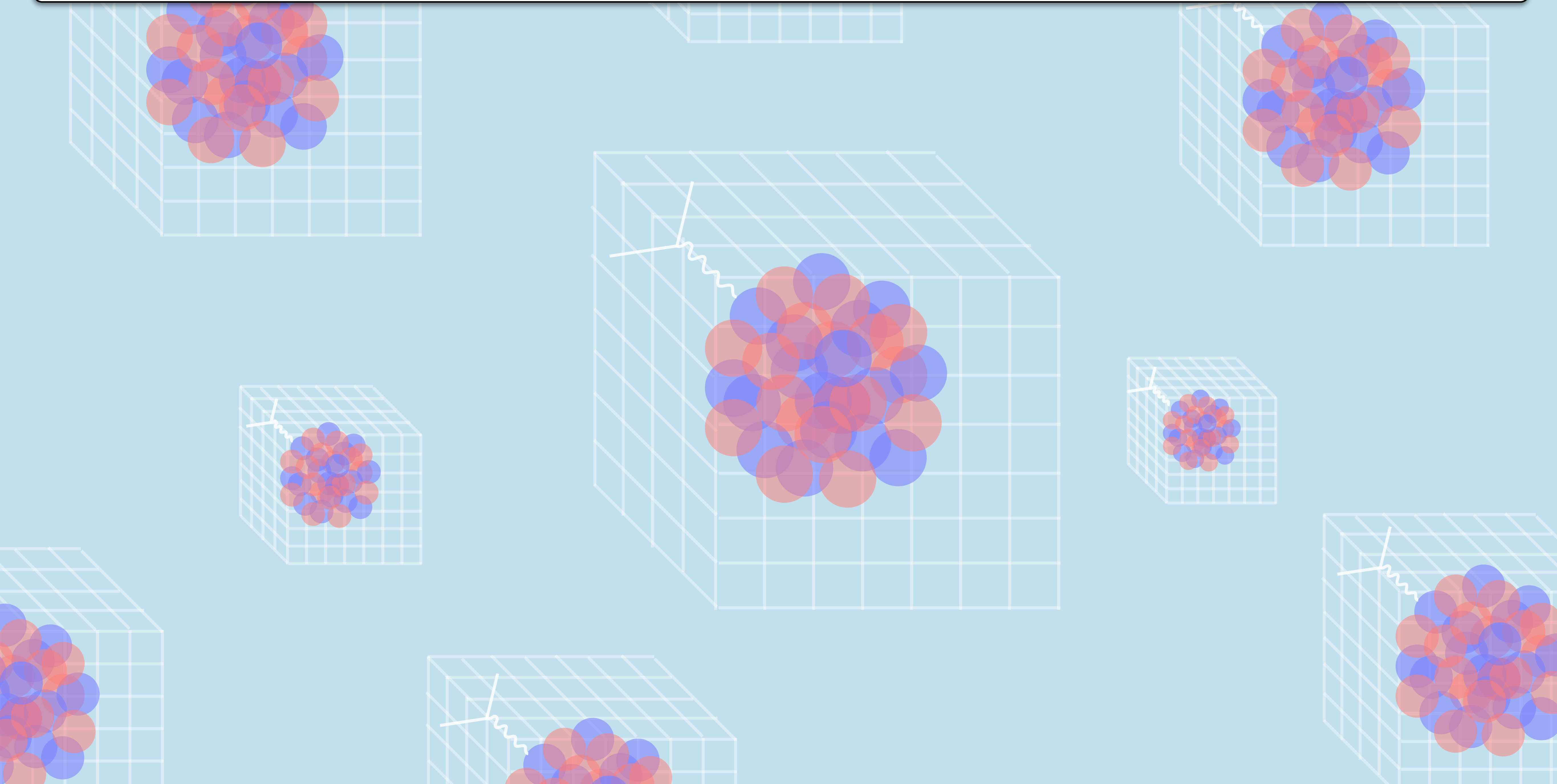
ZOHREH DAVOUDI
UNIVERSITY OF MARYLAND AND RIKEN FELLOW

LECTURE I:
LATTICE QCD FORMALISM AND METHODOLOGY

LECTURE II:
NUCLEON STRUCTURE FROM LATTICE QCD

LECTURE III:
TOWARDS NUCLEAR STRUCTURE FROM LATTICE QCD

LECTURE I:
LATTICE QCD FORMALISM AND METHODOLOGY

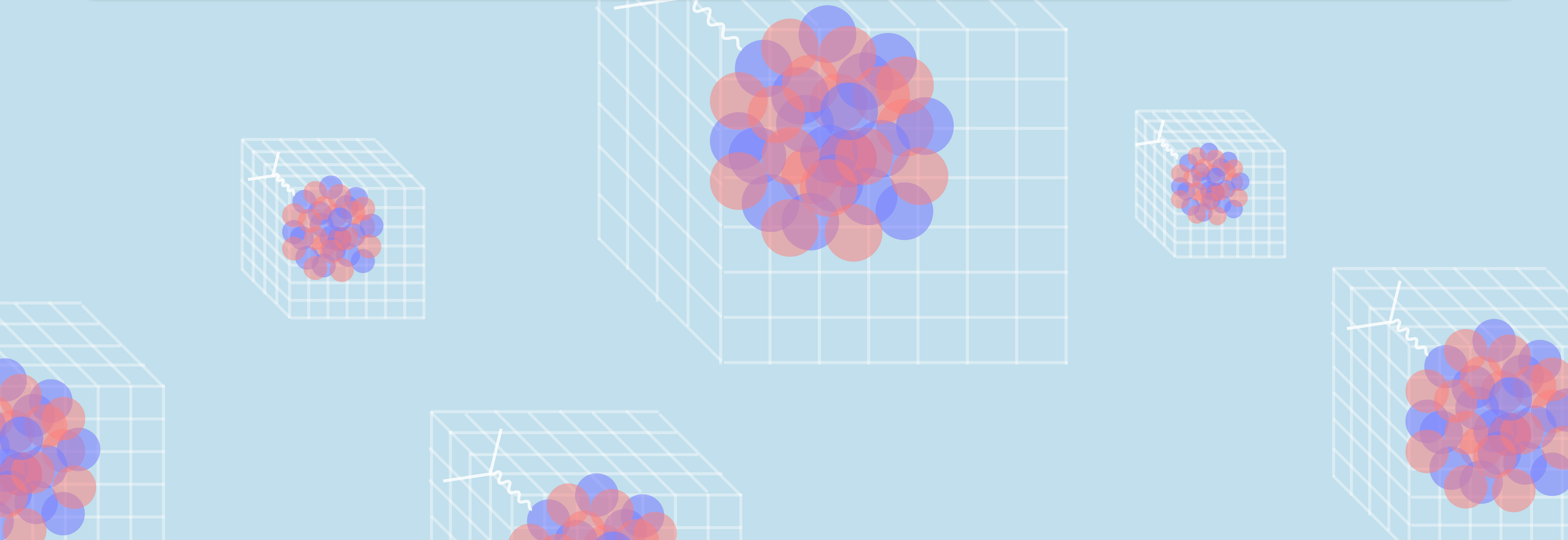


Quantum chromodynamics (QCD) in continuum:

QCD is a SU(3) Yang-Mills theory augmented with several flavors of massive quarks:

$$\mathcal{L}_{QCD} = \sum_{f=1}^{N_f}\left[\overbrace{\bar{q}_f(i\gamma^\mu\partial_\mu - m_f)q_f}^{\text{Quark kinetic and mass term}} - \overbrace{gA^i_\mu\bar{q}_f\gamma^\mu T^i q_f}^{\text{Quark/gluon interactions}}\right]$$

$$\underbrace{-\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} + \frac{g}{2}f_{ijk}F^i_{\mu\nu}A^{i\mu}A^{j\nu} - \frac{g^2}{4}f_{ijk}f_{klm}A^j_\mu A^k_\nu A^{l\mu}A^{m\nu}}_{\text{Gluons kinetic and interaction terms}}$$



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Quark kinetic and mass term

Quark/gluon interactions

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Gluons kinetic and interaction terms

Observe that:

i) There are only $1 + N_f$ input parameters plus QCD coupling. Fix them by a few quantities and all strongly-interacting aspects of nuclear physics is predicted (in principle)!

ii) QCD is asymptotically free such that: $\alpha_s(\mu') = \dfrac{1}{2b_0 \log \frac{\mu'}{\Lambda_{QCD}}}$

Positive constant for $N_f \leq 16$

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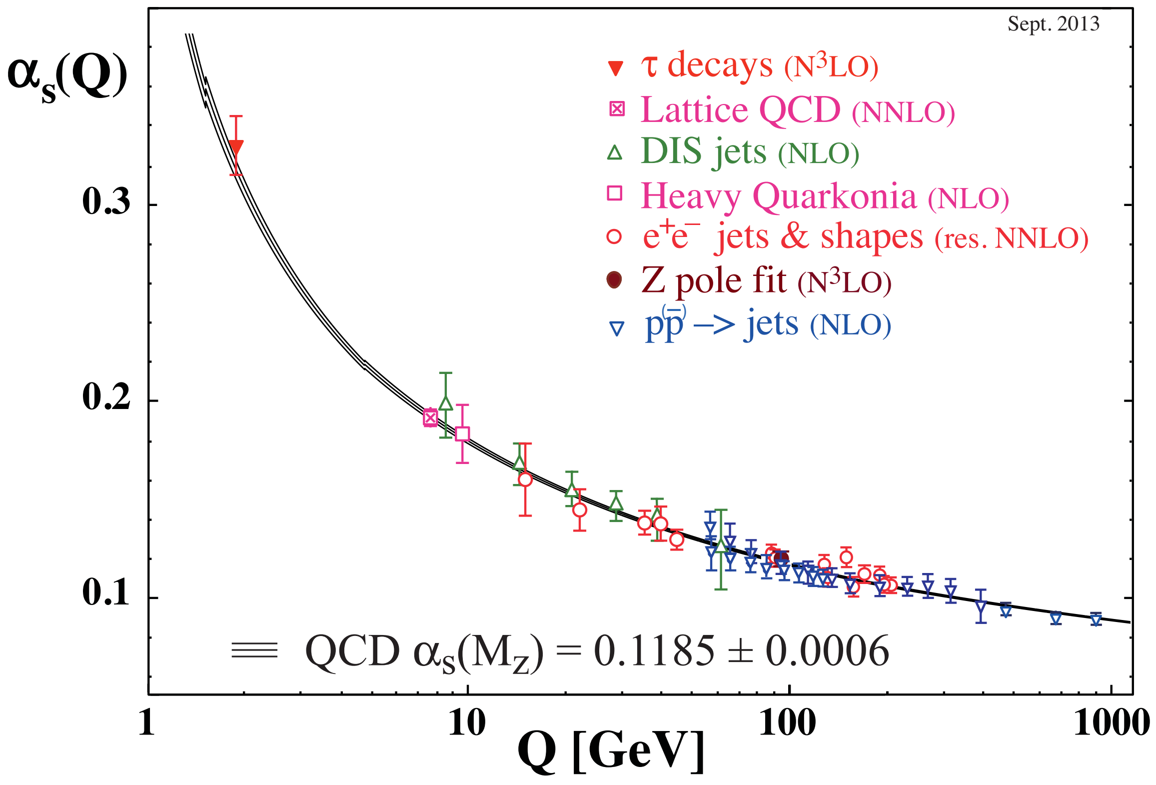

Let’s enumerate the steps toward numerically simulating this theory nonperturbatively…

Step I: Discretize the QCD action in both space and time. Consider a finite hypercubic lattice. Wick rotate to imaginary times.

Step II: Generate a large sample of thermalized decorrelated vacuum configurations.

Step III: Form the correlation functions by contracting the quark fields. Need to specify the interpolating operators for the state under study.

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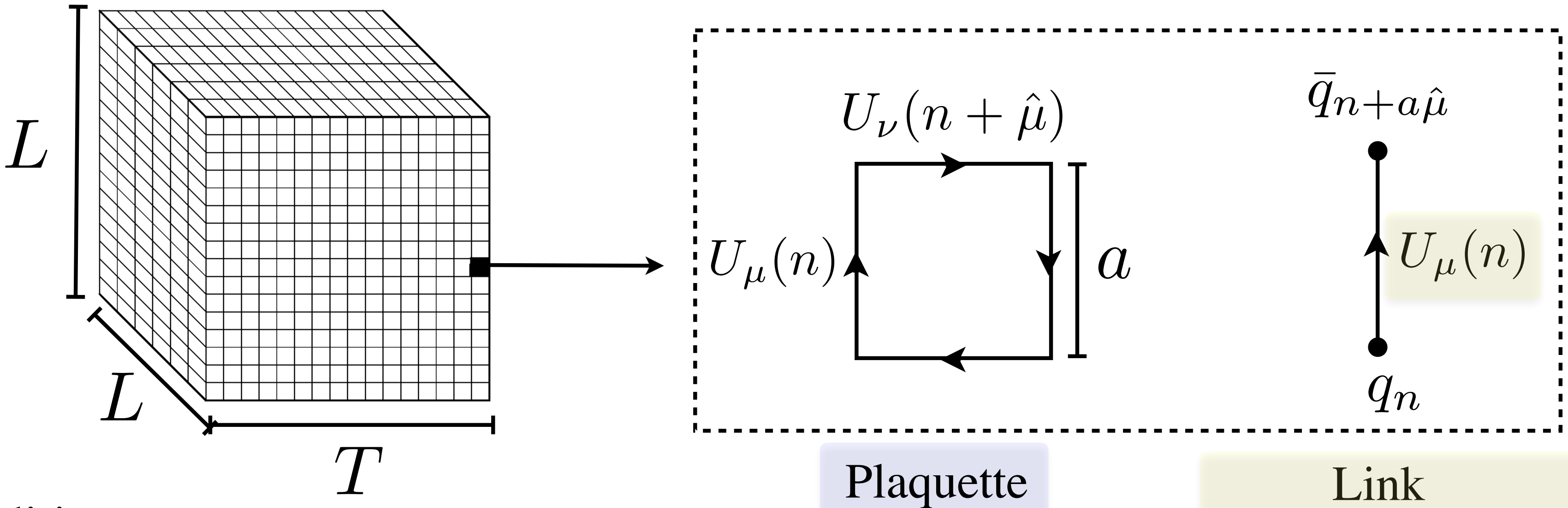

Plaquette

Link

$U_\mu(n) \equiv e^{igA_\mu(n)}$

Two conditions:

$T, L \gg m_\pi^{-1}$ $\quad$ $a \ll \Lambda_{QCD}^{-1}$

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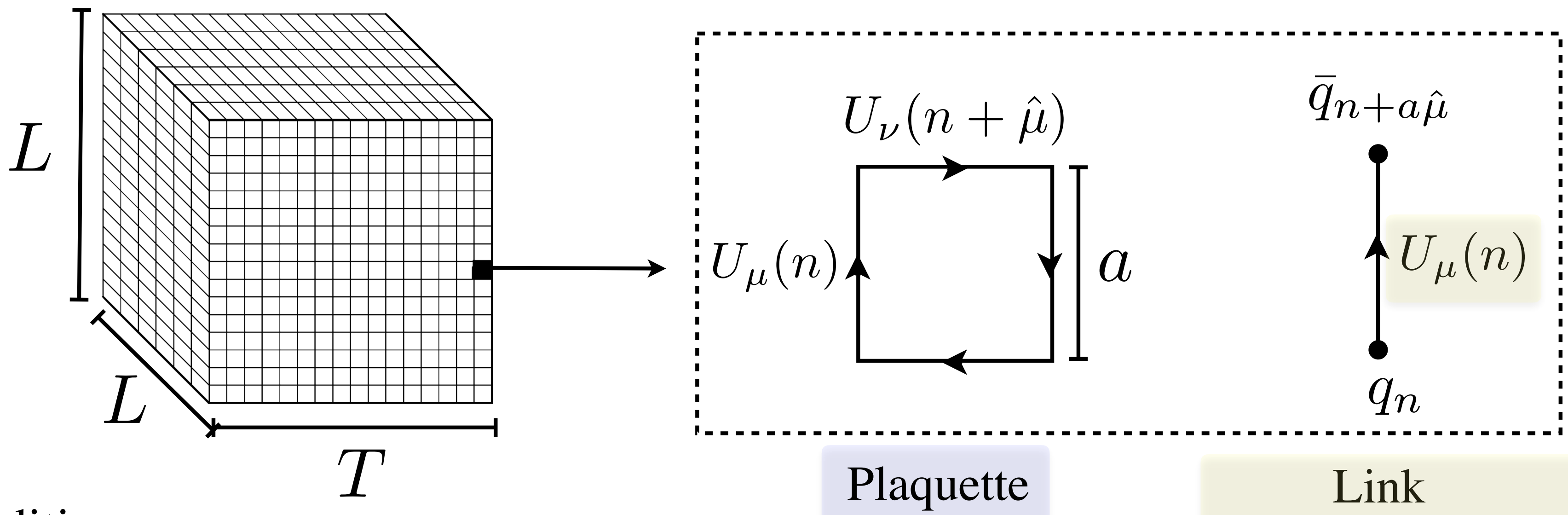

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An example of a discretized action by K. Wilson:

$$S^{(E)}_{\text{Wilson}} = \overset{= 2/g^2}{\frac{\beta}{N_c}} \sum_n \sum_{\mu<\nu} \Re\mathrm{Tr}[\mathbb{1} - P_{\mu\nu;n}]$$
$$- \sum_n \bar{q}_n[\overline{m}^{(0)} + 4]q_n + \sum_n \sum_\mu \left[\bar{q}_n \frac{r - \gamma_\mu}{2} U_\mu(n) q_{n+\hat{\mu}} + \bar{q}_n \frac{r + \gamma_\mu}{2} U^\dagger_\mu(n-\hat{\mu}) q_{n-\hat{\mu}}\right]$$

Wilson parameter. Gives the naive action if set to zero and has doublers problem.

`For discussions of actions consistent with chiral symmetry of continuum see: Kaplan, arXiv:0912.2560 [hep-lat].`

Step II: Generate a large sample of thermalized decorrelated vacuum configurations.

$$\langle \hat{\mathcal{O}} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U_\mu \mathcal{D}q \mathcal{D}\bar{q} \; e^{-S^{(G)}_{\mathrm{lattice}}[U] - S^{(F)}_{\mathrm{lattice}}[U,q,\bar{q}]} \; \hat{\mathcal{O}}[U, q, \bar{q}]$$

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Quark part of expectation values

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Define: $\langle\hat{\mathcal{O}}\rangle_F = \frac{1}{\mathcal{Z}_F}\int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S^{(F)}_{\text{lattice}}[U,q,\bar{q}]}\mathcal{O}[q,\bar{q},U]$

$\mathcal{Z}_F = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S^{(F)}_{\text{lattice}}[U,q,\bar{q}]} = \prod_f \det D_f$ Dirac matrix

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$$\Downarrow$$

$$\langle \hat{\mathcal{O}} \rangle = \frac{1}{N} \sum_i^N \langle \hat{\mathcal{O}} \rangle_F [U^{(i)}]$$

N number of $U^{(i)}$ sampled from the distribution: $\frac{1}{\mathcal{Z}} e^{-S^{(G)}_{\text{lattice}}[U]} \prod_f \det D_f$

Steps II is computationally costly…

Example: Consider a lattice with: $L/a = 48, \; T/a = 256$

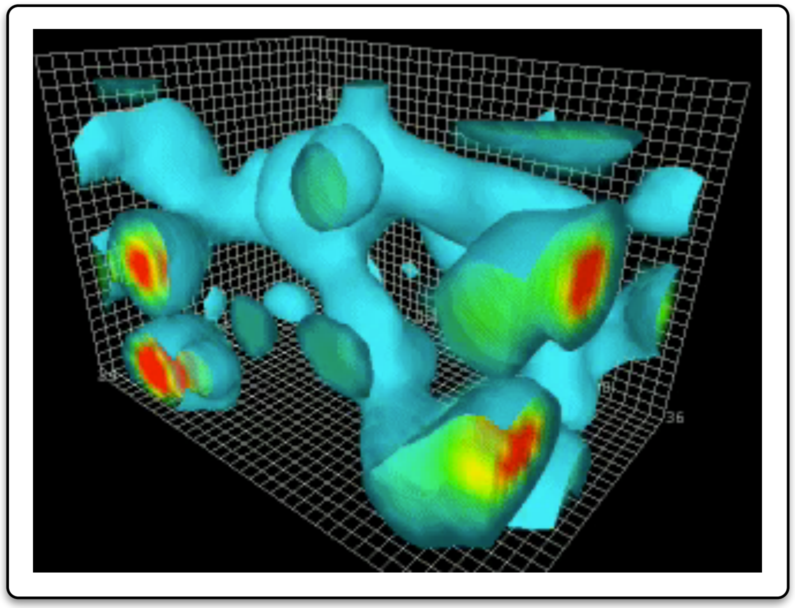
Sampling SU(3) matrices. Already for one sample requires storing

$$8 \times 48^3 \times 256 = 226,492,416$$

c-numbers in the computer!

Requires calculating determinant of a large matrix.

Requires tens of thousands of uncorrelated samples. Molecular-dynamics-inspired hybrid Monte Carlo sampling algorithms often used.



Step III: Form the correlation functions by contracting the quarks. Need to specify the interpolating operators for the state under study.

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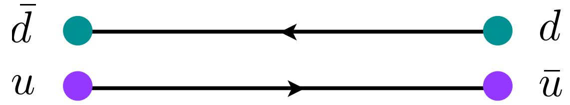
e.g., $\boxed{\hat{O} = \overline{u}\gamma_5 d}$



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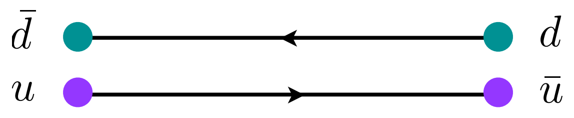
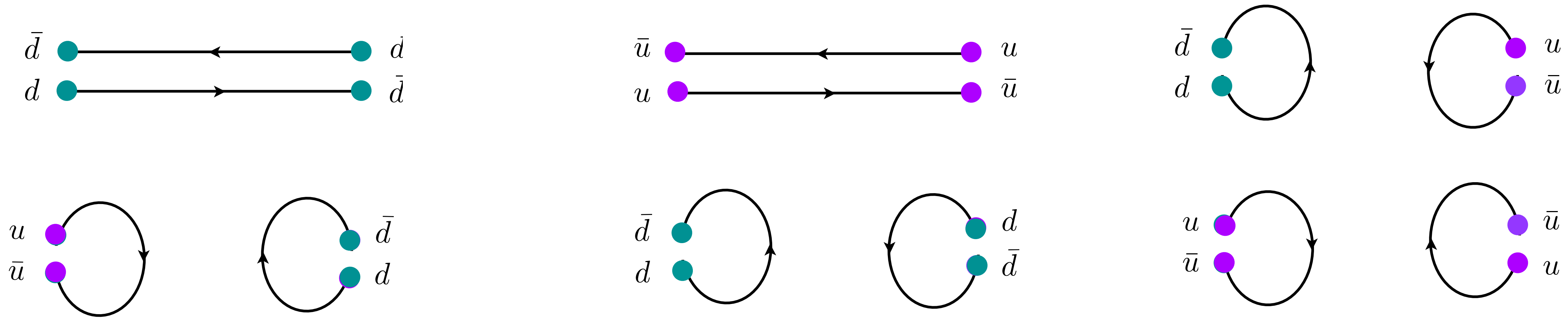


e.g., $\boxed{\hat{O} = \frac{1}{\sqrt{2}}(\bar{u}\gamma^5 u - \bar{d}\gamma^5 d)}$



Quark disconnected diagrams. Require expensive all-to-all propagators.

Steps III is computationally costly…

Example: Consider a lattice with: $L/a = 48, \ T/a = 256$

Solving

$$[D(U)]_{X,Y}[S(U)]_{Y,X_0} = G_{X,X_0}$$

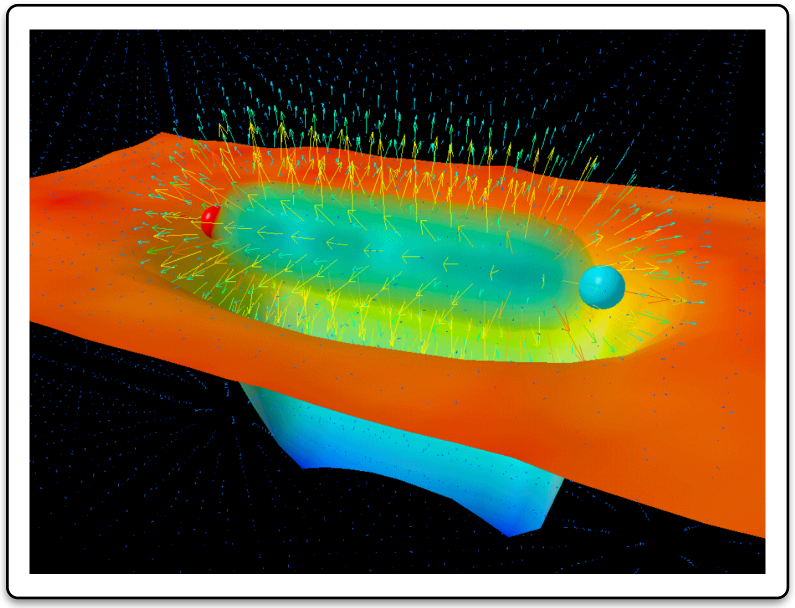
Dirac matrix

Quark propagator

Source

Requires taking determinant and inverting a matrix with dimensions:

$$(4 \times 3 \times 48^3 \times 256)^2 =$$
$$339,738,624 \times 339,738,624$$



EXERCISE 1

Show that for the correlation function of the charged pion:

$$\langle \hat{O}^{\pi^+}(n)\hat{O}^{\pi^+\dagger}(0)\rangle_F \quad = \quad -\mathrm{Tr}\left[D_u^{-1}(n,0)D_d^{-1}(n,0)\right]$$

where D_u^{-1} and D_d^{-1} denote the the inverse Dirac matrix (the quark propagator) for the *u* and *d* quarks, respectively. Trace is over spin and color degrees of freedom.

BONUS EXERCISE 1

Show that for the correlation function of the neutral pion:

$$\begin{aligned}\langle \hat{O}^{\pi^0}(n)\hat{O}^{\pi^0\dagger}(0)\rangle_F \quad = \quad & -\frac{1}{2}\mathrm{Tr}\left[\gamma^5 D_u^{-1}(n,0)\gamma^5 D_u^{-1}(0,n)\right] \\ & +\frac{1}{2}\mathrm{Tr}\left[\gamma^5 D_u^{-1}(n,n)\right]\,\mathrm{Tr}\left[\gamma^5 D_u^{-1}(0,0)\right] \\ & -\frac{1}{2}\mathrm{Tr}\left[\gamma^5 D_u^{-1}(n,n)\right]\,\mathrm{Tr}\left[\gamma^5 D_d^{-1}(0,0)\right] + \{u \leftrightarrow d\}\end{aligned}$$

Step IV: Extract energies and matrix elements from correlation functions

$$C_{\hat{\mathcal{O}},\hat{\mathcal{O}}'}(\tau;\mathbf{d}) = \sum_{\mathbf{x}} e^{2\pi i \mathbf{d}\cdot\mathbf{x}/L} \langle 0|\hat{\mathcal{O}}'(\mathbf{x},\tau)\hat{\mathcal{O}}^{\dagger}(\mathbf{0},0)|0\rangle = \mathcal{Z}_0'\mathcal{Z}_0^{\dagger} e^{-E^{(0)}\tau} + \mathcal{Z}_1'\mathcal{Z}_1^{\dagger} e^{-E^{(1)}\tau} + \ldots$$

Ground state and a tower of excited states are, in principle, accessible!

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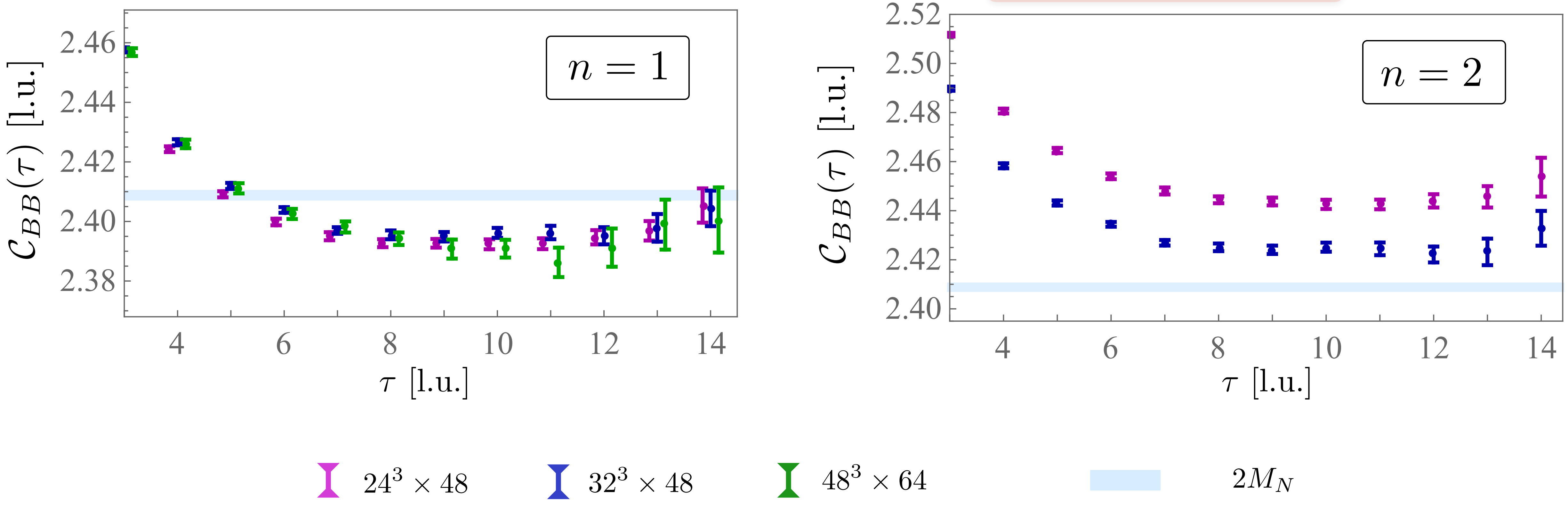
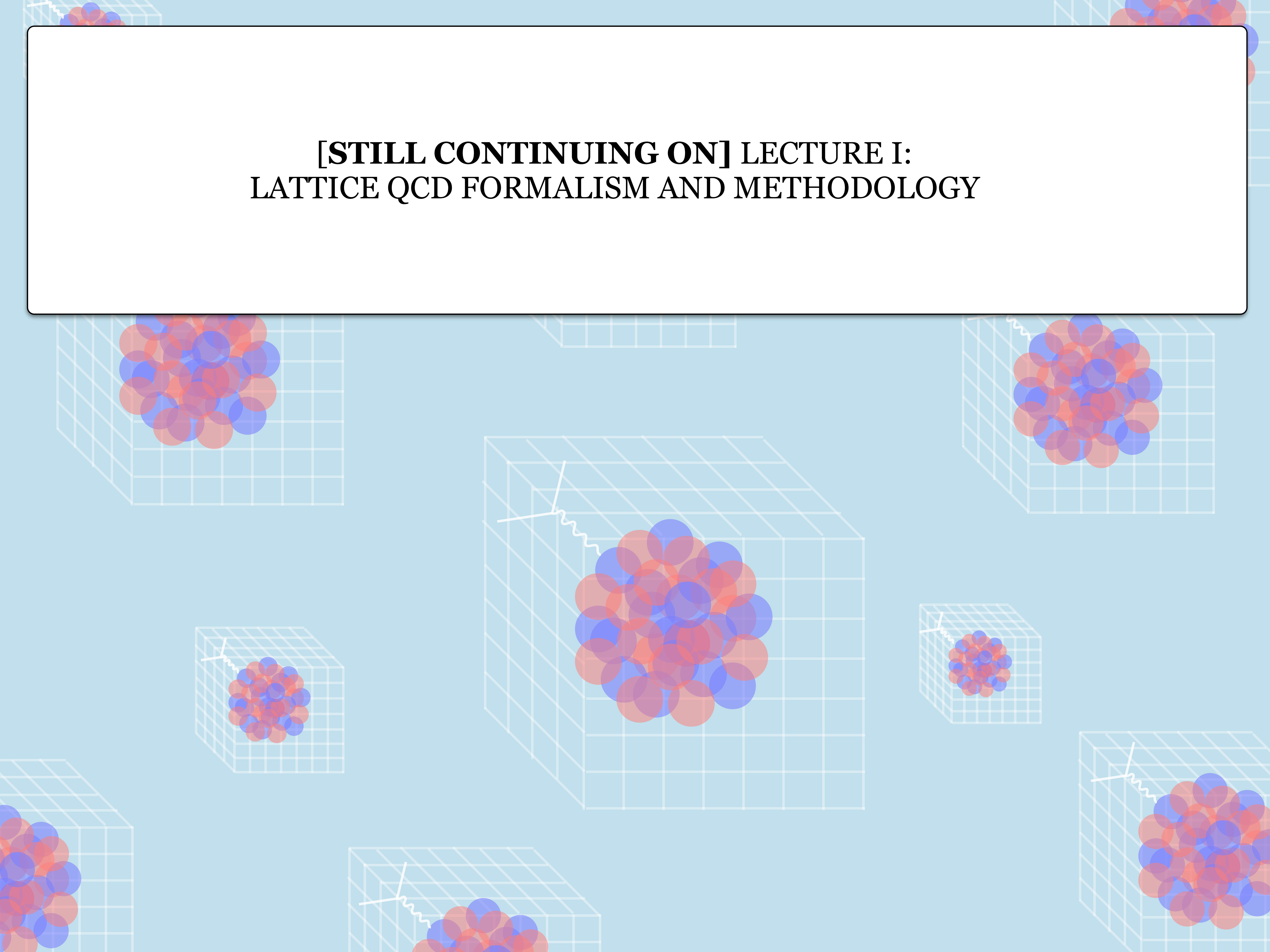
Example: $NN\ ({}^1S_0)$

What should we make of the volume dependence?



Beane et al (NPLQCD), arXiv:1705.09239, Wagman et al (NPLQCD), arXiv:1706.06550.

[STILL CONTINUING ON] LECTURE I: LATTICE QCD FORMALISM AND METHODOLOGY



[Recap] Steps involved in any lattice QCD calculation:

Step I: Discretize the QCD action in both space and time. Consider a finite hypercubic lattice. Wick rotate to imaginary times.

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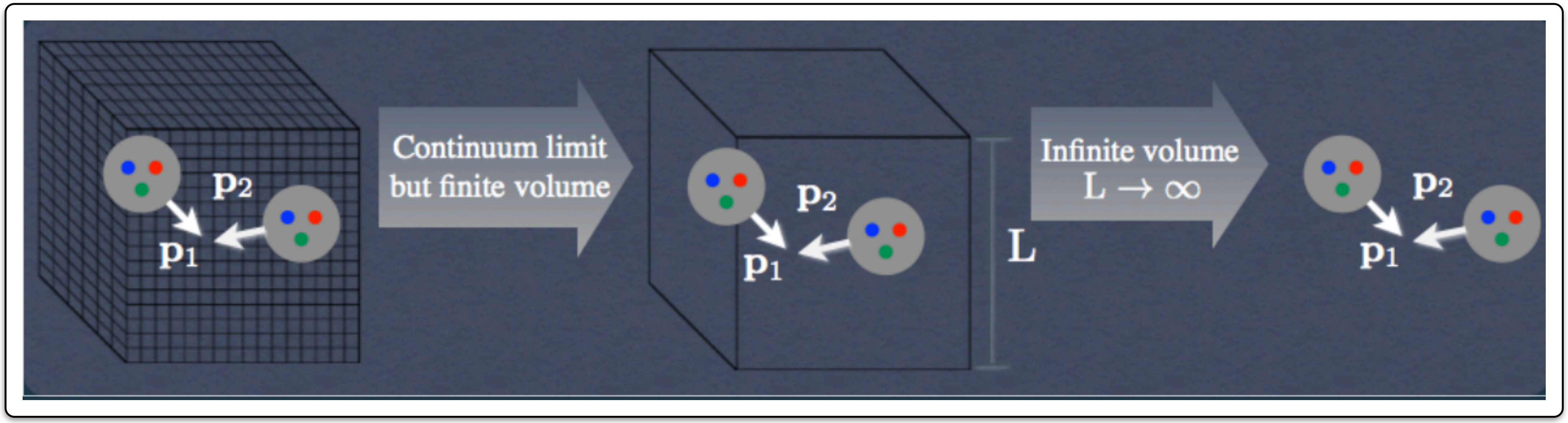
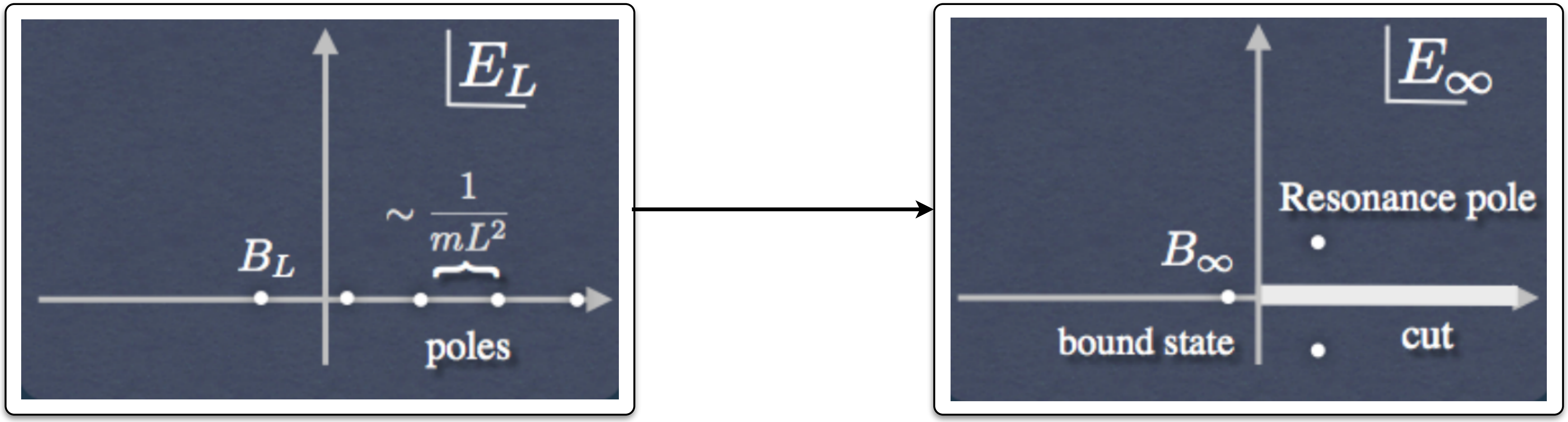
Example: two-hadron scattering





Let’s discuss in greater depth step V:

Step V: make the connection to physical observables, such as scattering amplitudes, decay rates, etc.

i) Finite-volume effects in the single-hadron sector

ii) Finite-volume formalism for two-hadron elastic scattering

iii) Finite-volume formalism for coupled-channel two-hadron inelastic scattering and resonances

iv) Finite-volume formalism for transition amplitudes and resonance form factors

v) Finite-volume formalism for three-hadron scattering and resonances and decays

vi) Finite-volume effects in lattice QED+QCD studies of hadrons

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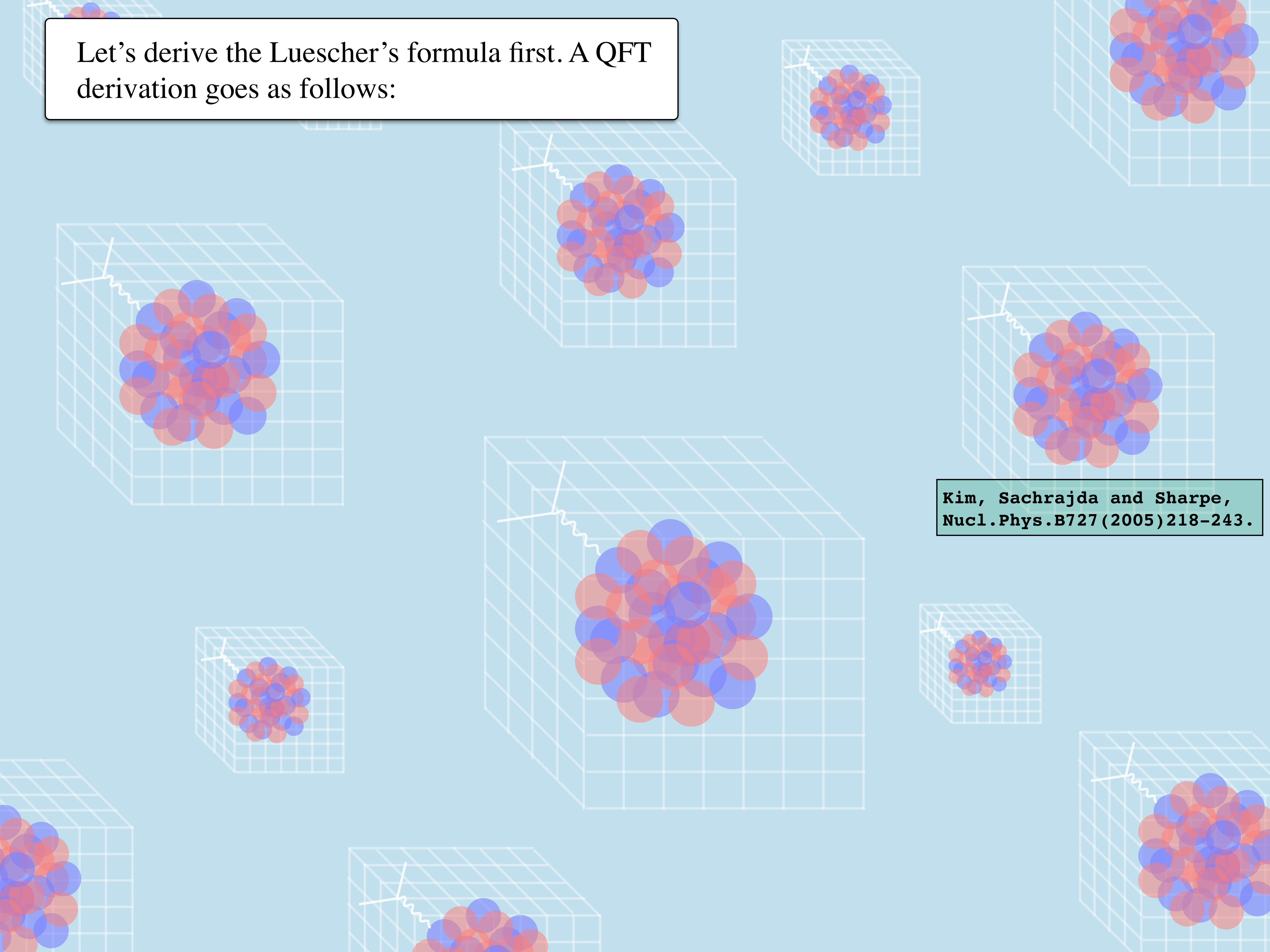

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`Kim, Sachrajda and Sharpe,`
`Nucl.Phys.B727(2005)218-243.`

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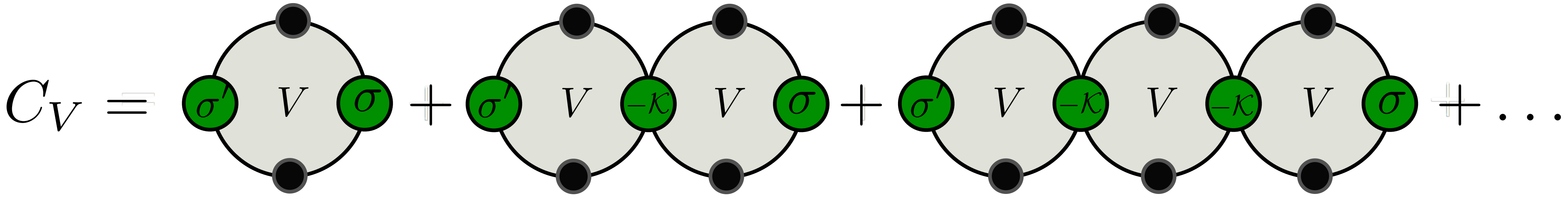

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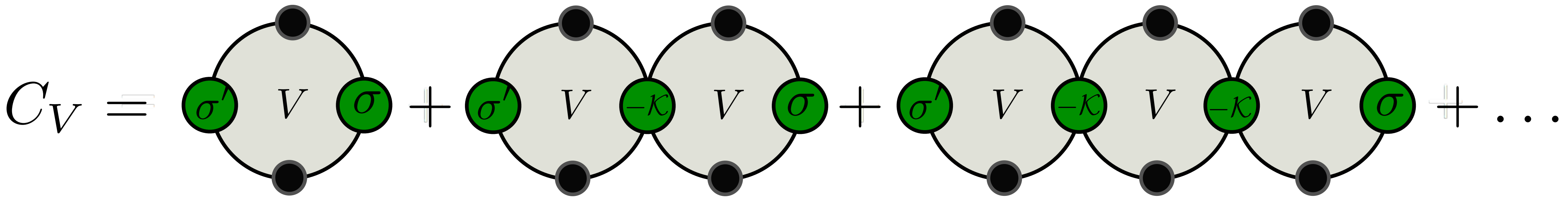

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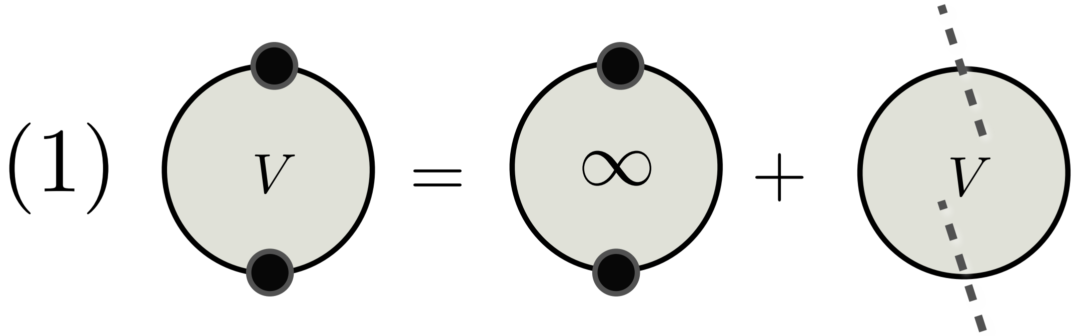

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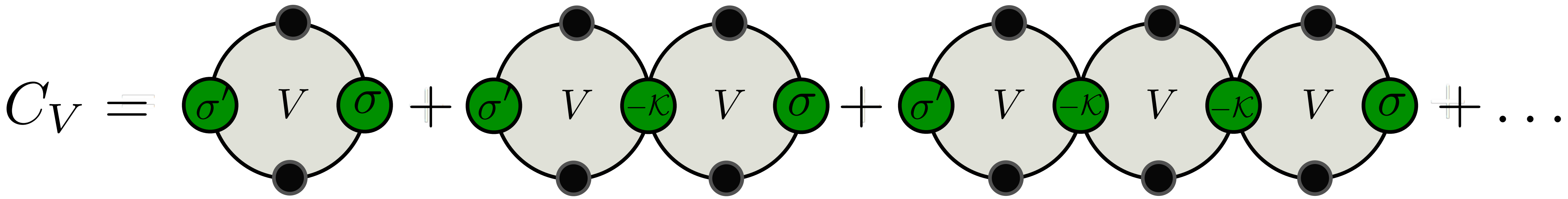

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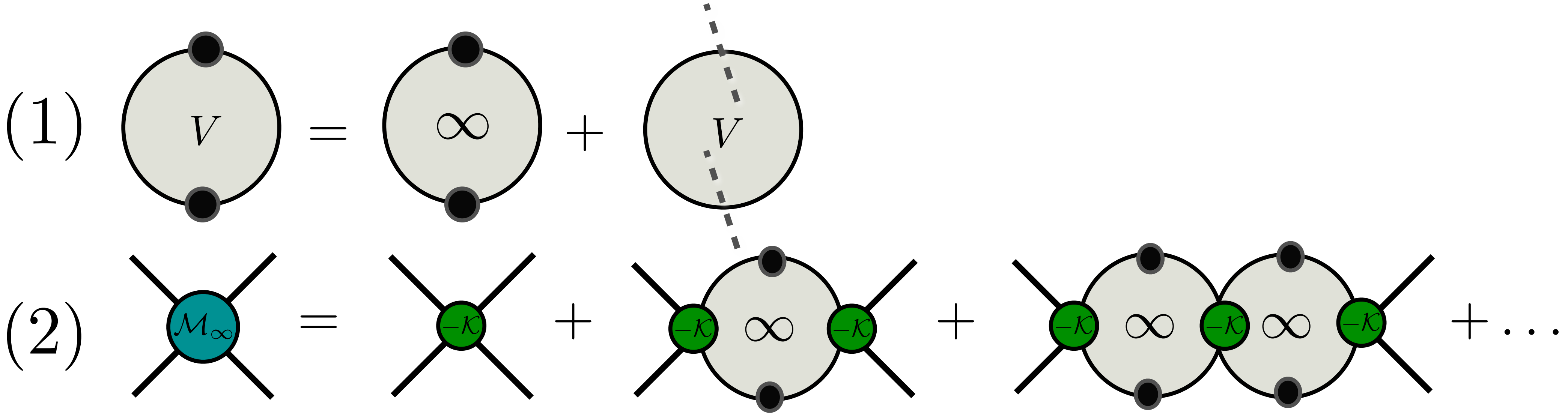

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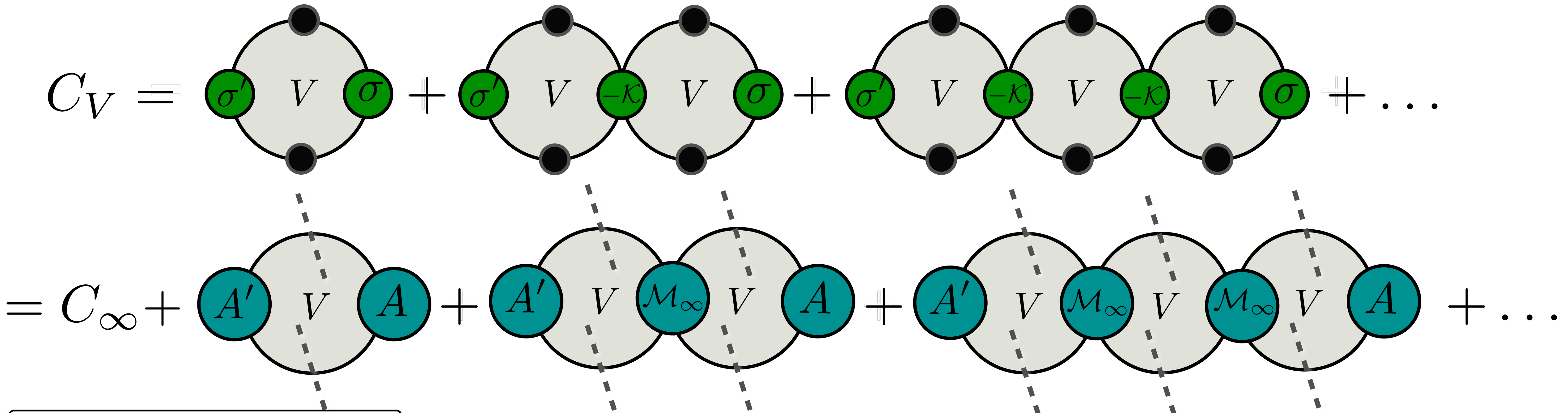

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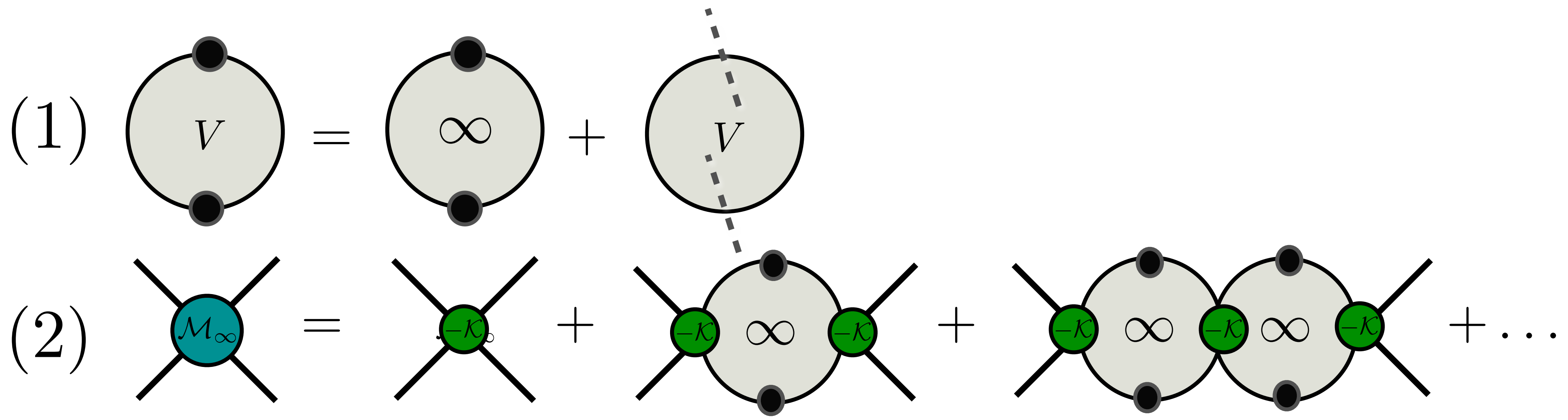

EXERCISE 2

By rearranging the diagrams in C_V (the first line in the upper panel) using the relations in the lower panel, verify the expansion in the second line in the upper panel. What is the relation between $\sigma(\sigma')$ and $A(A')$?

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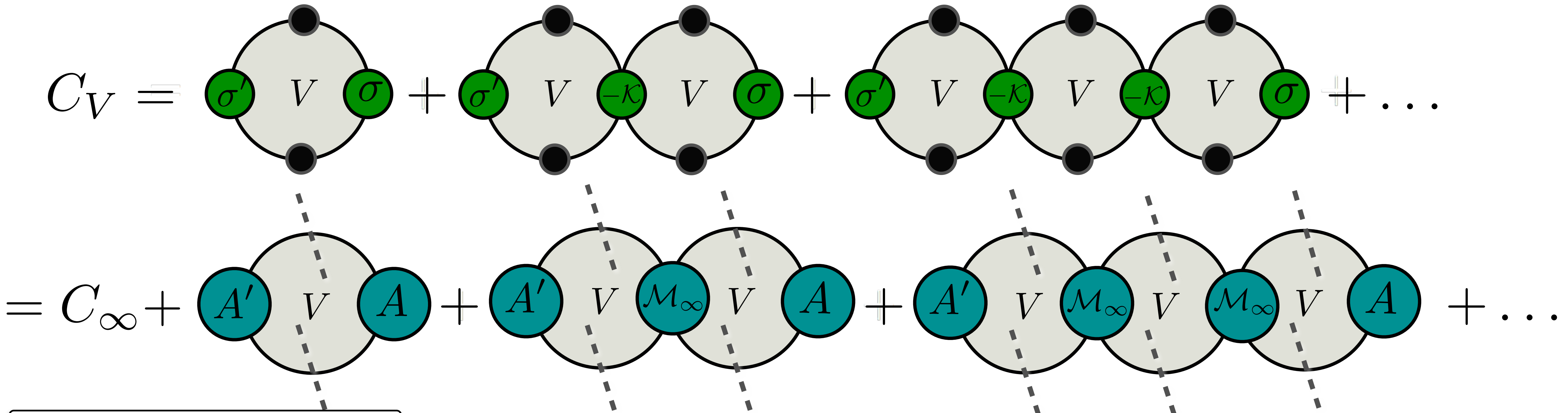

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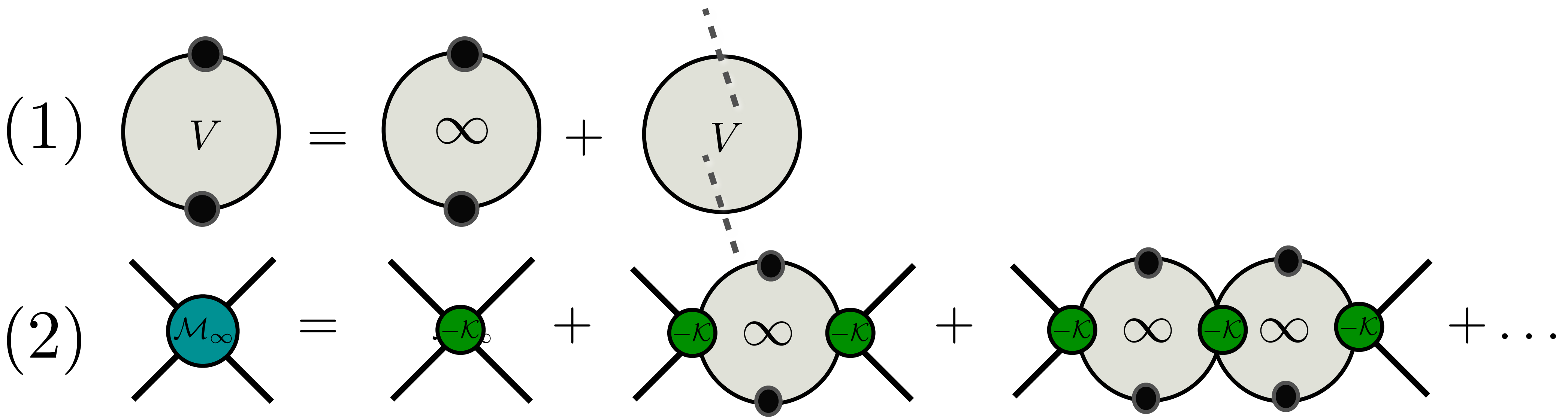

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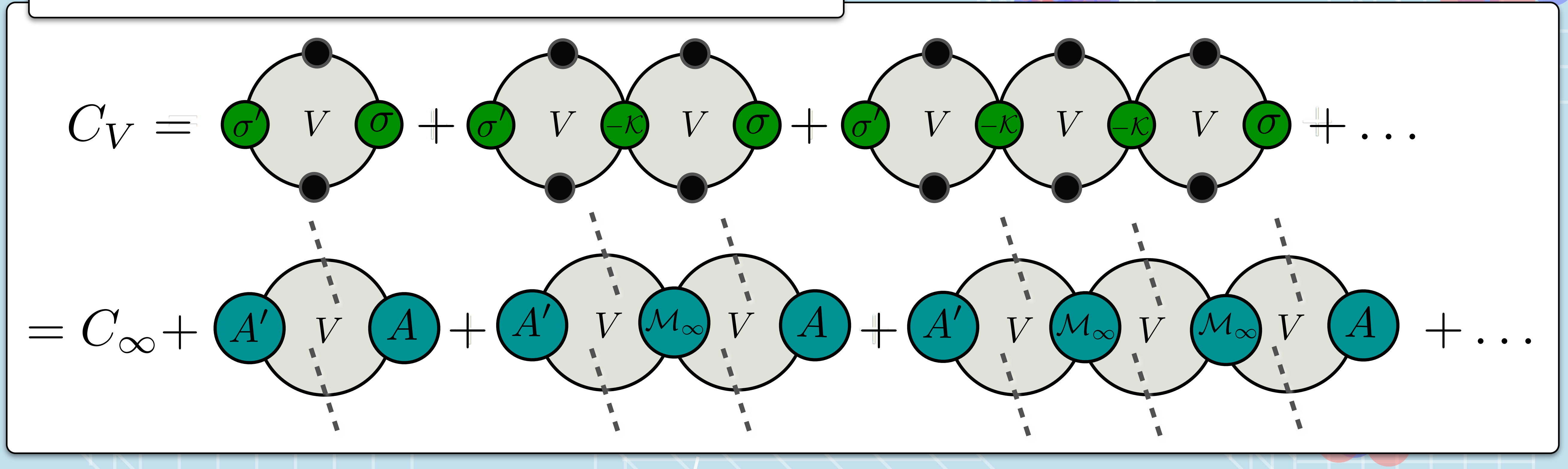


$$\det\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}_\infty^{-1}(E^*)\right] = 0$$

Finite-volume function

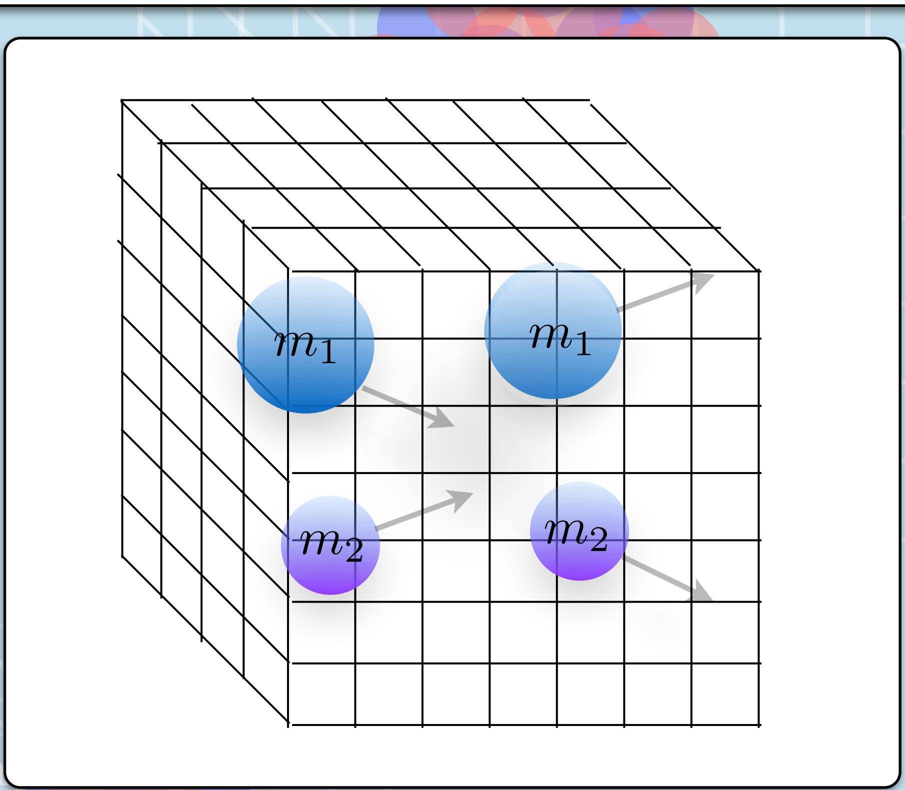
Scattering amplitude

Kim, Sachrajda and Sharpe, Nucl.Phys.B727(2005)218-243.

Poles of C_V which are the finite-volume CM energy eigenvalues.

$$\det\left[\delta\mathcal{G}^V(E^*)+\mathcal{M}_\infty^{-1}(E^*)\right]=0$$

`Luescher (1986, 1991).`



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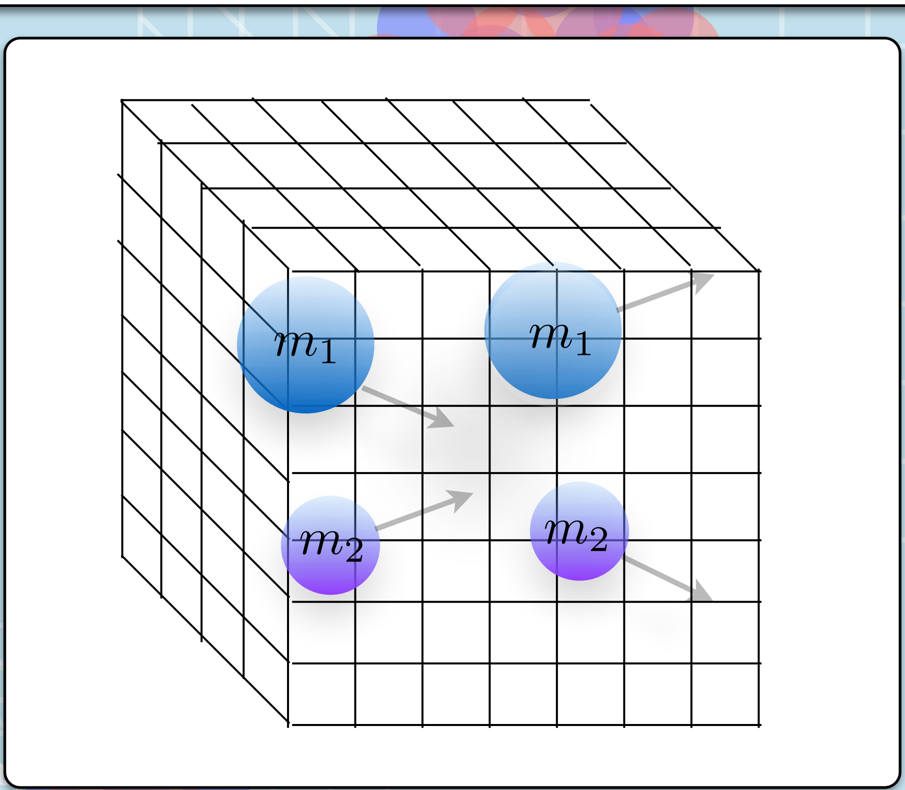


Elastic amplitude more closely…

CM energy

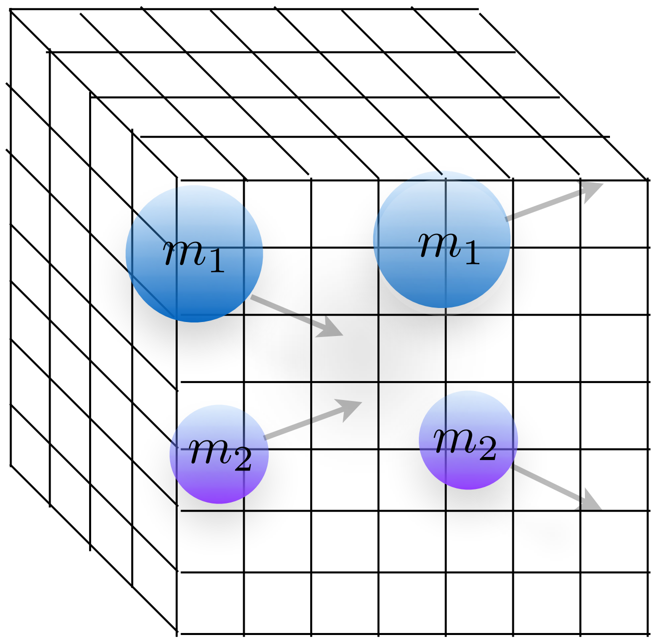
Phase shift

$$(\mathcal{M})_{l_1,m_1;l_2,m_2} = \delta_{l_1,l_2}\delta_{m_1,m_2}\frac{8\pi E^*}{nq^*}\frac{e^{2i\delta^{(l)}(q^*)}-1}{2i}$$

Symmetry factor

$$q^{*2} = \frac{1}{4}\left(E^{*2} - 2(m_1^2+m_2^2) + \frac{(m_1^2-m_2^2)^2}{E^{*2}}\right)$$

$$\det\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}_\infty^{-1}(E^*)\right] = 0$$



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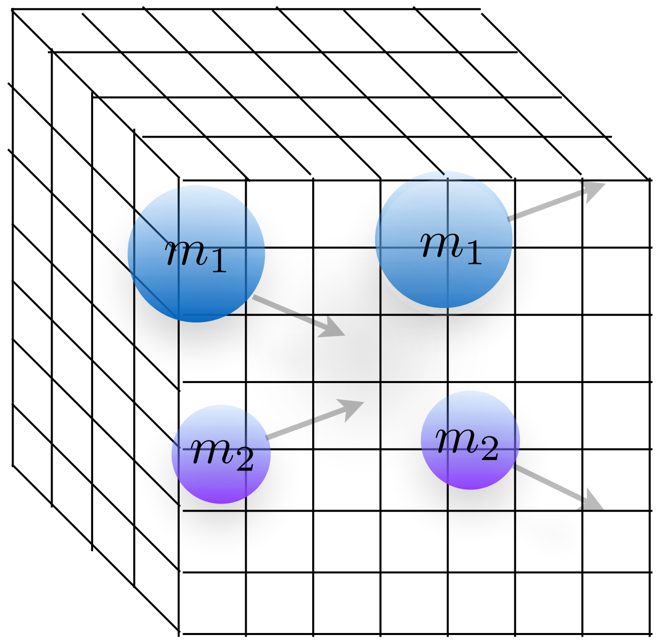


Finite-volume function more closely…

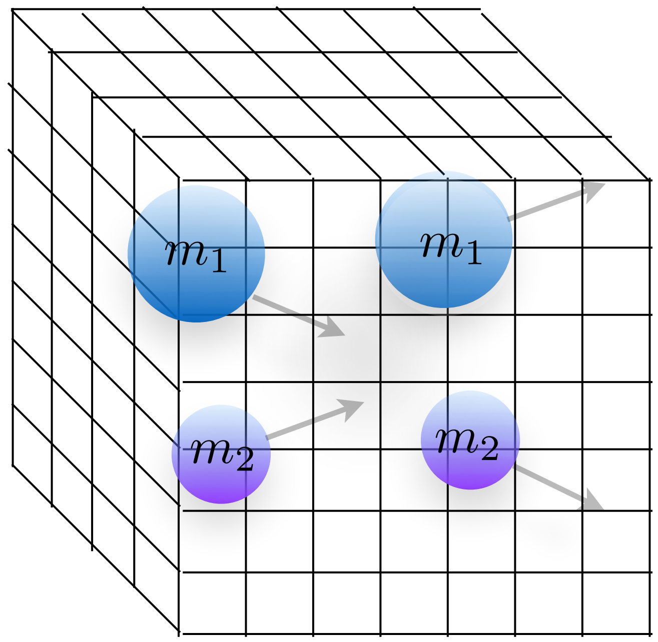
$$(\delta\mathcal{G}^V)_{l_1,m_1;l_2,m_2} = i\frac{q^* n}{8\pi E^*}\left(\delta_{l_1,l_2}\delta_{m_1,m_2} + i\frac{4\pi}{q^*}\sum_{l,m}\frac{\sqrt{4\pi}}{q^{*l}}\, c^{\mathbf{P}}_{lm}(q^{*2})\int d\Omega^* Y^*_{l_1m_1} Y^*_{lm} Y_{l_2m_2}\right)$$

$$c^{\mathbf{P}}_{lm}(x) = \frac{1}{\gamma}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \mathcal{P}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\right]\frac{\sqrt{4\pi}\,Y_{lm}(\hat{k}^*)\ k^{*l}}{k^{*2} - x}$$

$$\mathbf{k}^* = \gamma^{-1}\left[\mathbf{k}_\| - \frac{1}{2}(1 + \frac{m_1^2 - m_2^2}{E^{*2}})\mathbf{P}\right] + \mathbf{k}_\perp$$

`ZD and Savage, Phys. Rev.D84,114502(2011).`

$$\det\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}_\infty^{-1}(E^*)\right] = 0$$



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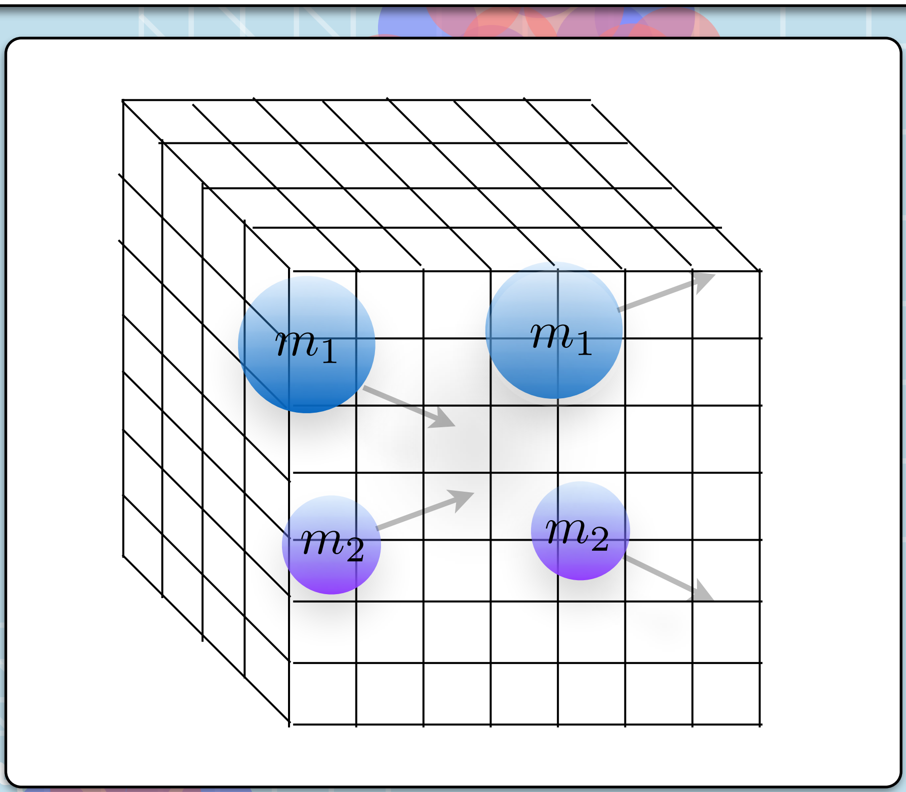


S-wave approximation,
valid at low energies:

$$q^* \cot \delta^{(0)} = 4\pi c^{\mathbf{0}}_{00}(q^{*2})$$

S-wave phase shift

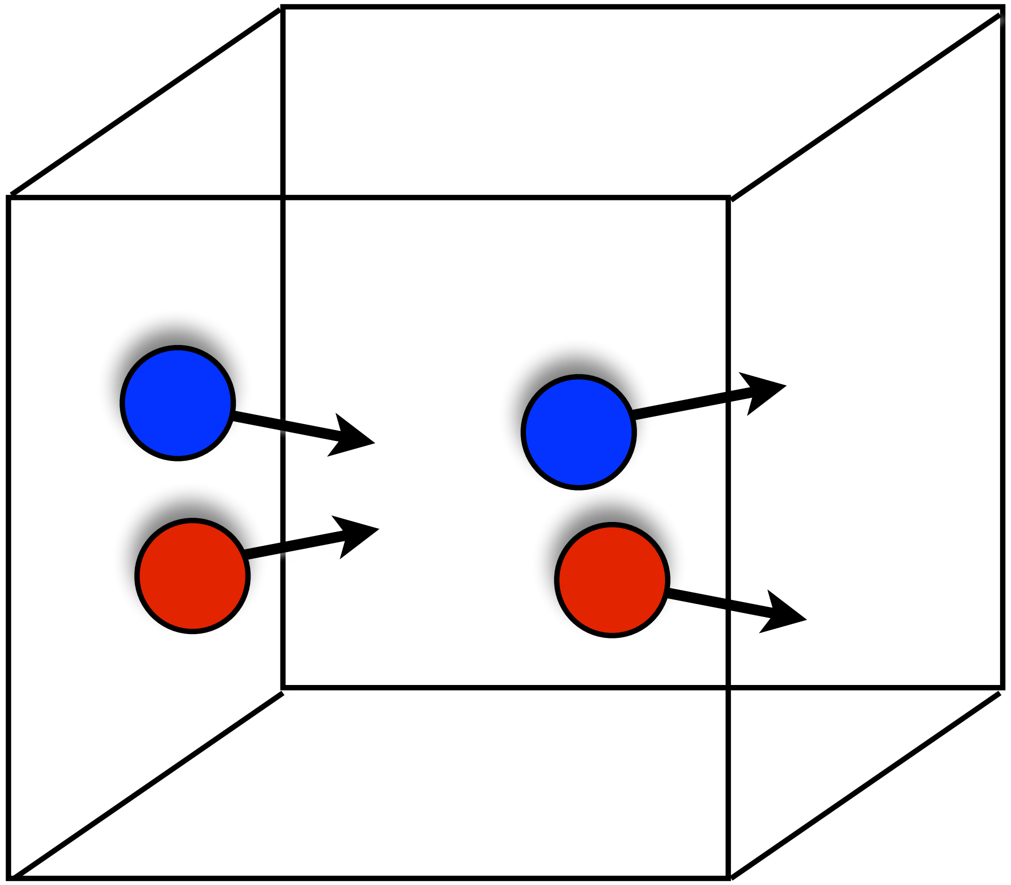
EXERCISE 3

Derive the S-wave limit of Luescher's quantization condition from the master relation.

BONUS EXERCISE 2

Plot the S-wave finite-volume function $c_{00}^{\mathbf{0}}$ for a range of momenta q^{*2}, including negative values. At what values of q^{*2} do you observe singularities? What do these momenta correspond to?

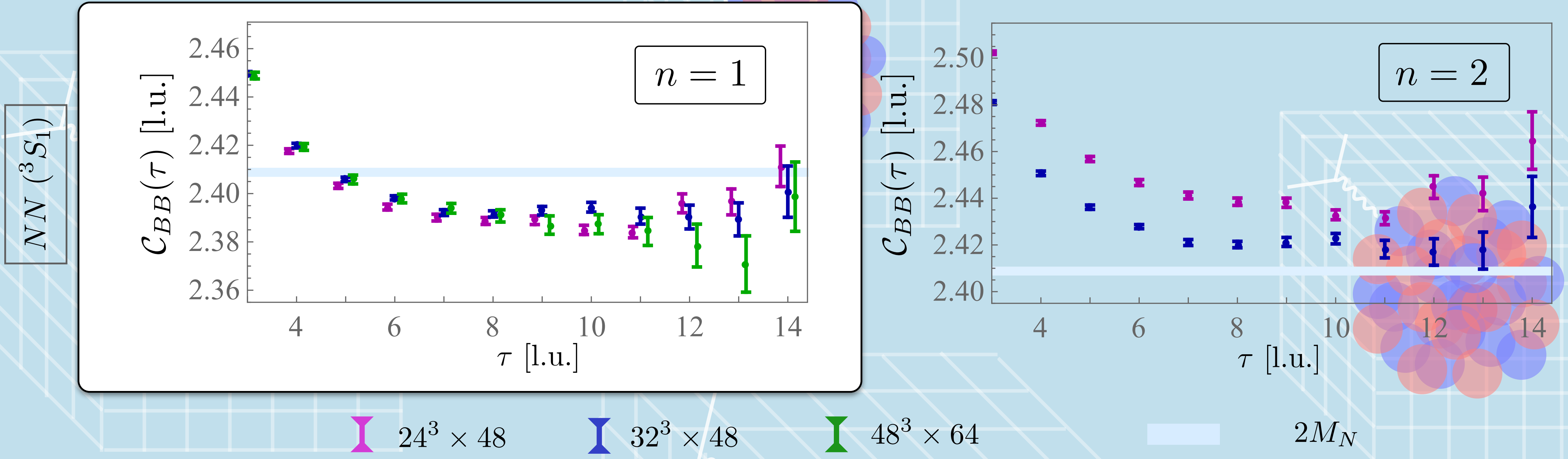
Now let’s see an application of Luescher’s method to obtain elastic scattering amplitudes of two nucleon from lattice QCD (at a large quark mass!):



Wagman et al.(NPLQCD), Phys.Rev.D 96,114510(2017).

Step 1: Obtain the lowest-lying spectra

$N_f = 3,\ m_\pi = 0.806\ \mathrm{GeV},\ a = 0.145(2)\ \mathrm{fm}$



Beane et al (NPLQCD), arXiv:1705.09239, Wagman et al (NPLQCD), arXiv:1706.06550.

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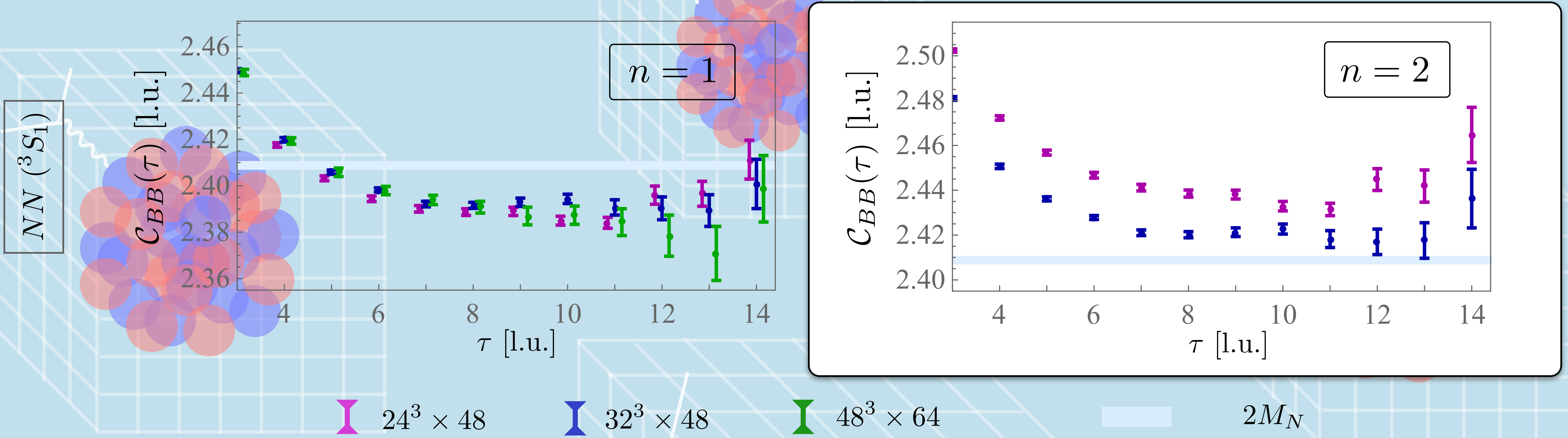

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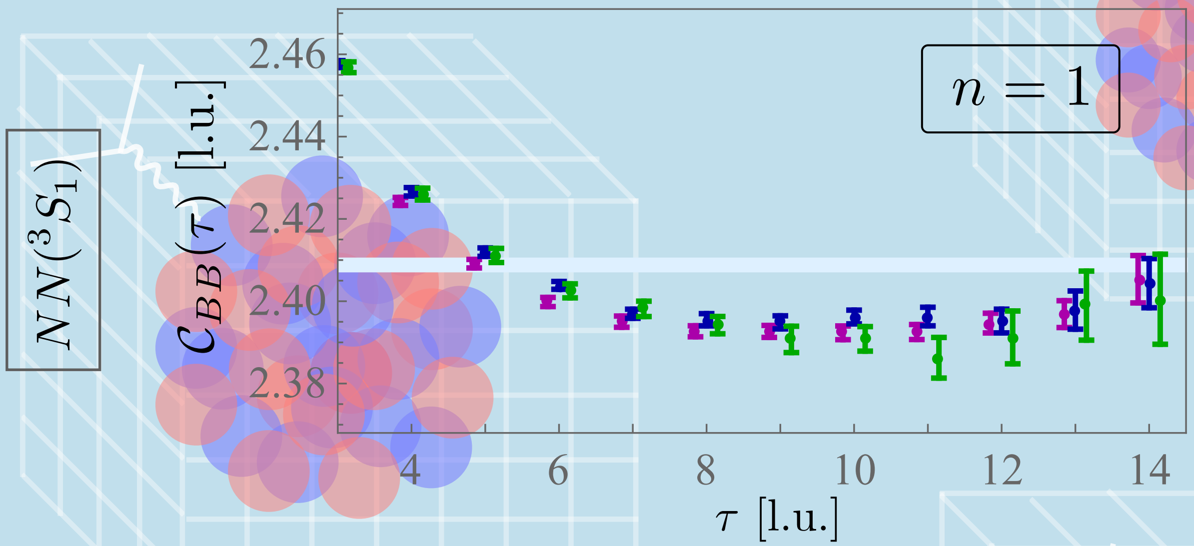

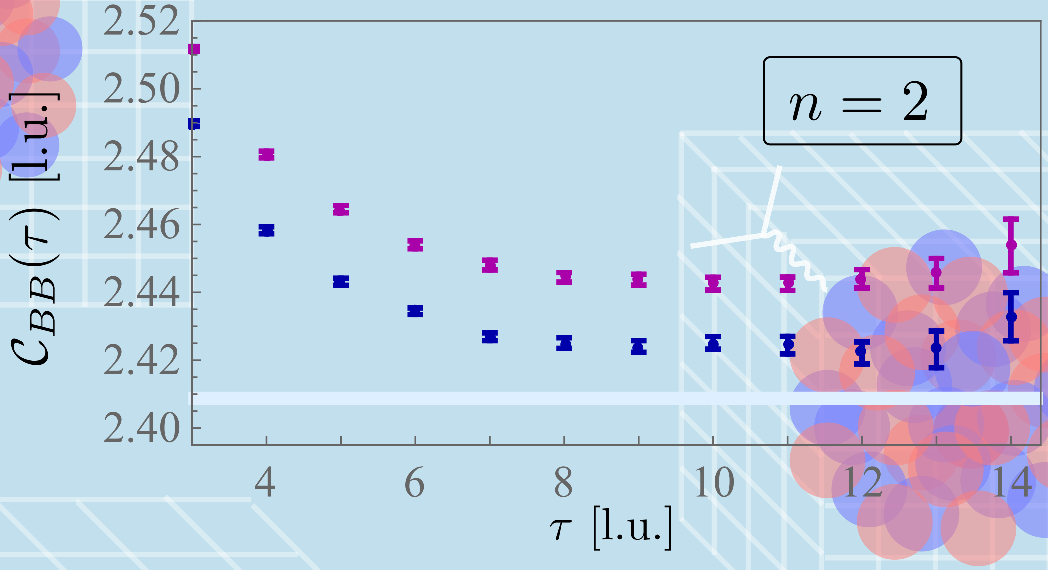

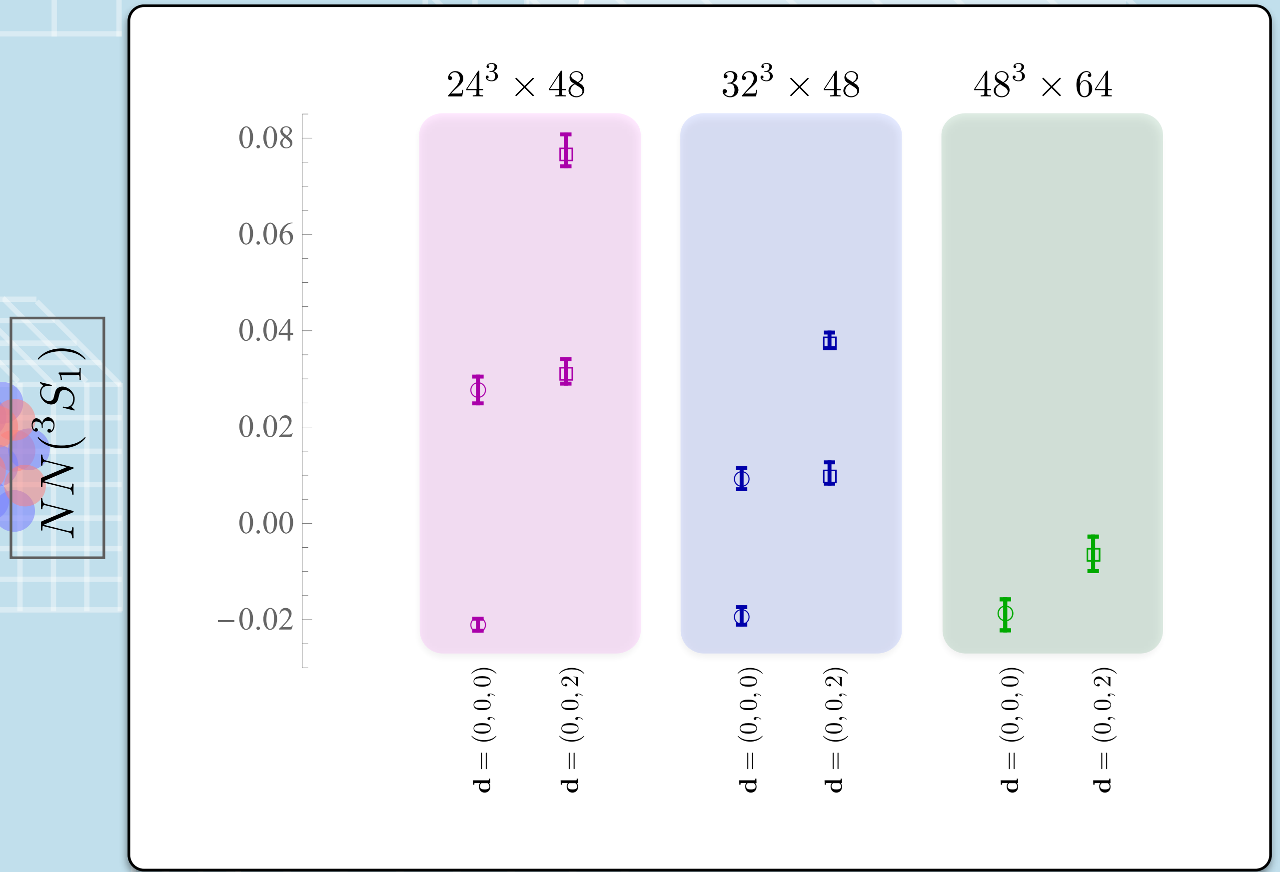

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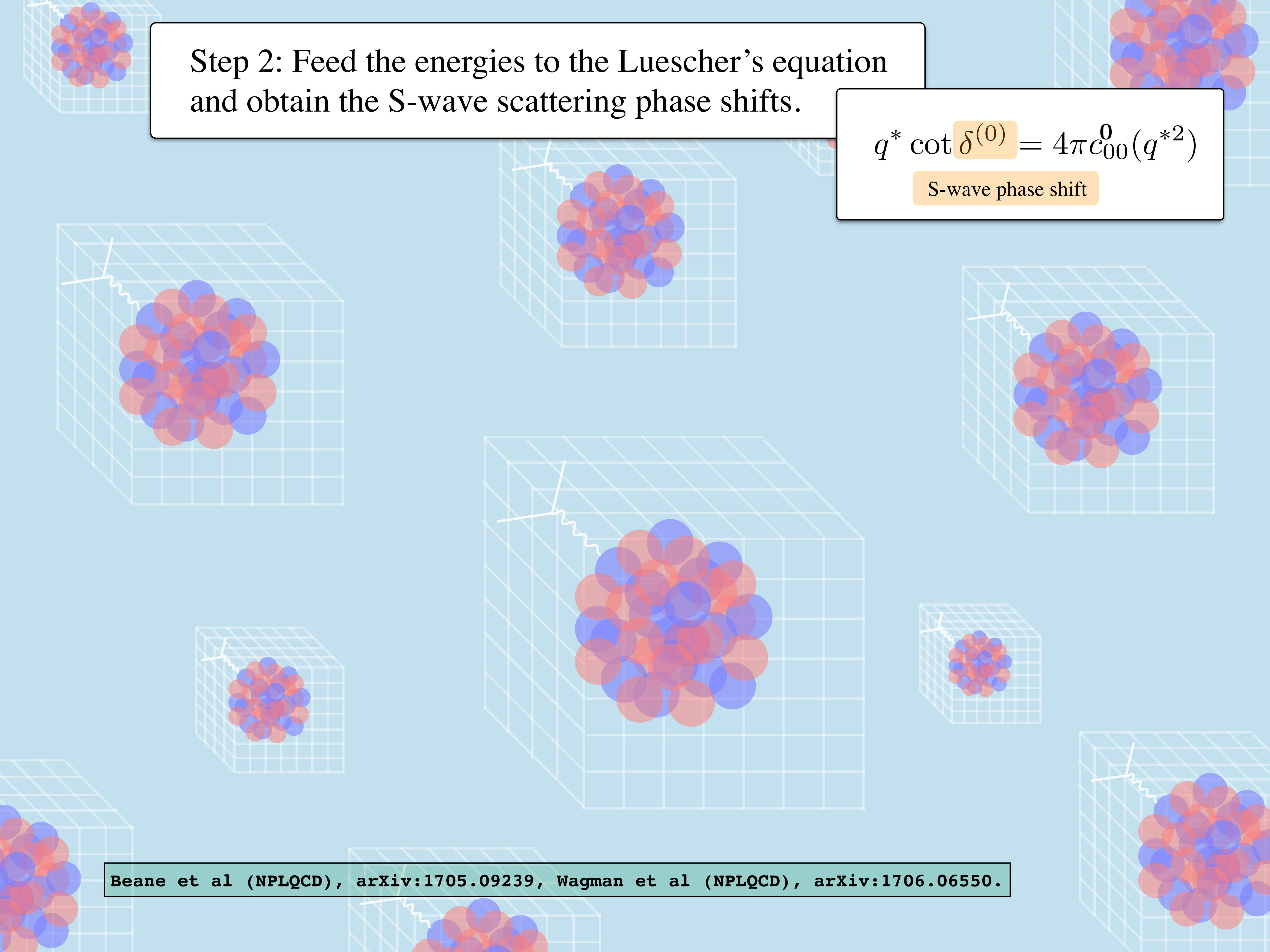


Step 2: Feed the energies to the Luescher's equation and obtain the S-wave scattering phase shifts.

$$q^* \cot \delta^{(0)} = 4\pi c^{\mathbf{0}}_{00}(q^{*2})$$

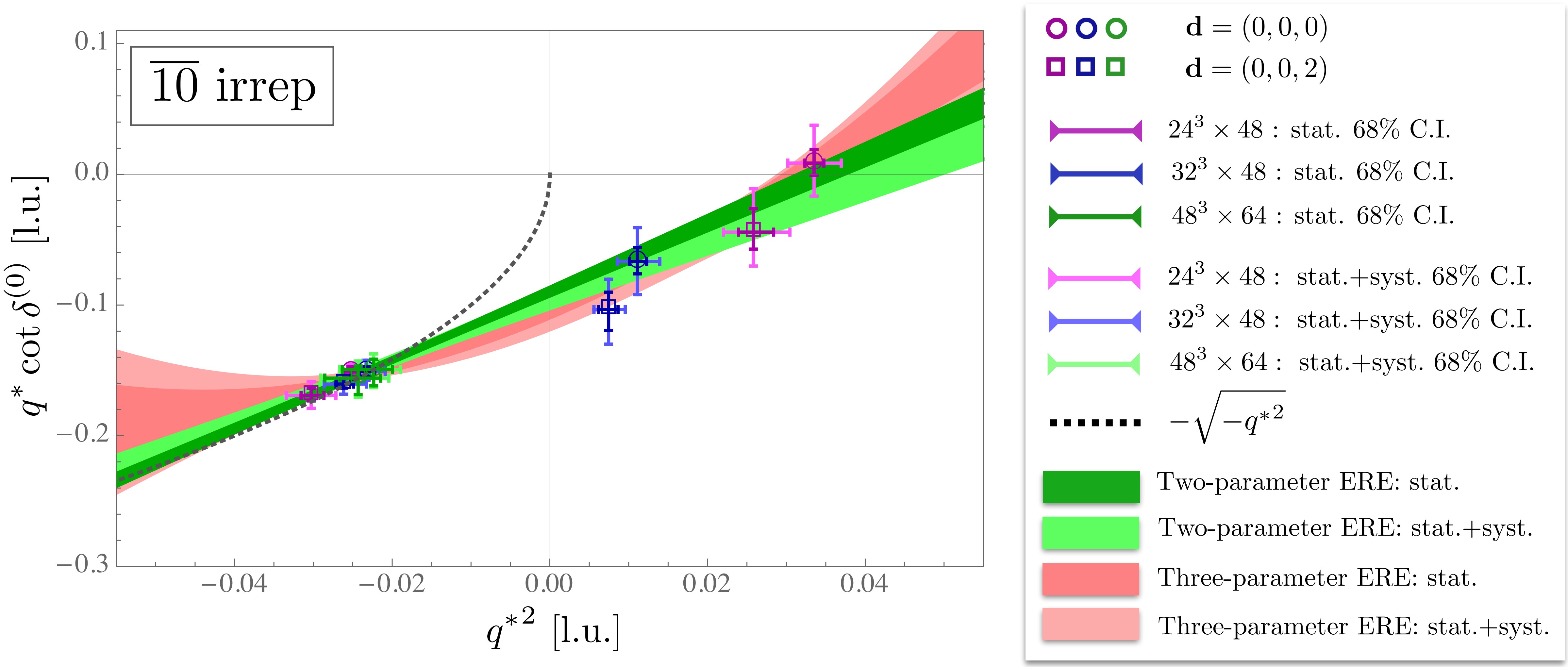
S-wave phase shift

Beane et al (NPLQCD), arXiv:1705.09239, Wagman et al (NPLQCD), arXiv:1706.06550.

Step 2: Feed the energies to the Luescher's equation and obtain the S-wave scattering phase shifts.

$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$

$$q^* \cot \delta^{(0)} = -\frac{1}{a} + \frac{1}{2} r q^{*2} + \cdots$$



$$B = 27.9^{(+3.1)(+2.2)}_{(-2.3)(-1.4)}\ \text{MeV}$$

Beane et al (NPLQCD), arXiv:1705.09239, Wagman et al (NPLQCD), arXiv:1706.06550.

Let’s discuss in greater depth step V:

Step V: make the connection to physical observables, such as scattering amplitudes, decay rates, etc.

i) Finite-volume effects in the single-hadron sector

ii) Finite-volume formalism for two-hadron elastic scattering

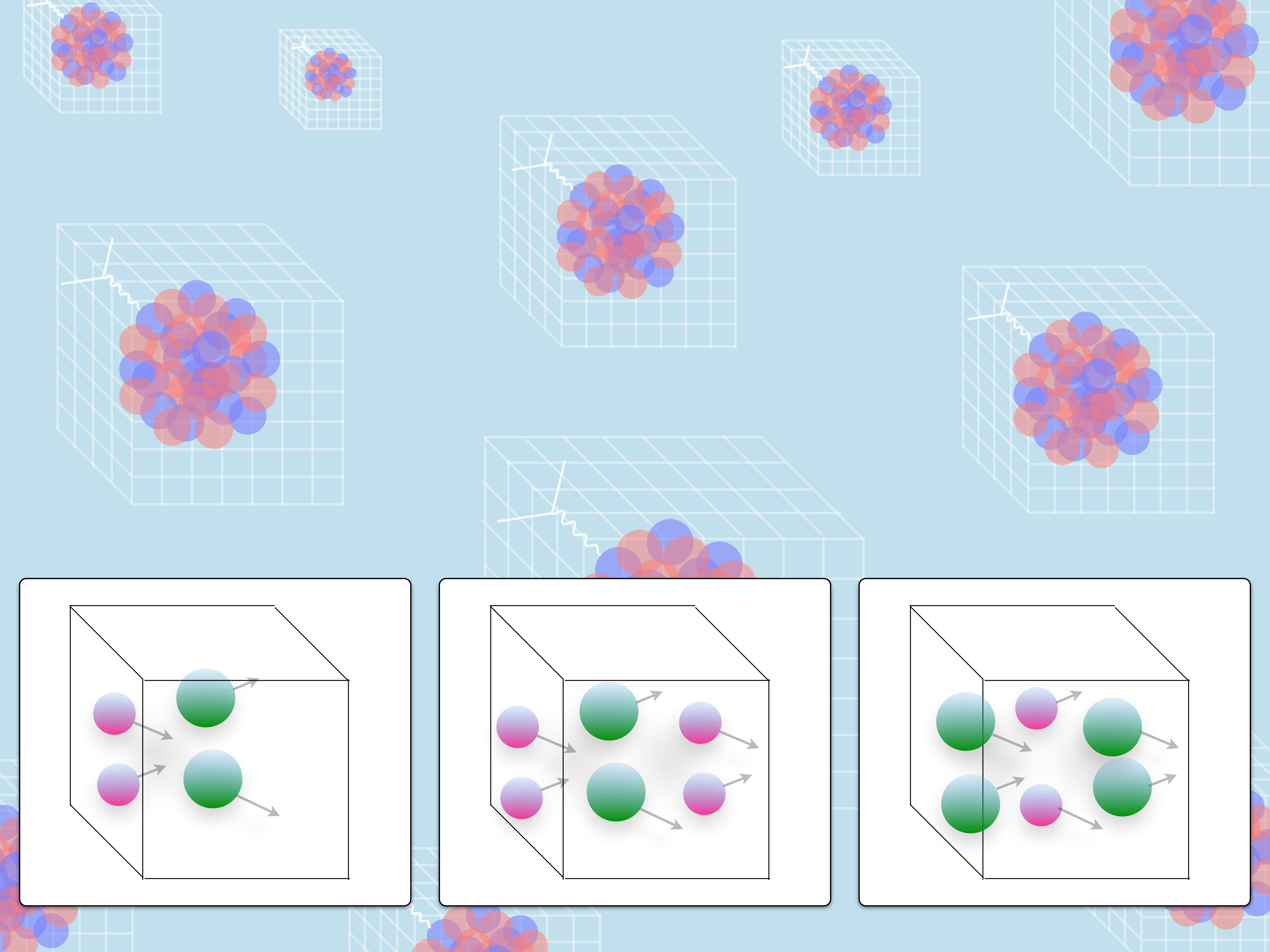
⇨ iii) Finite-volume formalism for coupled-channel two-hadron elastic scattering and resonances

iv) Finite-volume formalism for transition amplitudes and resonance form factors

v) Finite-volume formalism for three-hadron scattering and resonances

vi) Finite-volume effects in lattice QED+QCD studies of hadrons

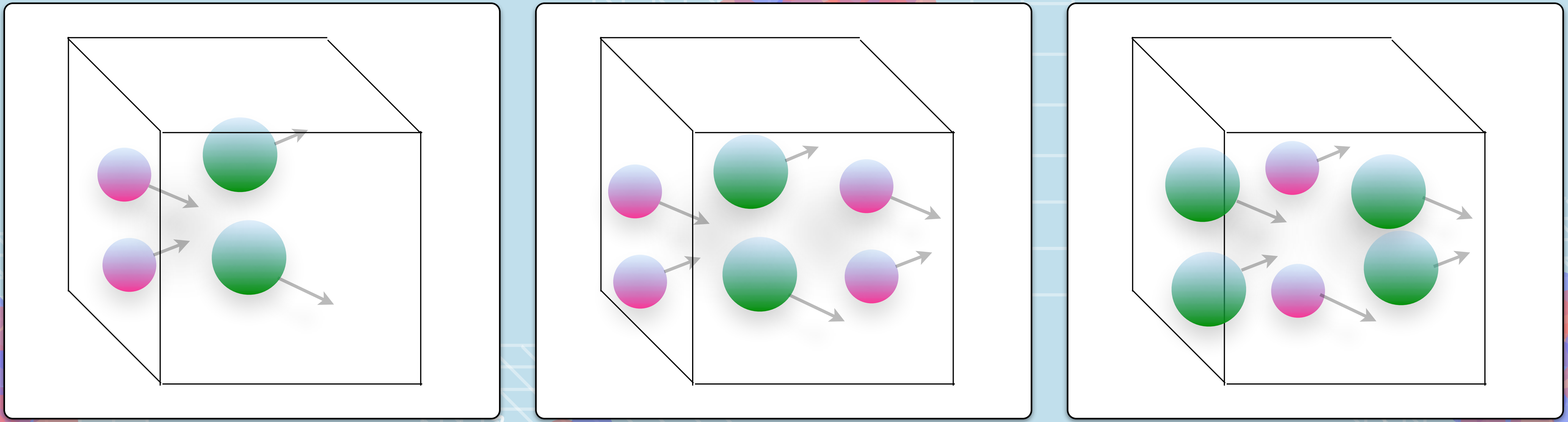
`See e.g., ZD, arXiv:1409.1966 [hep-lat, Briceno, Dudek and Young, Rev. Mod. Phys. 90.025001, Ann. Rev. Nucl. Part. Sci. 69 (2019).`



Briceno and ZD, Phys. Rev.D88,094507(2013).

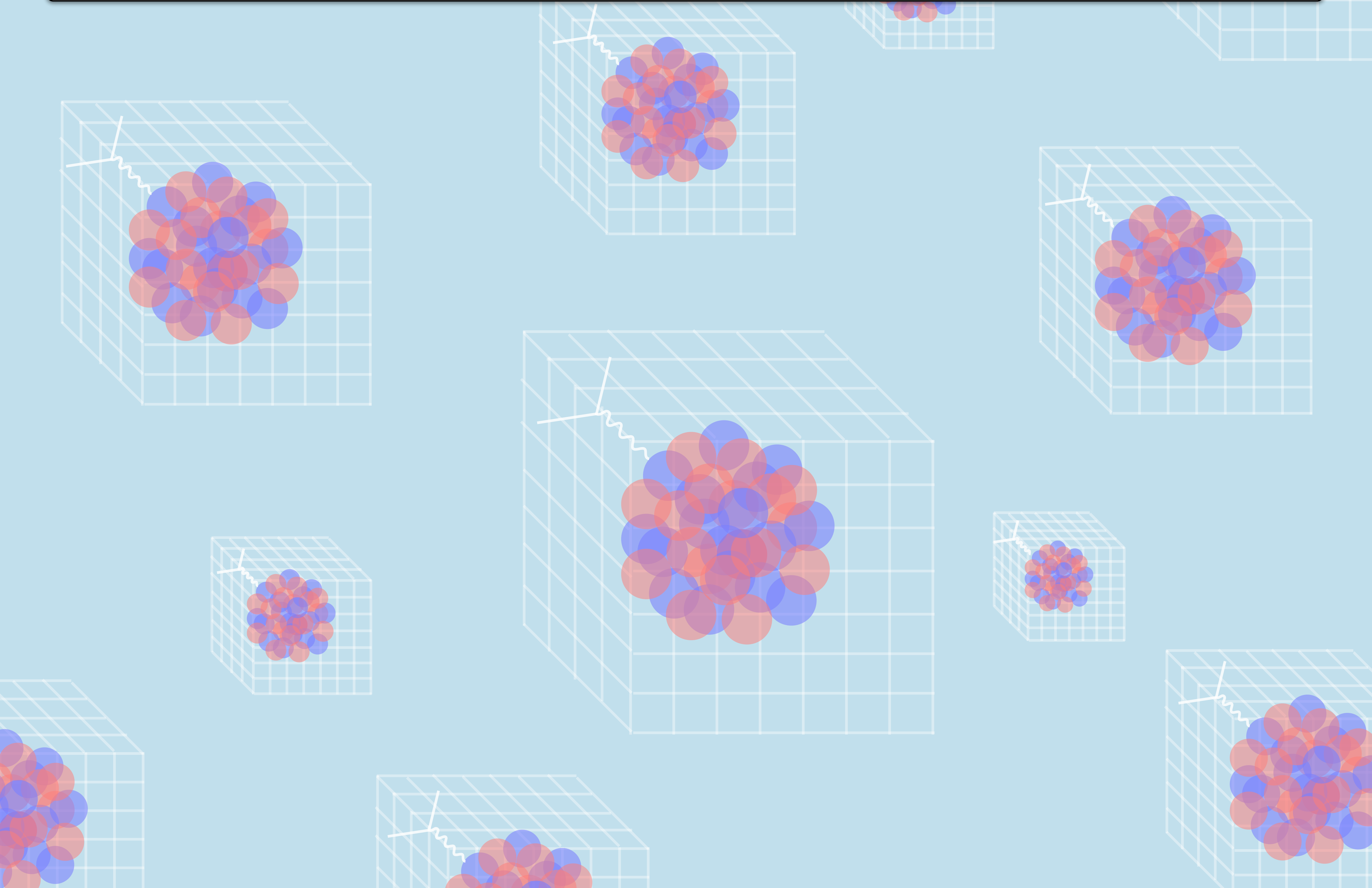
Hansen and Sharpe, Phys. Rev. D86, 016007(2012).

$$\mathrm{Det}\left[\delta\mathcal{G}^{V}(E^{*})+\mathcal{M}_{\infty}^{-1}(E^{*})\right]=0$$



Now let's see an application of the coupled-channel formalism: Hunting resonances using lattice QCD in the P-wave coupled $\pi\pi - K\overline{K}$ channel

Wilson et al.(HadSpec), Phys.Rev. D92 (2015), 094502



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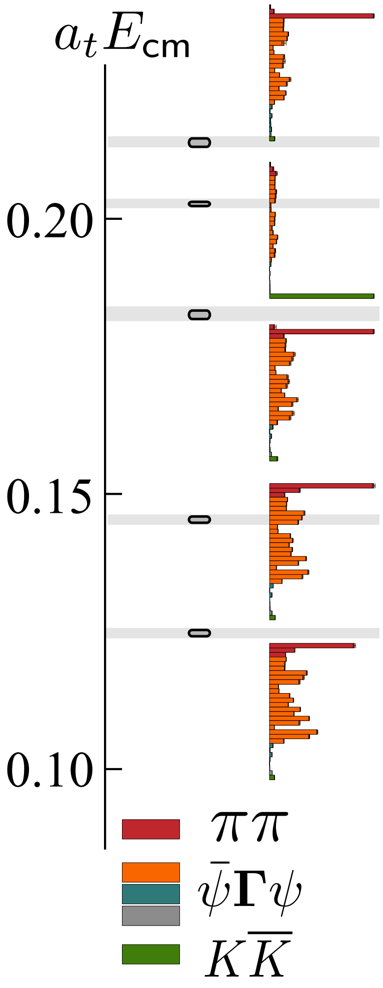

Example: T1 irrep energies

$$N_f = 2 + 1, m_\pi = 236 \text{ MeV}, \; V \approx (4 \text{ fm})^3$$

Wilson et al.(HadSpec),
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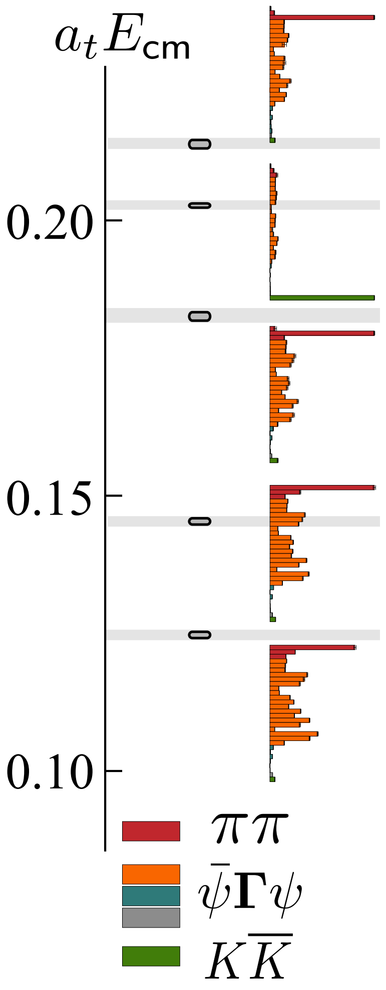

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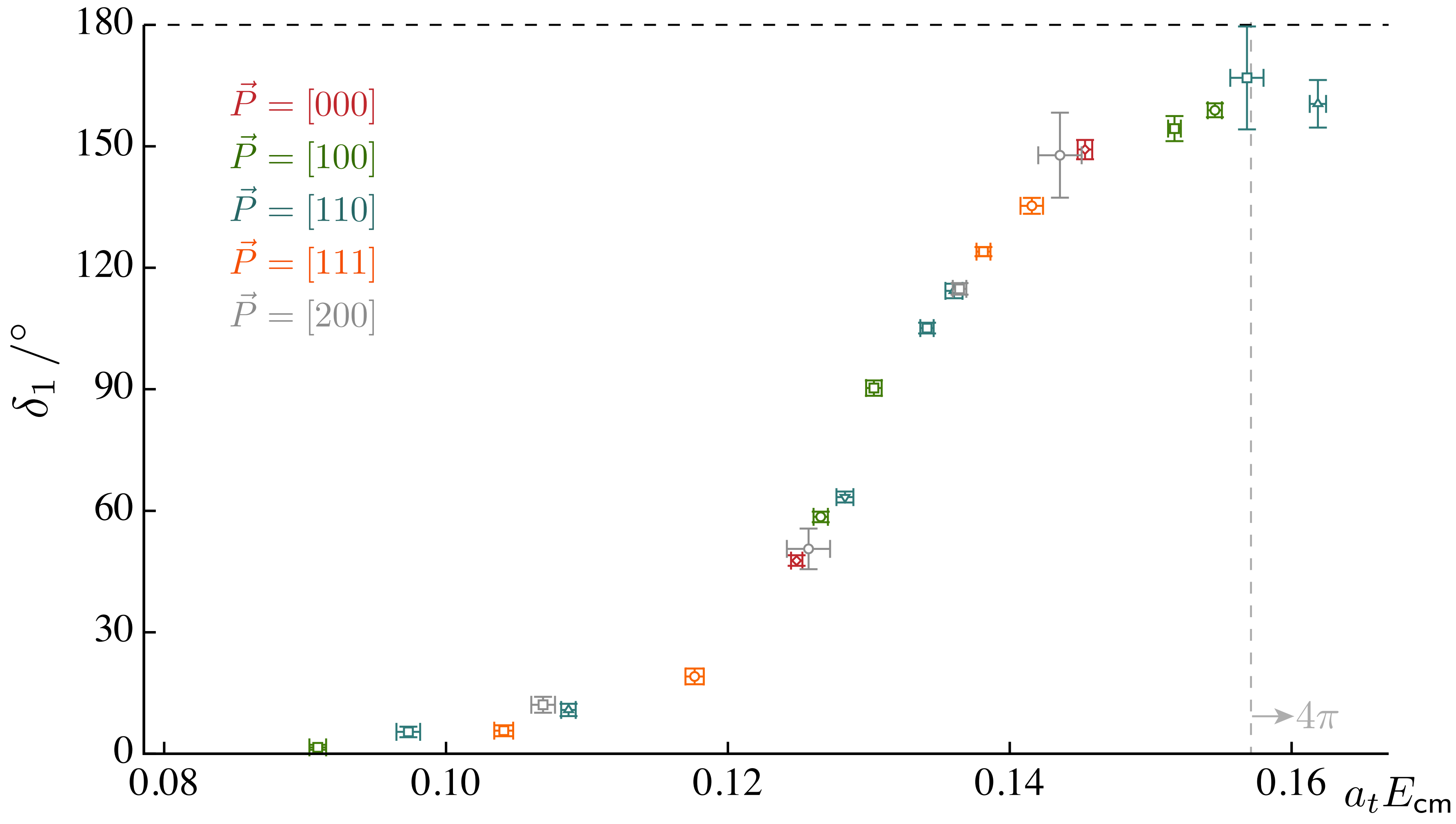
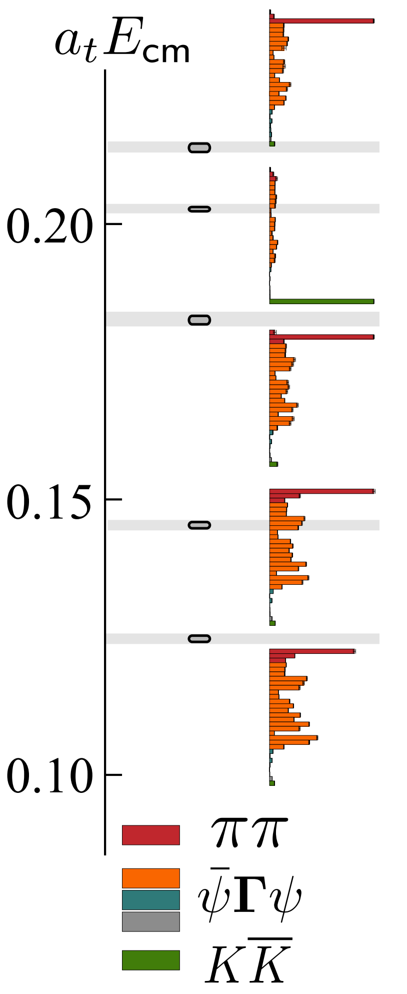


P-wave $\pi\pi$ phase shift as a function of energy

$$N_f = 2 + 1, m_\pi = 236 \text{ MeV}, \ V \approx (4 \text{ fm})^3$$

`Wilson et al.(HadSpec),`
`Phys.Rev. D92 (2015), 094502`



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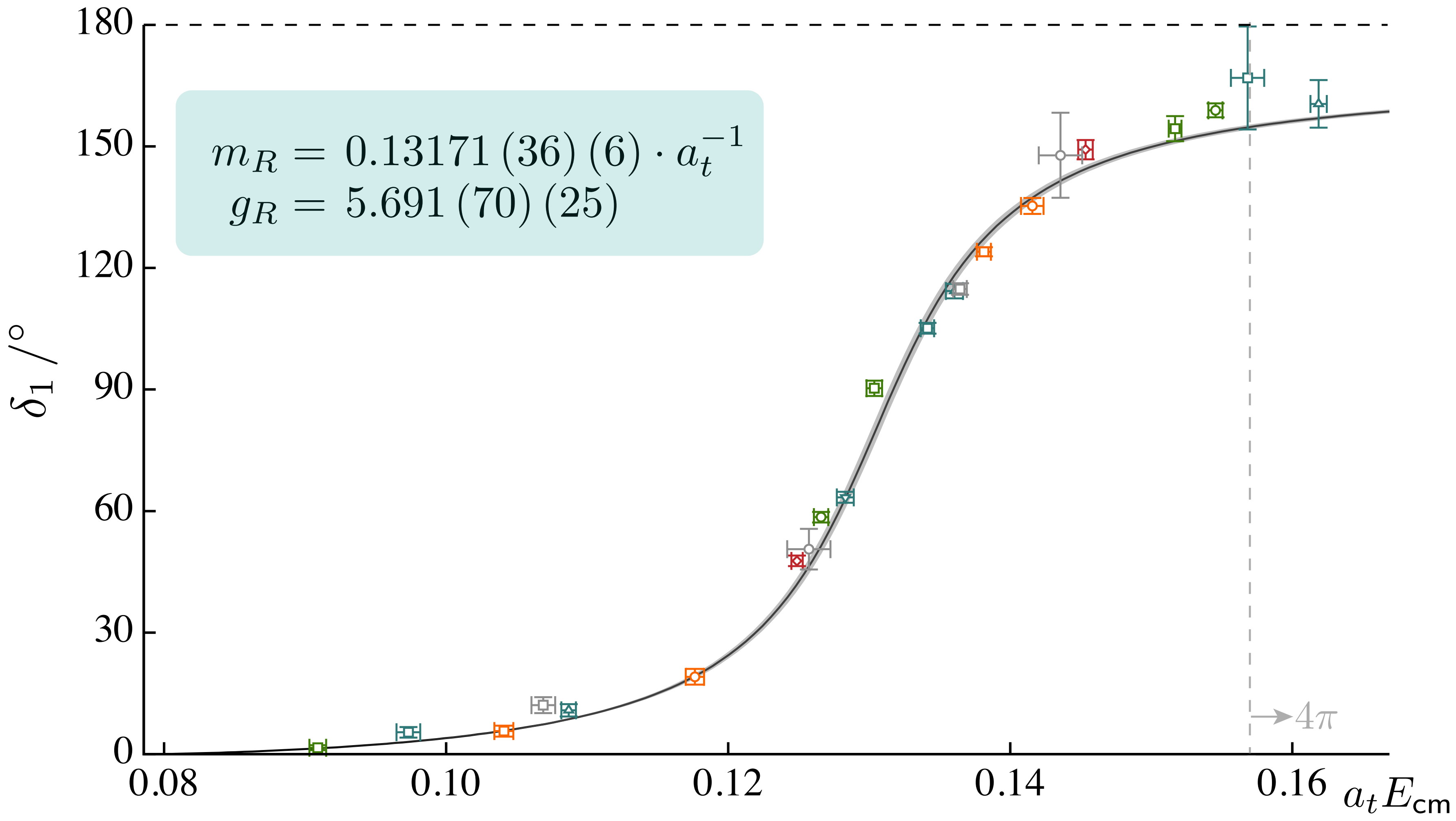


$$m_R = 0.13171\,(36)\,(6)\cdot a_t^{-1}$$
$$g_R = 5.691\,(70)\,(25)$$

Fit to a Breit-Wigner form

$$\mathcal{M}(s) = \frac{1}{\rho(s)}\,\frac{\sqrt{s}\,\Gamma(s)}{m_R^2 - s - i\sqrt{s}\,\Gamma(s)}$$

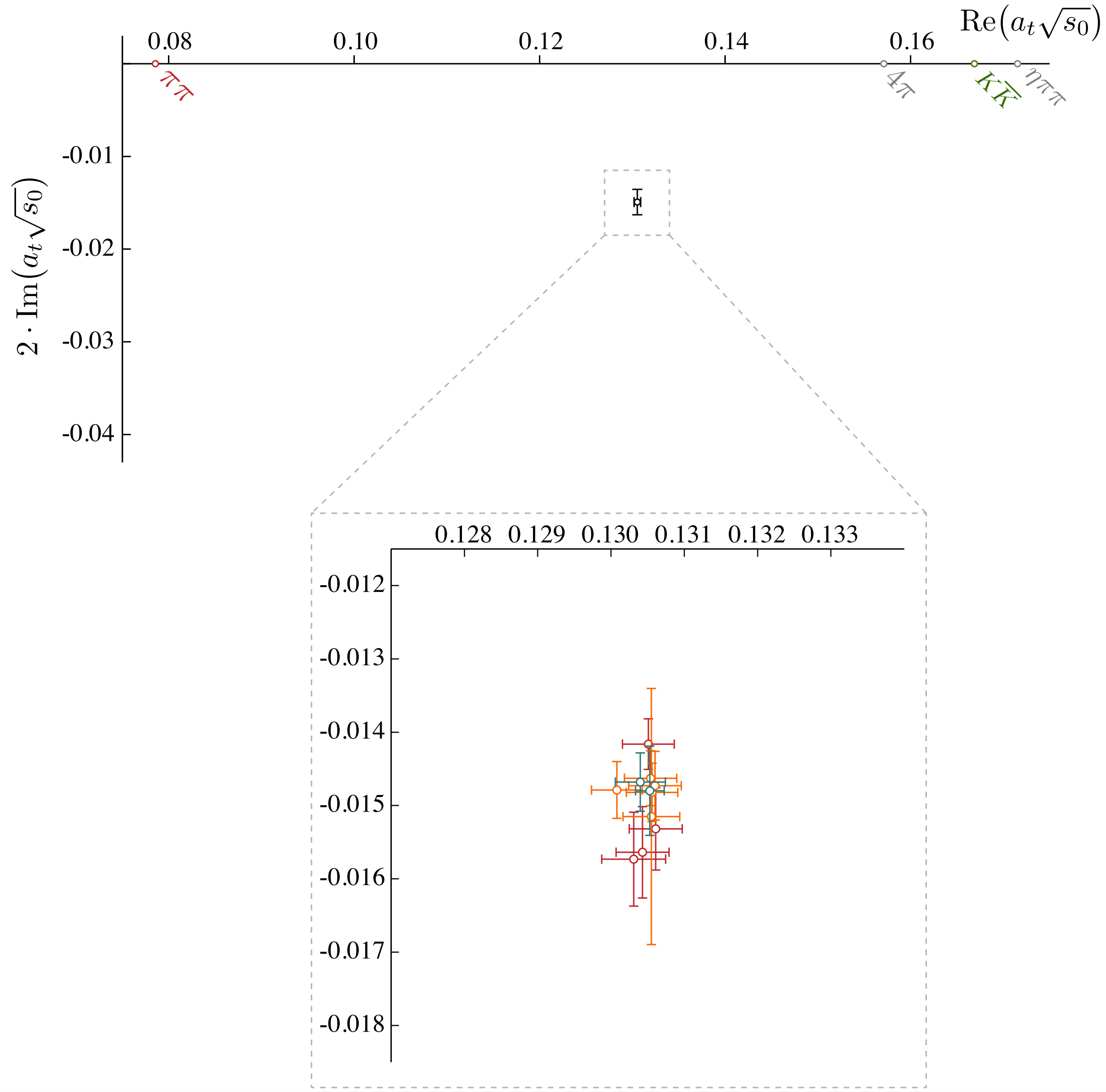
$$\rho_i(E_{\text{cm}}) = 2k_i/E_{\text{cm}}$$
$$s = E_{\text{cm}}^2$$
$$\Gamma(s) = \frac{g_R^2}{6\pi}\,\frac{k^3}{s}$$

$$N_f = 2+1, m_\pi = 236\ \text{MeV},\ V \approx (4\ \text{fm})^3$$

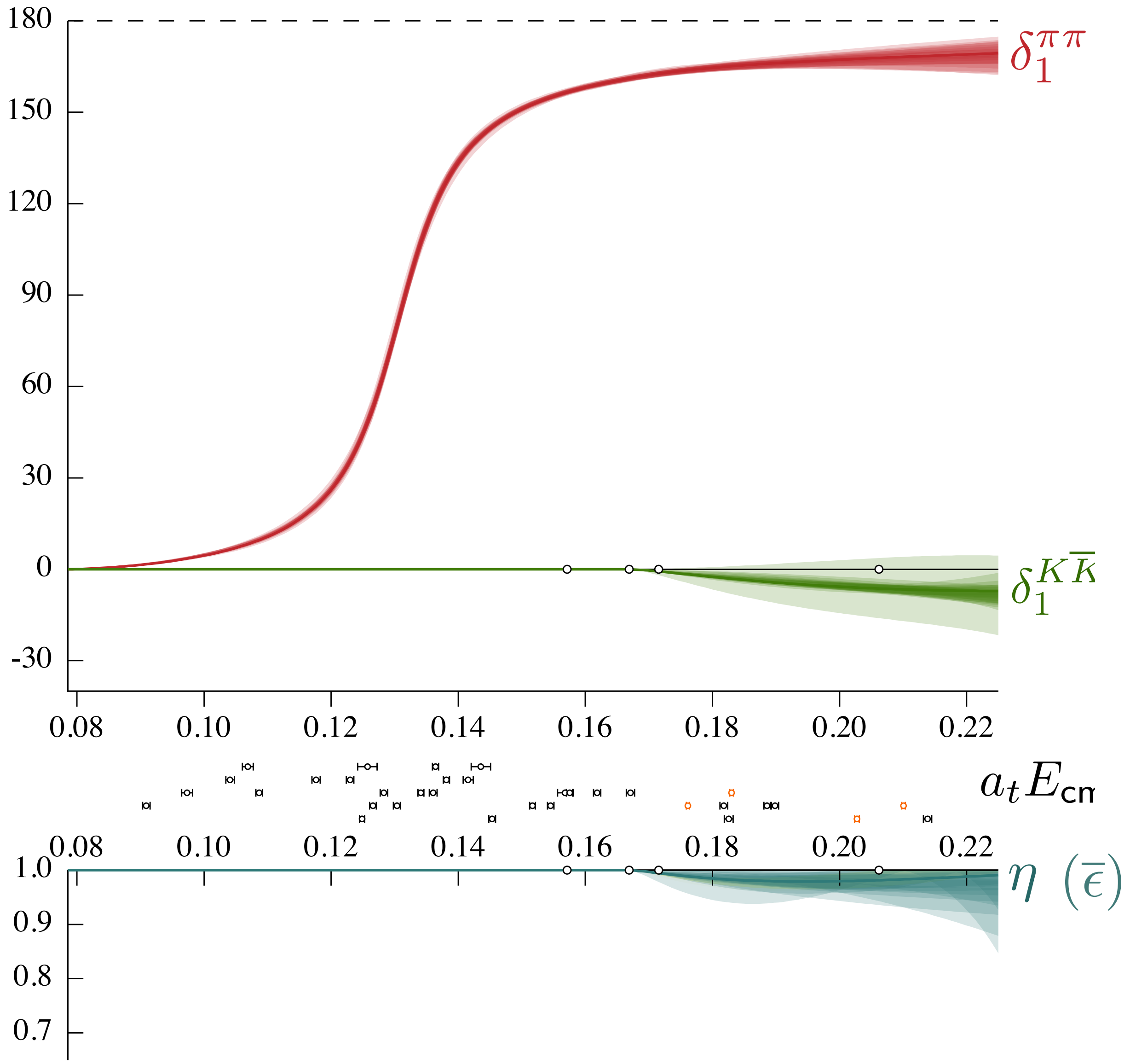
Wilson et al.(HadSpec),
Phys.Rev. D92 (2015), 094502

Using a range of parametrizations:

Pole position:



All three scattering parameters:



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SUMMARY OF LECTURE I

Lattice QCD workflow

GENERATE A SAMPLE OF VACUUM CONFIGURATIONS

COMPUTE EUCLIDEAN CORRELATION FUNCTIONS

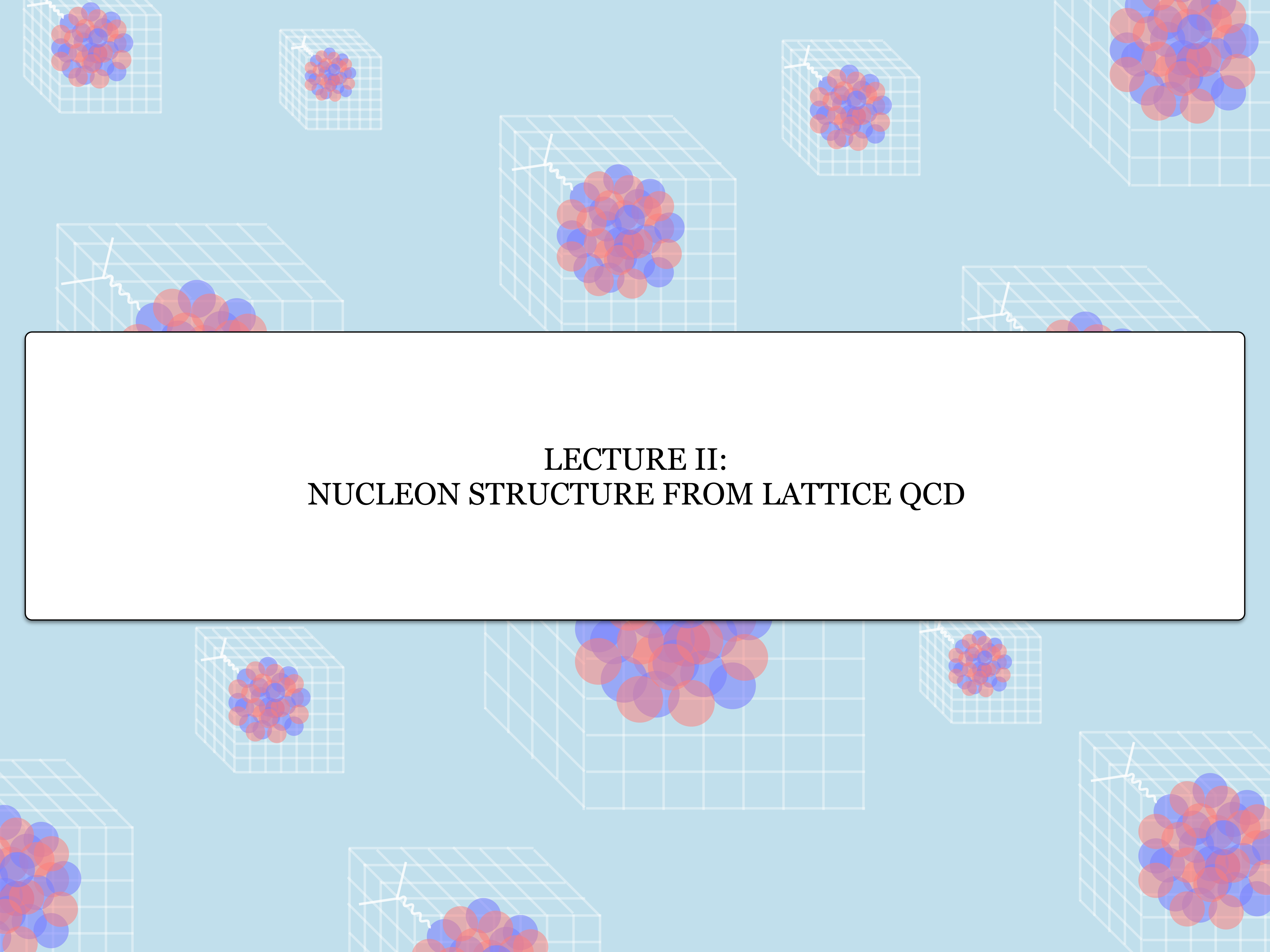
ANALYZE CORRELATION FUNCTIONS: NUMERICS AND ANALYTICAL WORK

— Hybrid Monte Carlo to sample gauge configurations
— Determinant of a high-dimensional matrix required

— Quark contractions
— Inverting a high-dimensional matrix required (to get the quark propagators)

— Assess stat. and sys. uncertainties (take the continuum and infinite-volume limits)
— Connect to physical observables

LECTURE II: NUCLEON STRUCTURE FROM LATTICE QCD…



LECTURE II:
NUCLEON STRUCTURE FROM LATTICE QCD

Let’s enumerate some of the methods that give access to structure quantities in general:

Three(four)-point functions

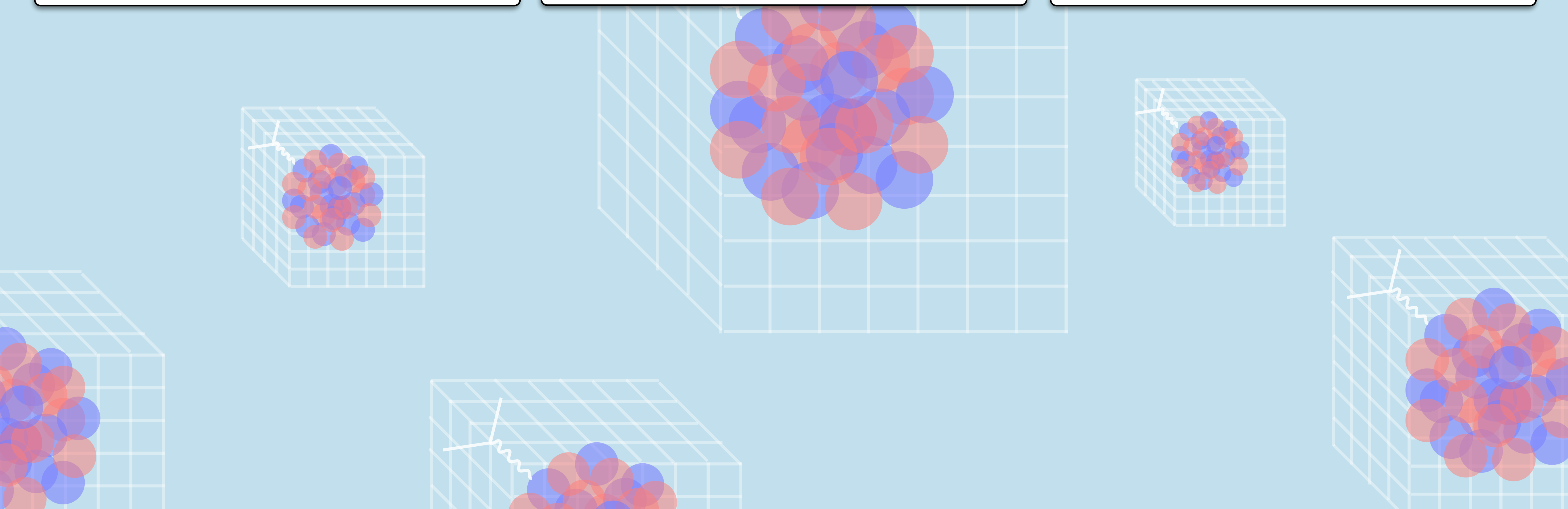
For e.g., form factors, moments of structure functions, Compton amplitude, transition amplitudes

Background-field methods

For e.g., EM moments and polarizabilities, charge radius, form factors and transition amplitudes.

Feynman-Hellmann inspired methods

Similar to background fields. For e.g., axial charge, form factors, EM moments, transition amplitudes



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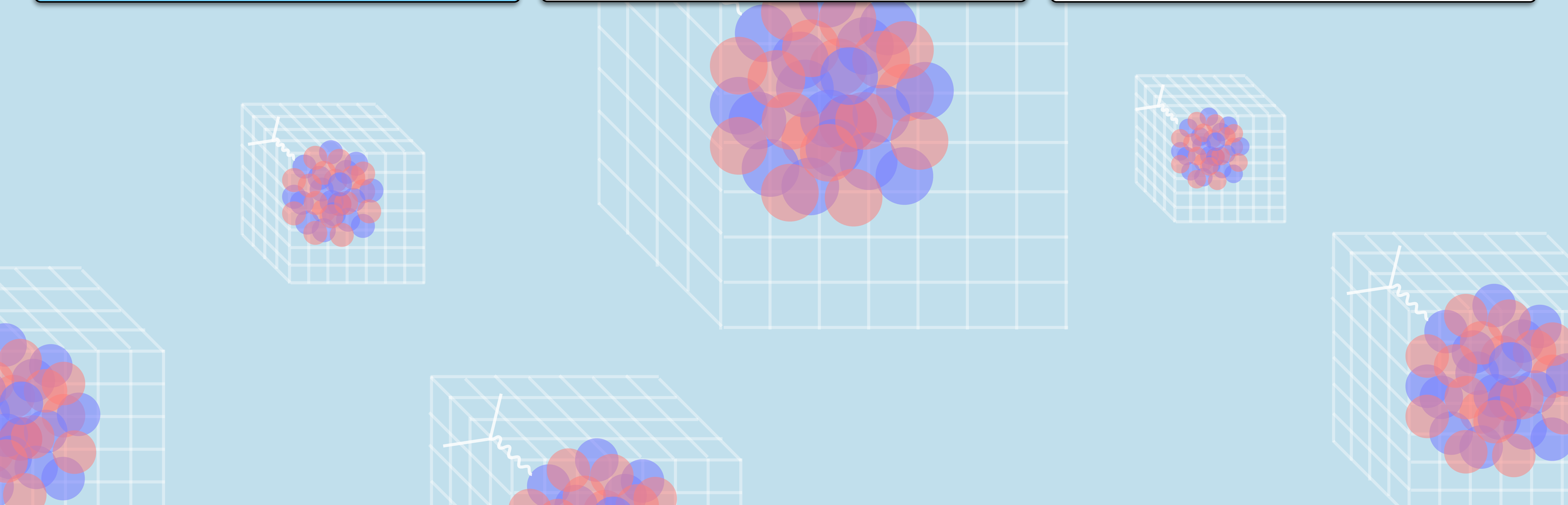

A three-point (3pt) function:

$$C_{\widetilde{\chi}\mathcal{O}\chi}(x', y, x) \equiv \langle \chi(x') \mathcal{O}(y) \widetilde{\chi}(x) \rangle$$

Chambers, http://inspirehep.net/record/1744874/

Annihilate the state

Insert the operator

Create the state

A three-point (3pt) function:

$$C_{\tilde{\chi}\mathcal{O}\chi}(x', y, x) \equiv \langle \underbrace{\chi(x')}_{\text{Annihilate the state}} \underbrace{\mathcal{O}(y)}_{\text{Insert the operator}} \underbrace{\tilde{\chi}(x)\rangle}_{\text{Create the state}}$$

Chambers, http://inspirehep.net/record/1744874/

Spectral decomposition of the 3pt function in Euclidean spacetime

$$G_{\chi\mathcal{O}\tilde{\chi}}(\mathbf{p}', \mathbf{p}; t', \tau, t) = \sum_{\mathrm{X,Y}} \frac{e^{-E_{\mathrm{X}}(\mathbf{p}')(t'-\tau)}}{2E_{\mathrm{X}}(\mathbf{p}')} \frac{e^{-E_{\mathrm{Y}}(\mathbf{p})(\tau-t)}}{2E_{\mathrm{Y}}(\mathbf{p})} \langle\Omega|\chi(0)\underbrace{|\mathrm{X}(\mathbf{p}')\rangle\ \langle\mathrm{X}(\mathbf{p}')|}_{\text{A complete set of states}}\mathcal{O}(0)\underbrace{|\mathrm{Y}(\mathbf{p})\rangle\ \langle\mathrm{Y}(\mathbf{p})|}_{\text{Another complete set of states}}\tilde{\chi}(0)|\Omega\rangle$$

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Spectral decomposition of the 3pt function in Euclidean spacetime

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$$\langle\Omega|\chi(0)\underbrace{|\mathrm{X}(\mathbf{p}')\rangle\;\langle\mathrm{X}(\mathbf{p}')|}_{\text{A complete set of states}}\mathcal{O}(0)\underbrace{|\mathrm{Y}(\mathbf{p})\rangle\;\langle\mathrm{Y}(\mathbf{p})|}_{\text{Another complete set of states}}\widetilde{\chi}(0)|\Omega\rangle$$

Long-separation behavior dominated by ground states

$$G_{\chi\mathcal{O}\widetilde{\chi}}(\mathbf{p}',\mathbf{p};t',\tau,t) \xrightarrow{\text{large } t'-\tau,\tau-t} \frac{e^{-E_{\mathrm{X}_0}(\mathbf{p}')(t'-\tau)}}{2E_{\mathrm{X}_0}(\mathbf{p}')}\,\frac{e^{-E_{\mathrm{X}_0}(\mathbf{p})(\tau-t)}}{2E_{\mathrm{X}_0}(\mathbf{p})}$$

$$\sum_{\underbrace{r',r}_{\text{If there are degenerate ground states}}} \langle\Omega|\chi(0)|\mathrm{X}_0(\mathbf{p}',r')\rangle\;\underbrace{\langle\mathrm{X}_0(\mathbf{p}',r')|\mathcal{O}(0)|\mathrm{X}_0(\mathbf{p},r)\rangle}_{\text{Desired ground state to ground state matrix element (unrenormalized and in a finite volume)}}\;\langle\mathrm{X}_0(\mathbf{p},r)|\widetilde{\chi}(0)|\Omega\rangle$$

A three-point (3pt) function:

$$C_{\widetilde{\chi}\mathcal{O}\chi}(x', y, x) \equiv \langle \underbrace{\chi(x')}_{\text{Annihilate the state}} \underbrace{\mathcal{O}(y)}_{\text{Insert the operator}} \underbrace{\widetilde{\chi}(x)\rangle}_{\text{Create the state}}$$

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Taking a proper ratio to 2pt functions

$$R_{\chi\mathcal{O}\widetilde{\chi}}(\mathbf{p}', \mathbf{p}; t', \tau, t) \overset{\text{large } t'-\tau, \tau-t}{\propto} \langle \mathrm{X}_0(\mathbf{p}', r')|\mathcal{O}(0)|\mathrm{X}_0(\mathbf{p}, r)\rangle$$

EXERCISE 4

If the computational resources do not allow large source, operator and sink time separations to be achieved, one should worry about the effect of excited states. One way to have more confidence over the extracted ground state to ground state matrix element is to perform a multi-exponential fits to the ratio of 3pt to 2pt functions as a function of both the source-sink and the source-operator separations. Assume that both the ground state and the first excited states contribute significantly to such a ratio. Write down a generic form for such a multi-exponential function.

BONUS EXERCISE 3

In the above exercise, sum over the time insertions of the operator and write down a new form for the ratio of 3pt to 2pt functions, which now is only a function of the source-sink time separation. This is referred to as the summation method in literature.

Example: The application of 3pt function method to obtain the axial charge/form factors of the nucleon

`Constantinou, arXiv:1411.0078 [hep-lat].`

$$\langle N(p',s')|\bar{\psi}(x)\,\gamma_\mu\,\gamma_5\,\psi(x)|N(p,s)\rangle = i\left(\frac{m_N^2}{E_N(\mathbf{p}')E_N(\mathbf{p})}\right)^{1/2}\bar{u}_N(p',s')\left[G_A(q^2)\gamma_\mu\gamma_5+\frac{q_\mu\gamma_5}{2m_N}G_p(q^2)\right]u_N(p,s)$$

Axial-vector current

Nucleon spinor

Axial and pseudo scalar form factors

$$G_A(0) = g_A$$

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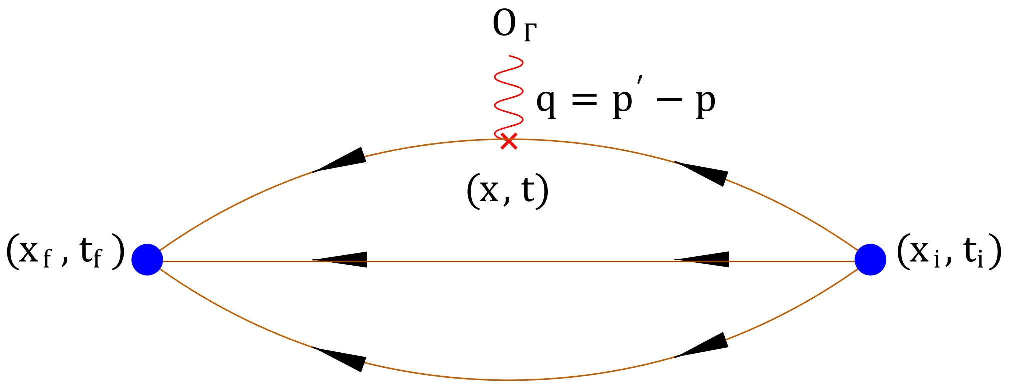
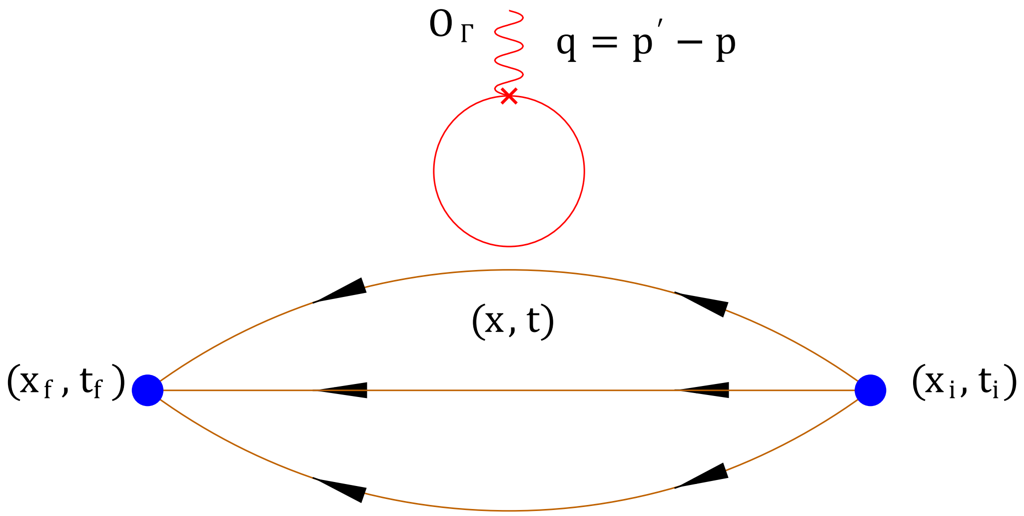


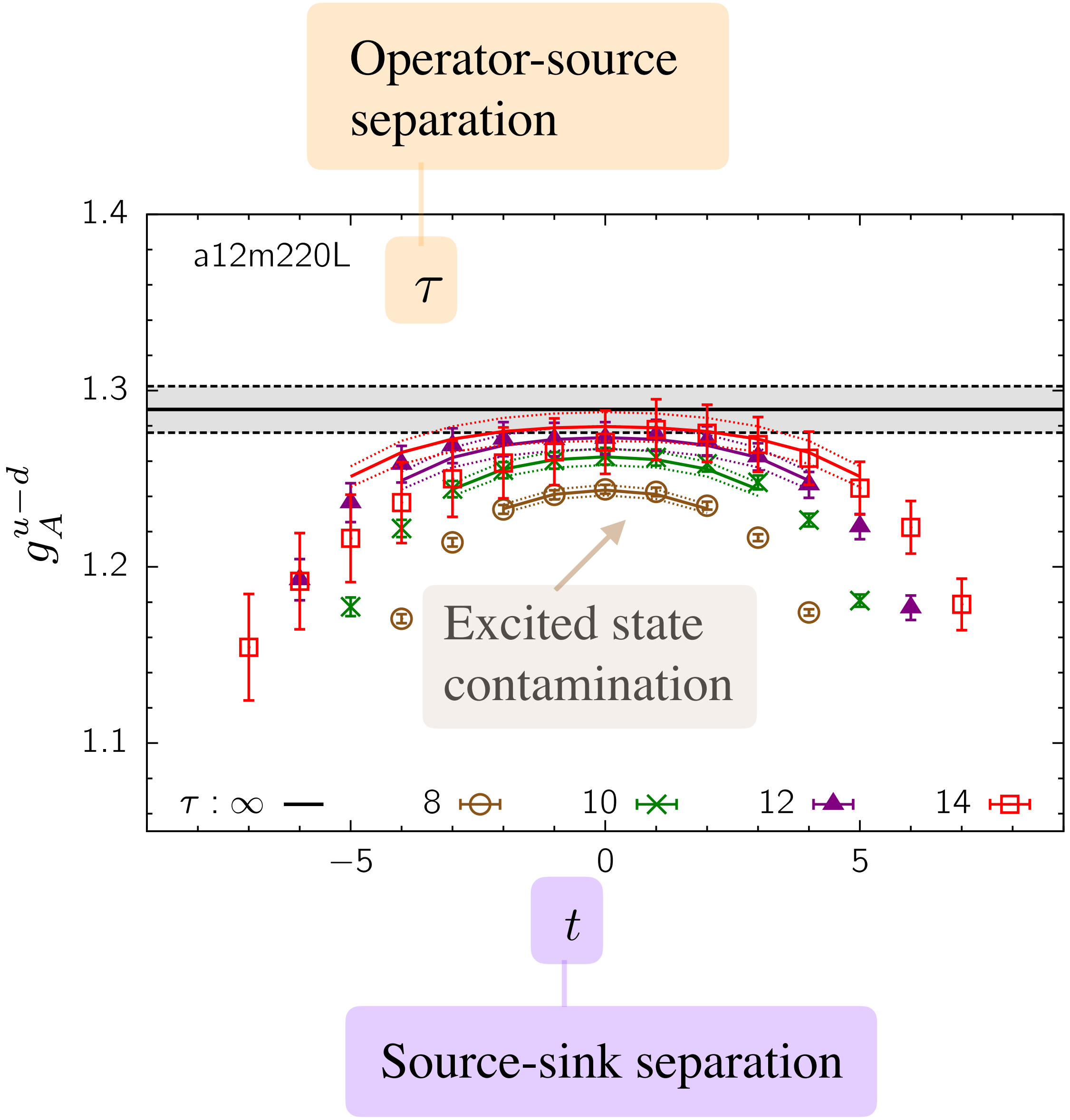
Connected contribution



Disconnected contribution
(vanishes at isospin limit for isovector quantities)

Example: The application of 3pt function method to obtain the axial charge/form factors of the nucleon

`Gupta et al (PNDME), Phys. Rev. D 98, 034503 (2018)`



Operator-source separation

τ

Excited state contamination

t

Source-sink separation

3pt function for a single lattice spacing, volume and quark masses

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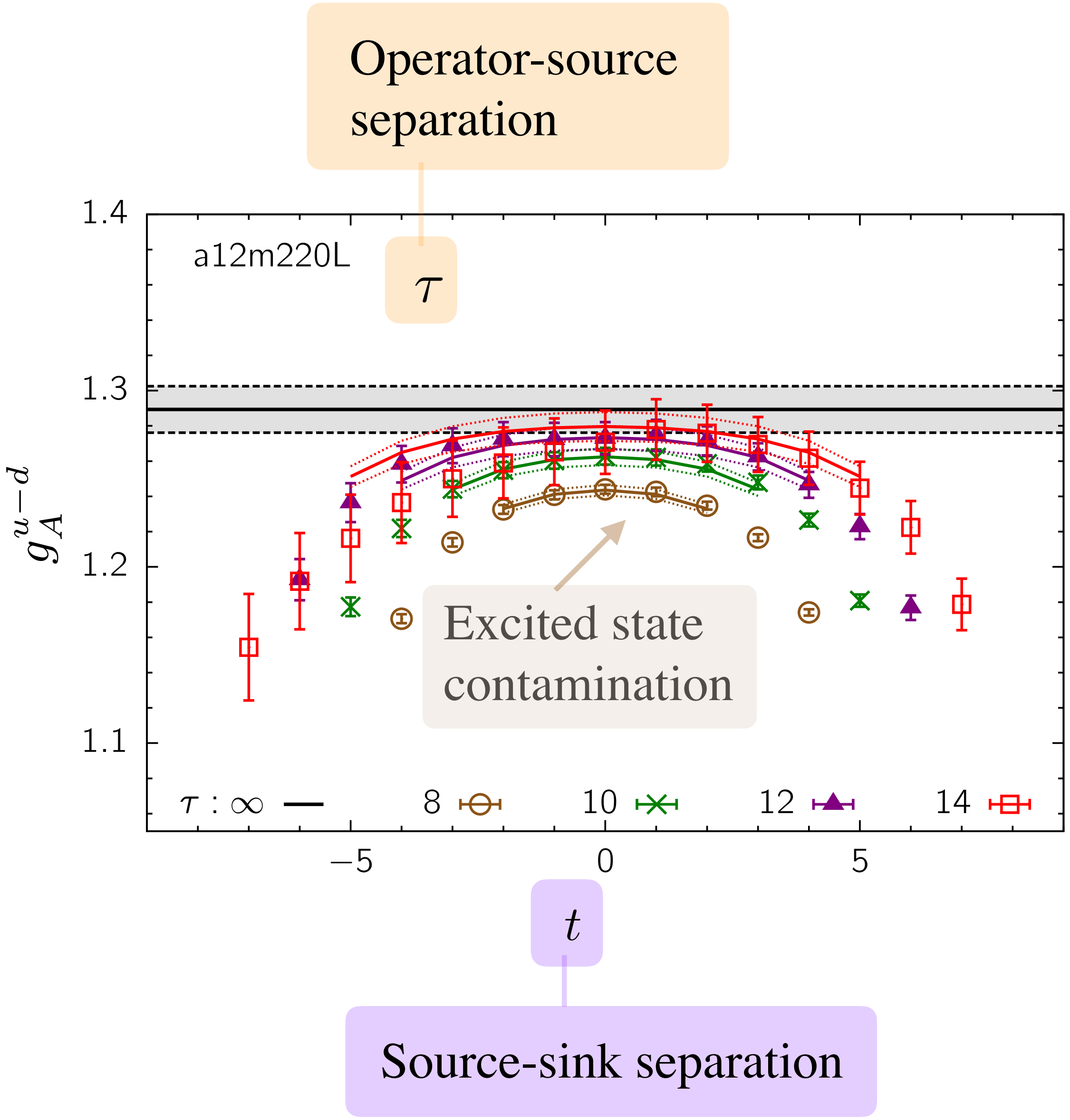

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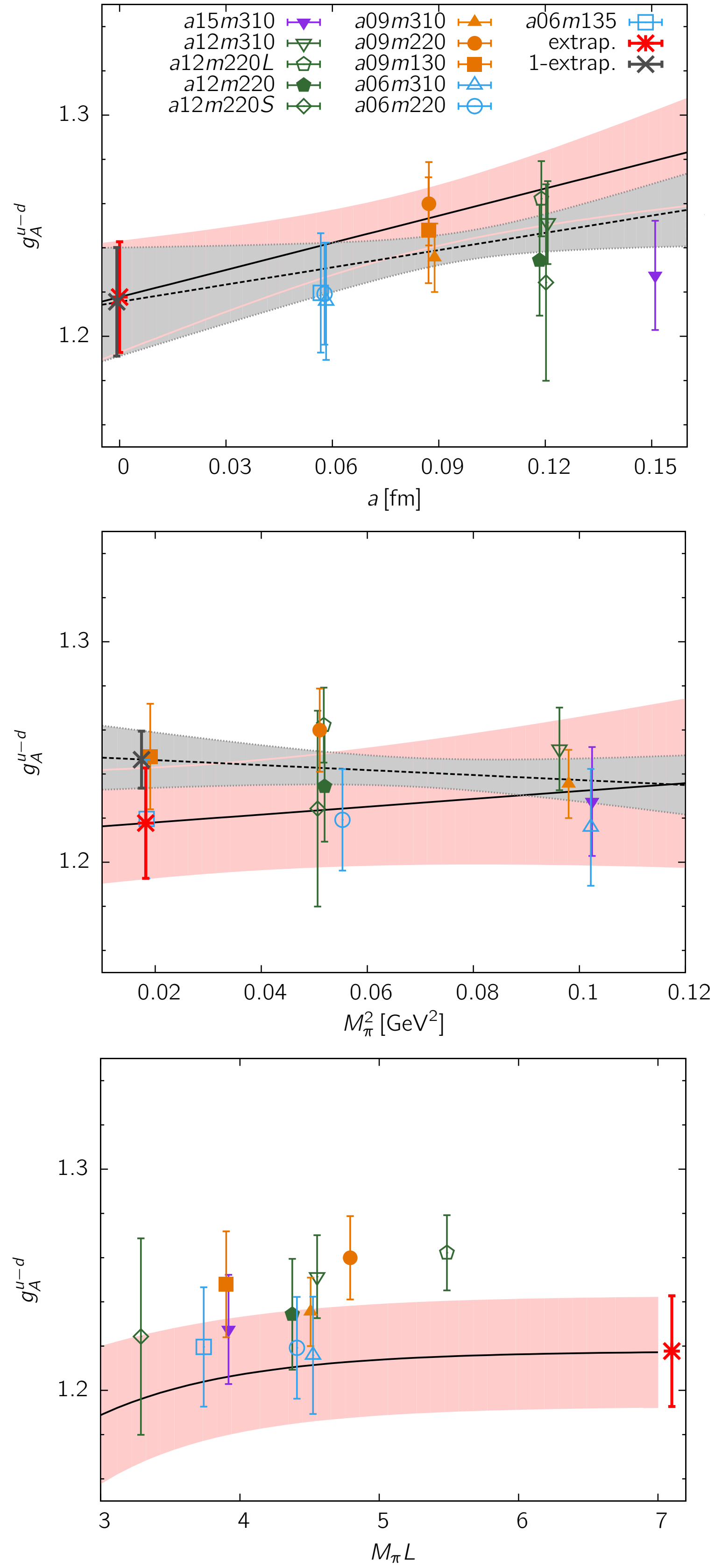

Extrapolation to continuum, infinite volume and physical quark masses

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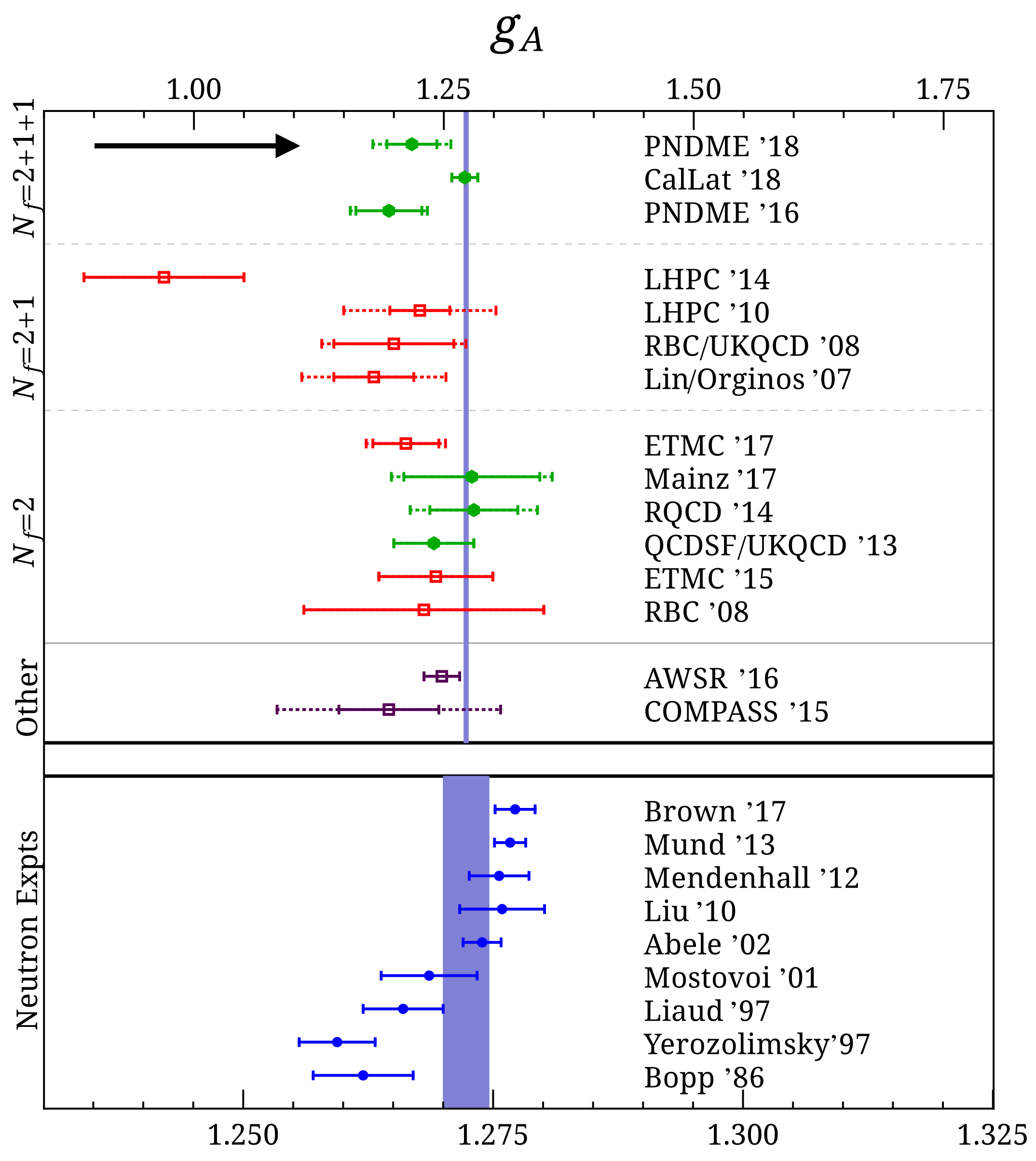
Gupta et al (PNDME), Phys. Rev. D 98, 034503 (2018)



Compilation of results

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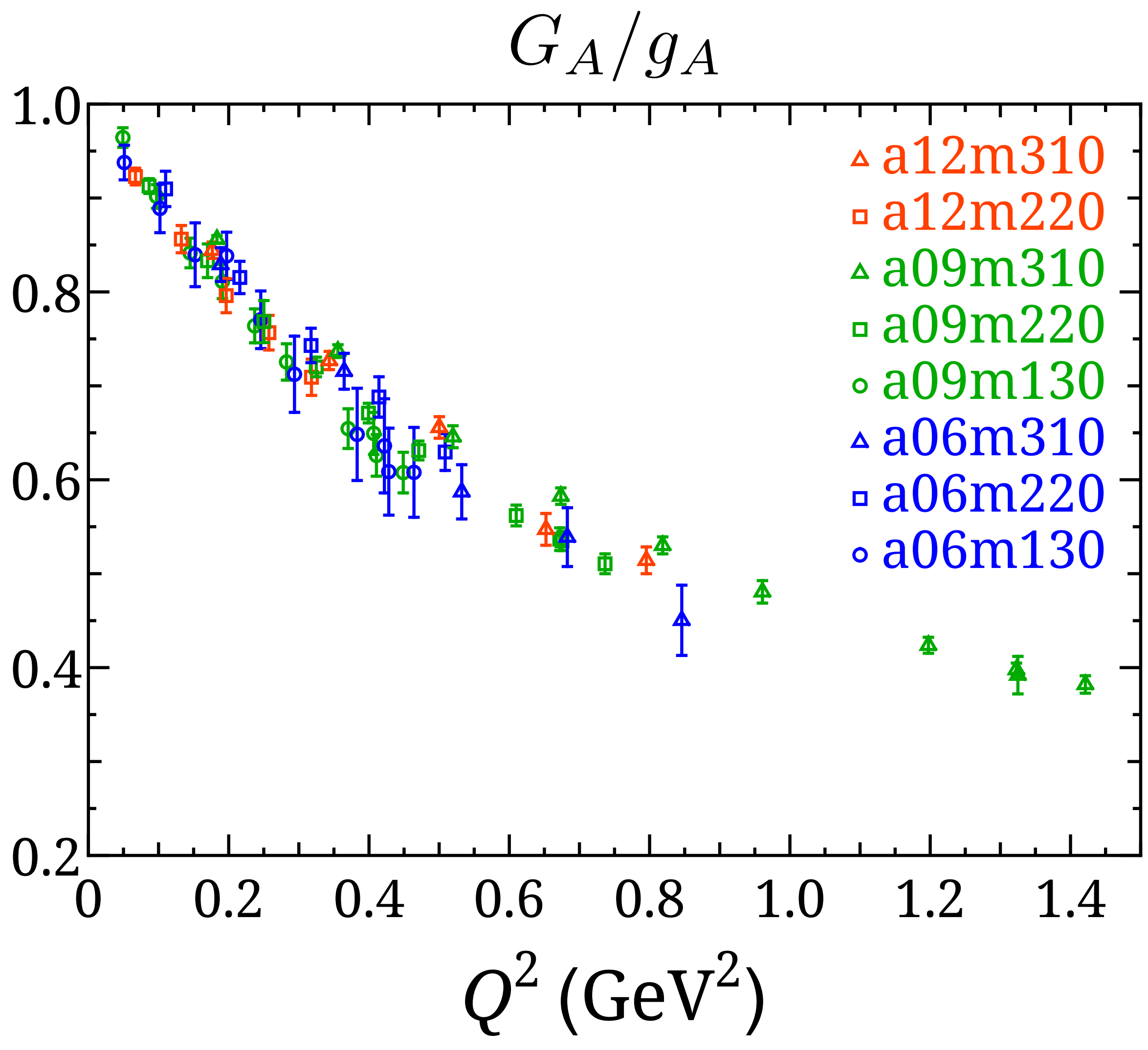
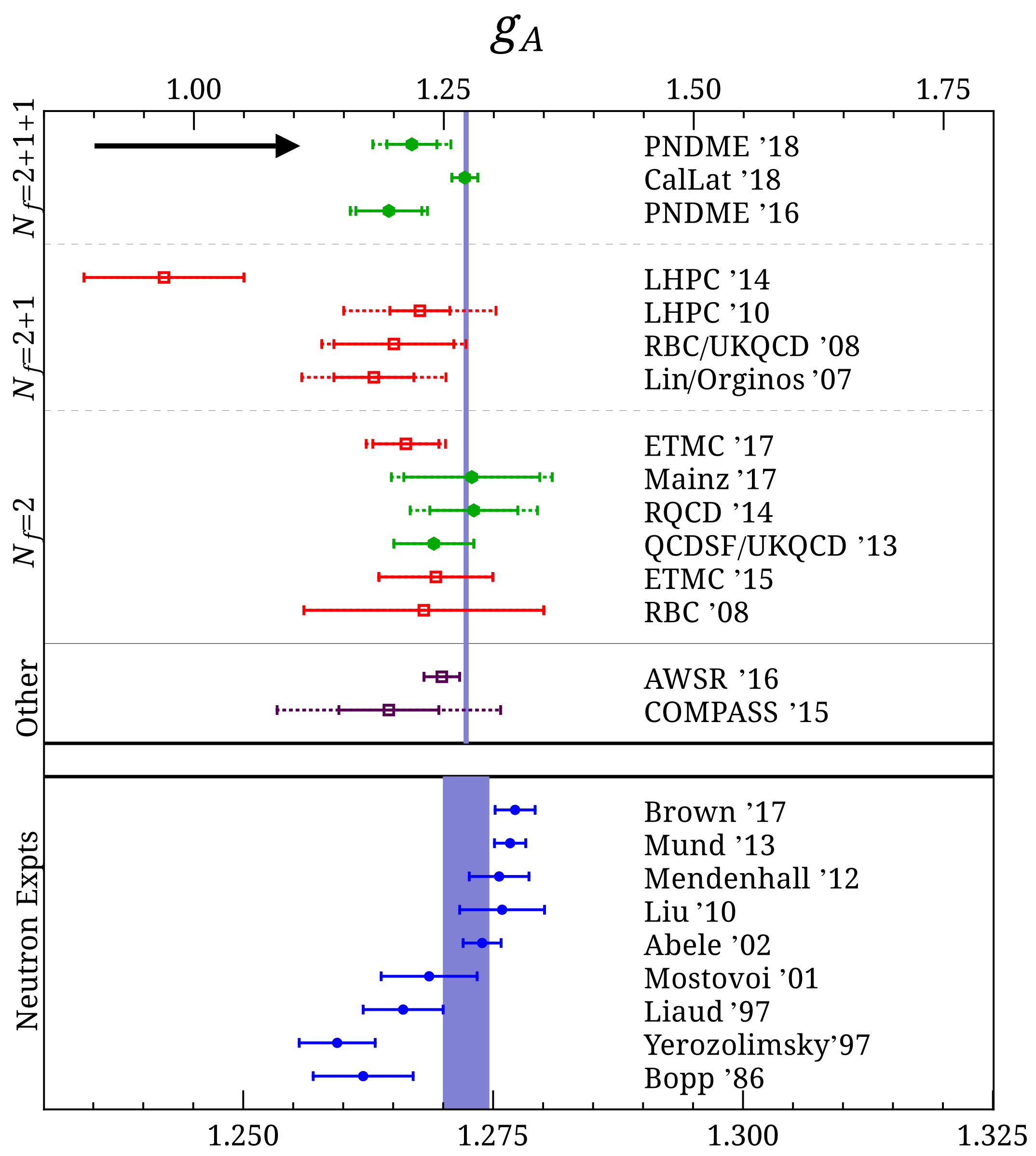
Gupta et al (PNDME), Phys. Rev. D 98, 034503 (2018)

Jang et al, EPJ Web Conf. 175, 06033 (2018)



Axial form factor results



Compilation of results

Example: The spin decomposition of the nucleon

Alexandru, Phys. Rev. Lett. 119, 142002 (2017).

$$J_N = \sum_{q=u,d,s,c\cdots} \left(\underbrace{\frac{1}{2}\Delta\Sigma_q}_{\text{Quark spin}} + \underbrace{L_q}_{\text{Quark orbital angular momentum}} \right) + \underbrace{J_g}_{\text{Total gluon angular momentum}}$$

Ji, Phys. Rev. Lett. 78, 610 (1997).

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Quark spin

Quark orbital angular momentum

Total gluon angular momentum

Ji, Phys. Rev. Lett. 78, 610 (1997).

Matrix elements needed

$$\langle N(p,s')|\mathcal{O}_A^\mu|N(p,s)\rangle = \bar{u}_N(p,s')\left[g_A^q\gamma^\mu\gamma_5\right]u_N(p,s)$$

$$\langle N(p',s')|\mathcal{O}_V^{\mu\nu}|N(p,s)\rangle = \bar{u}_N(p',s')\Lambda_{\mu\nu}^q(Q^2)u_N(p,s)$$

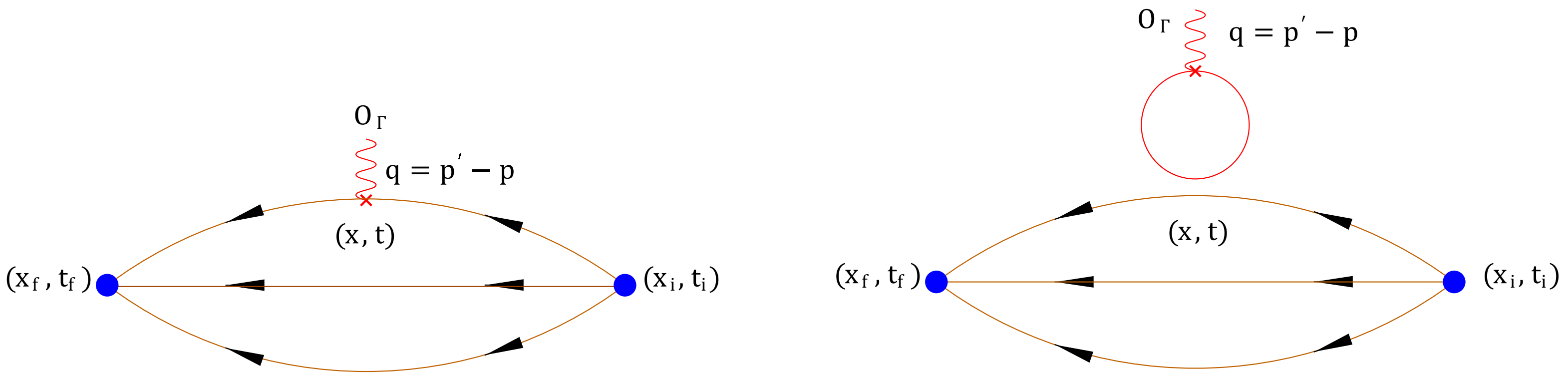
$$\langle N(p',s')|\mathcal{O}_g^{\mu\nu}|N(p,s)\rangle = \bar{u}_N(p',s')\Lambda_{\mu\nu}^g(Q^2)u_N(p,s)$$

With operators

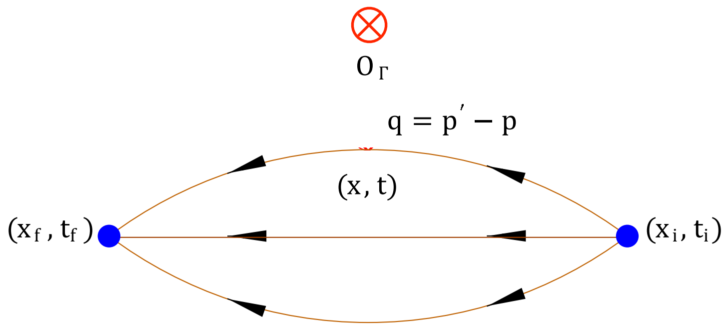
$$\mathcal{O}_A^\mu = \bar{q}\gamma_\mu\gamma_5 q$$

$$\mathcal{O}_V^{\mu\nu} = \bar{q}\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}}q$$

$$\mathcal{O}_g^{\mu\nu} = 2\mathrm{Tr}[G_{\mu\sigma}G_{\nu\sigma}]$$



Quark contributions



Qluonic contribution

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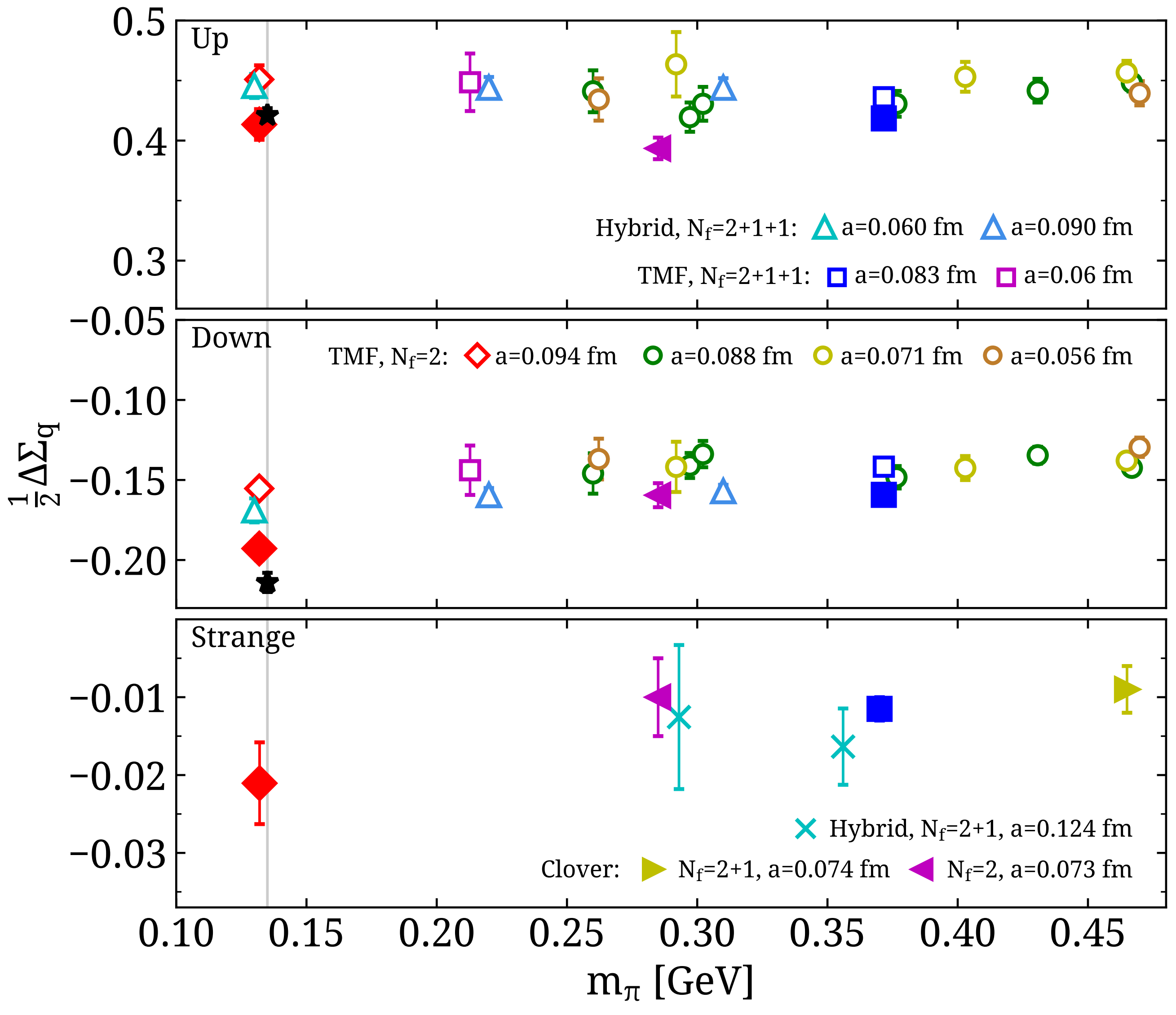
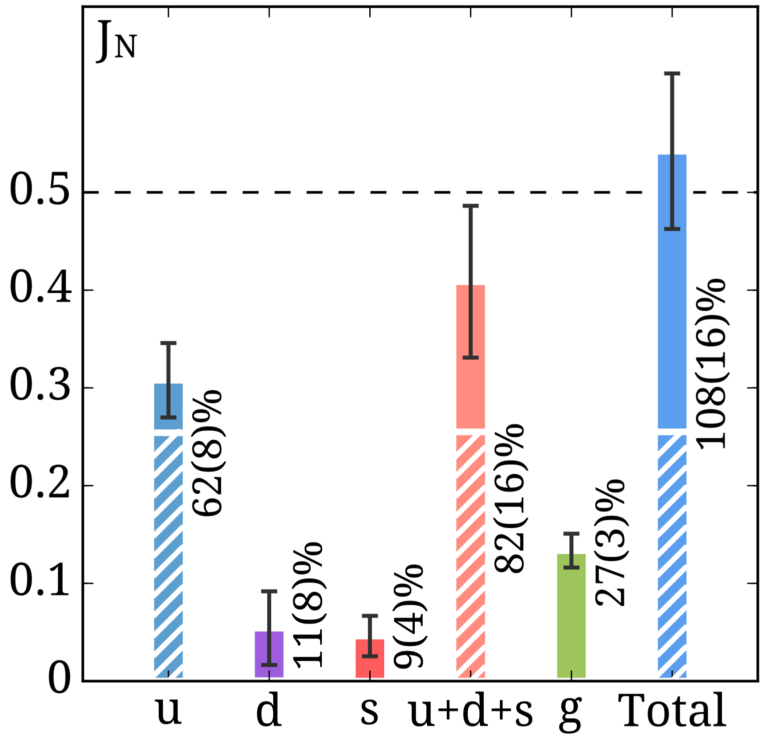
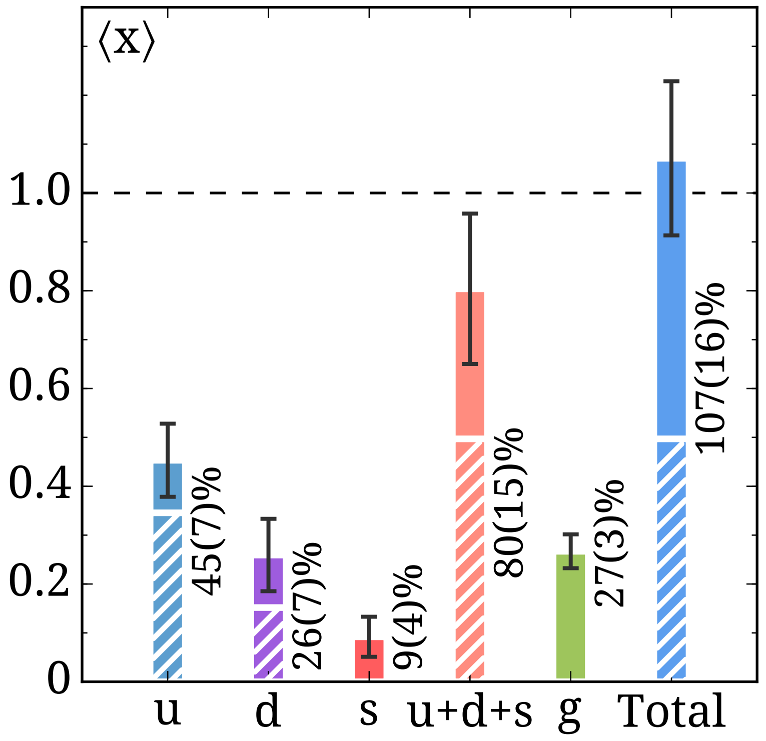


Quark spin contributions



Nucleon spin decomposition



Longitudinal momentum decomposition

Let’s enumerate a some of the methods that give access to structure quantities in general:

Three(four)-point functions

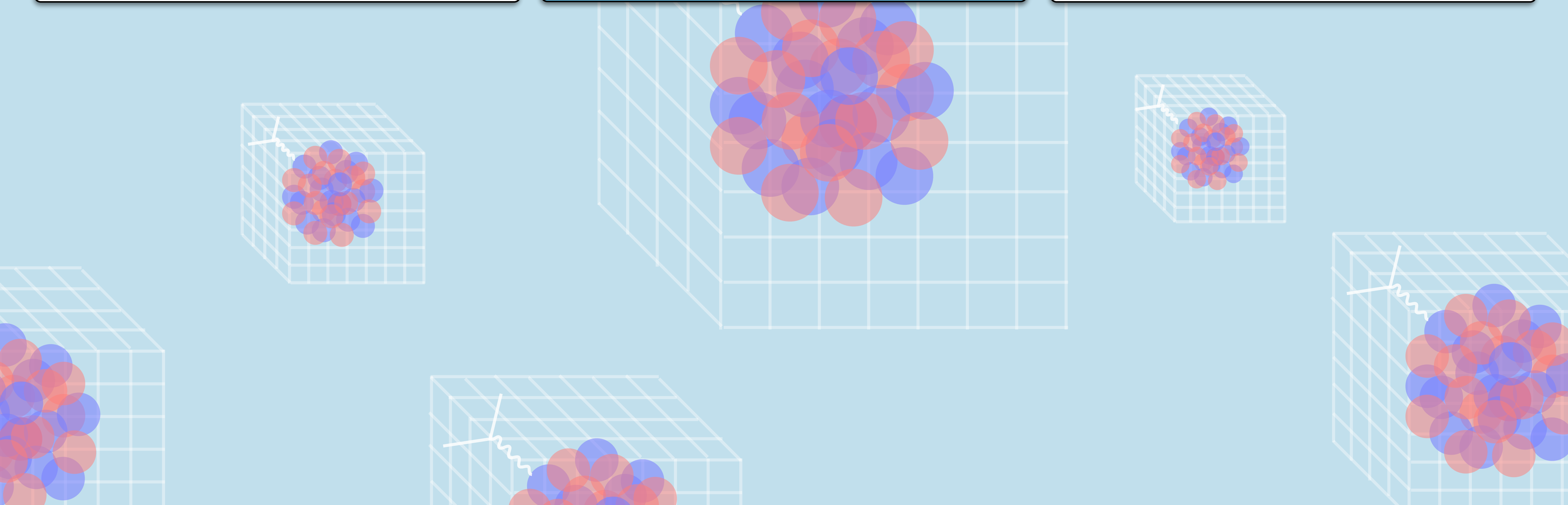
For e.g., form factors, moments of structure functions, Compton amplitude, transition amplitudes

Background-field methods

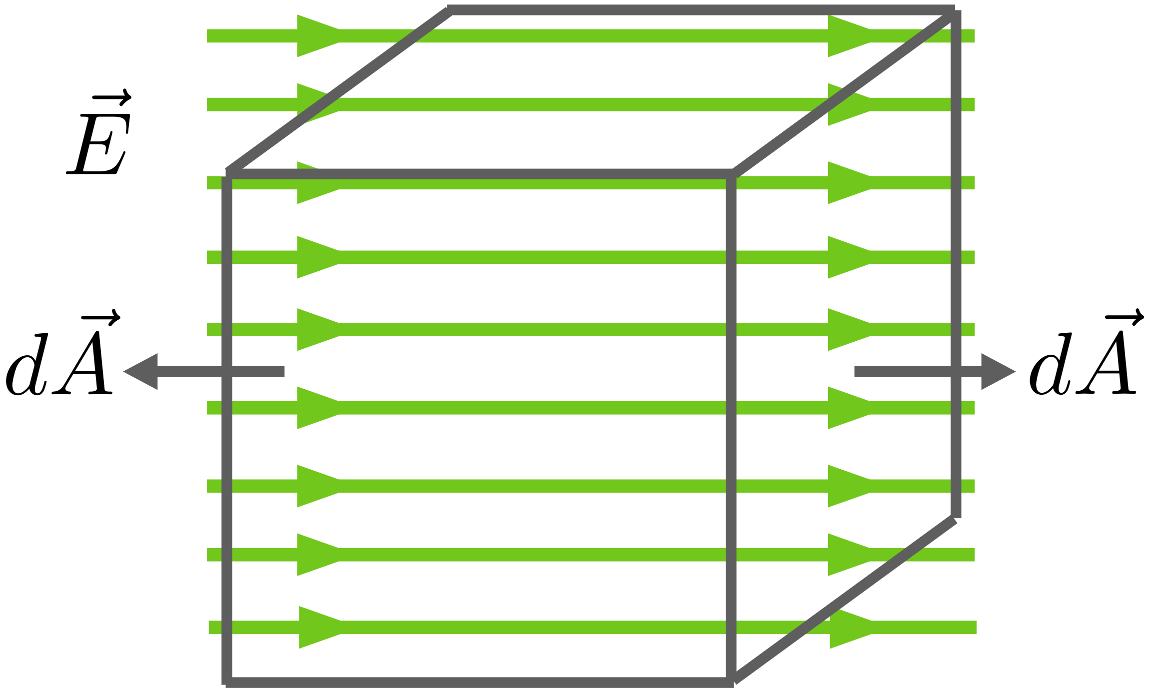
For e.g., EM moments and polarizabilities, charge radius, form factors and transition amplitudes.

Feynman-Hellmann inspired methods

Similar to background fields. For e.g., axial charge, form factors, EM moments, transition amplitudes



Background fields are non-dynamical, i.e., there will be no pair creation and annihilation in vacuum with a classical EM background field. This mean the photon zero mode is no problem: it is absent in the calculation!



$$U^{(\mathrm{QCD})} \to U^{(\mathrm{QCD})} \times U^{(\mathrm{QED})}$$

Modify the links when forming the quark propagators (quench approx).

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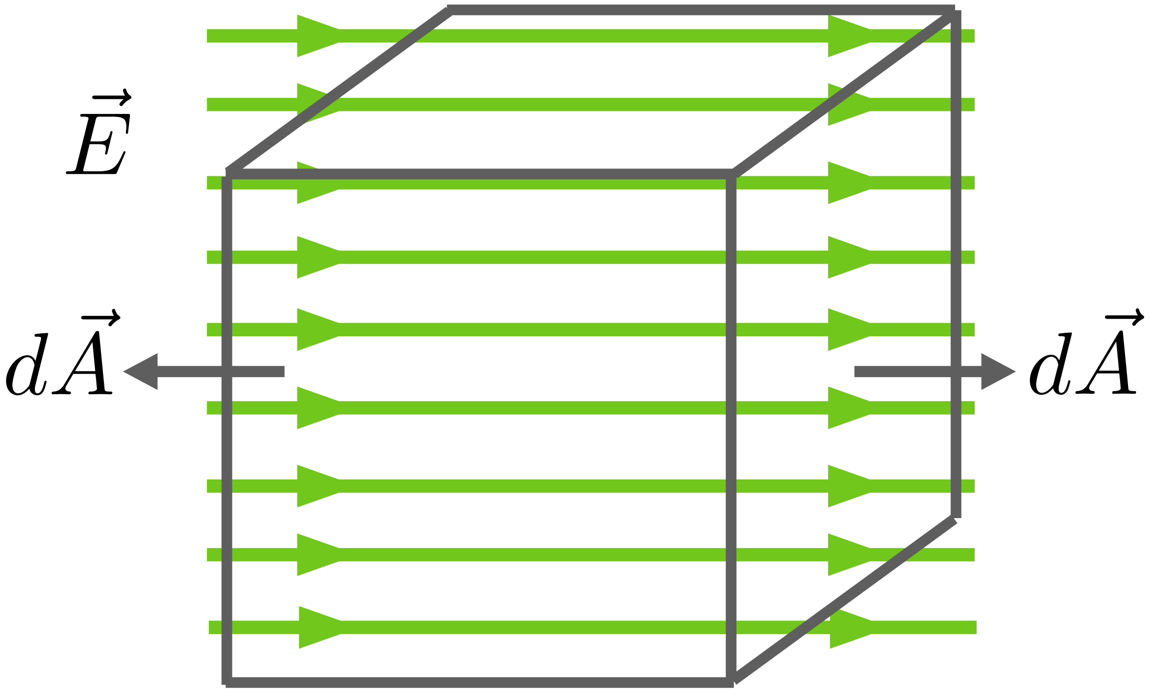
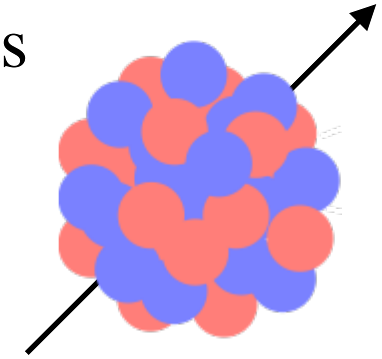


$$U^{(\mathrm{QCD})} \to U^{(\mathrm{QCD})} \times U^{(\mathrm{QED})}$$

Modify the links when forming the quark propagators (quench approx).

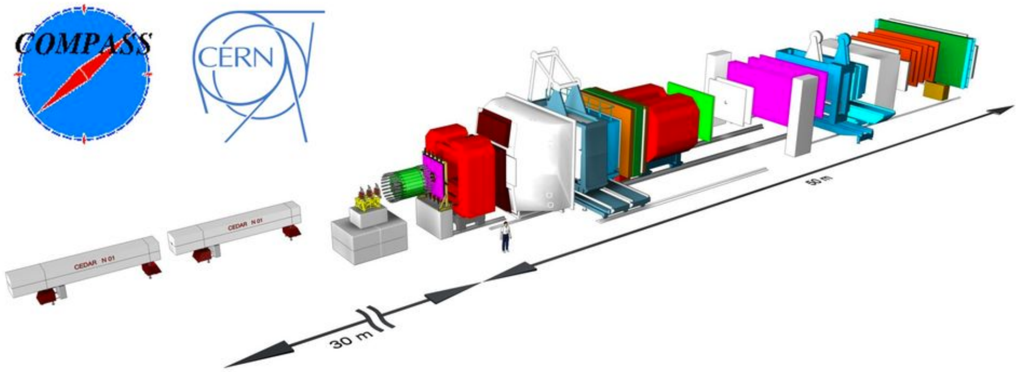
Traditionally they are used for constraining the response of hadrons/nuclei to external probes:

Magnetic moments



Electric and magnetic polarizabilities

COMPASS CERN



`See e.g., BEANE et al (NPLQCD), Phys.Rev.Lett. 113 (2014) 25, 252001 and Phys.Rev. D92 (2015) 11, 114502. for nuclear-physics calculations.`

Various other structure properties of hadrons and nuclei, as well as their transitions, can be studied using more complex background fields:

1) EM charge radius

ZD and Detmold, Phys. Rev. D 93, 014509 (2016).

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SHRINKING THE PROTON
New value from exotic atom trims radius by four per cent

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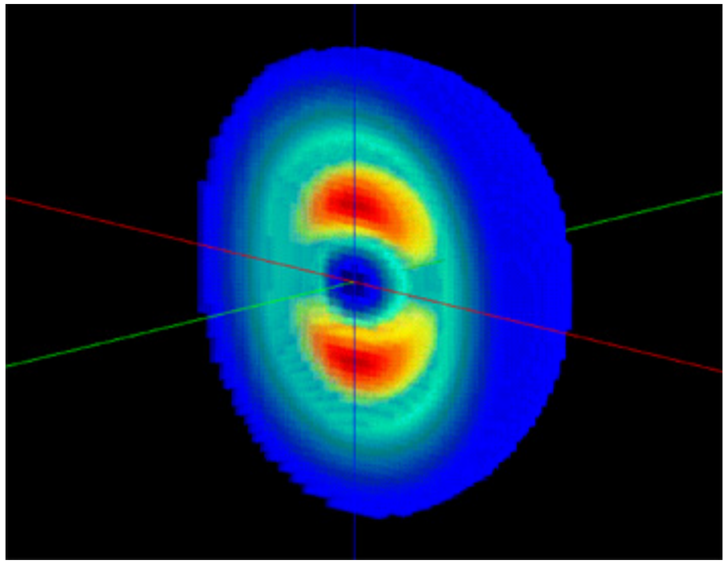
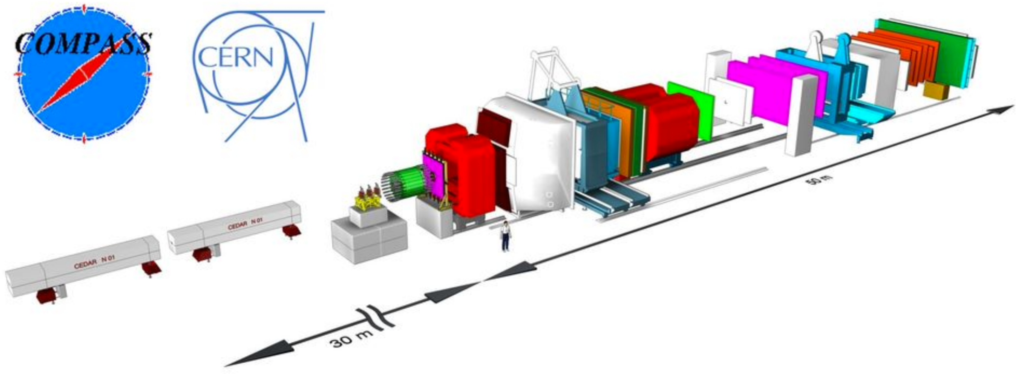
2) Electric quadrupole moment

ZD and Detmold, Phys. Rev. D 93, 014509 (2016).



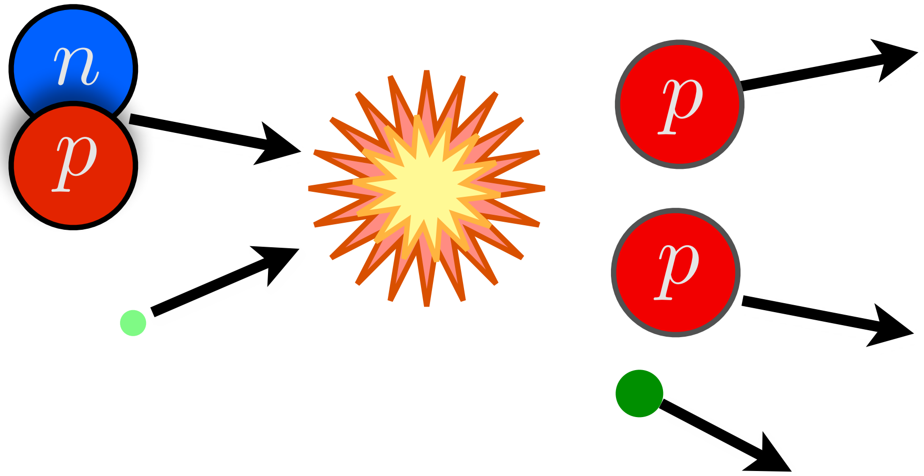
3) Form factors

Detmold, Phys.Rev. D71, 054506 (2005).



4) Axial background fields

Beane at al, Phys.Rev. Lett, 115 132001 (2015).



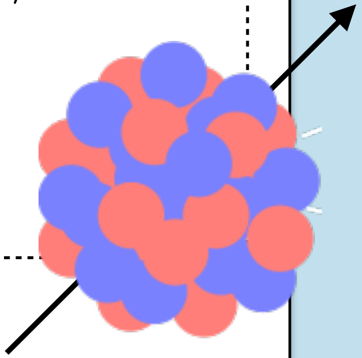
Here's an application of the background-field technique to obtain magnetic moment and polarizabilities of the nucleon:

$$E_{h;j_z}(\mathbf{B}) \;=\; \sqrt{M_h^2 + P_\parallel^2 + (2n_L+1)|Q_h e\mathbf{B}|} - \boldsymbol{\mu}_h \cdot \mathbf{B} - 2\pi\beta_h^{(M0)}|\mathbf{B}|^2 - 2\pi\beta_h^{(M2)}\langle \hat{T}_{ij} B_i B_j \rangle + \ldots$$

Landau levels for charged particles

Magnetic moment

Magnetic polarizabilities



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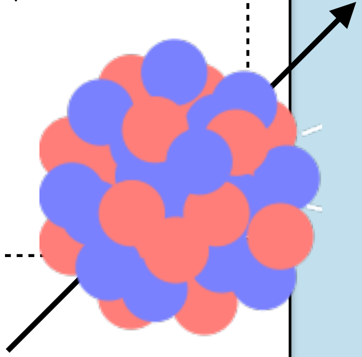
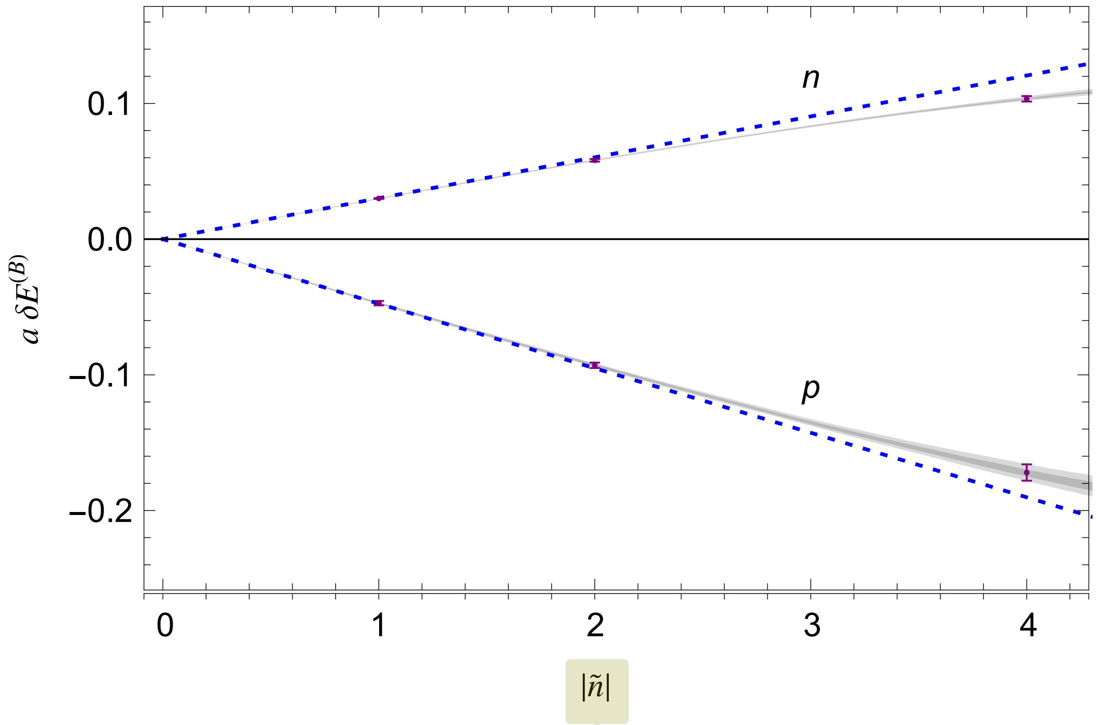
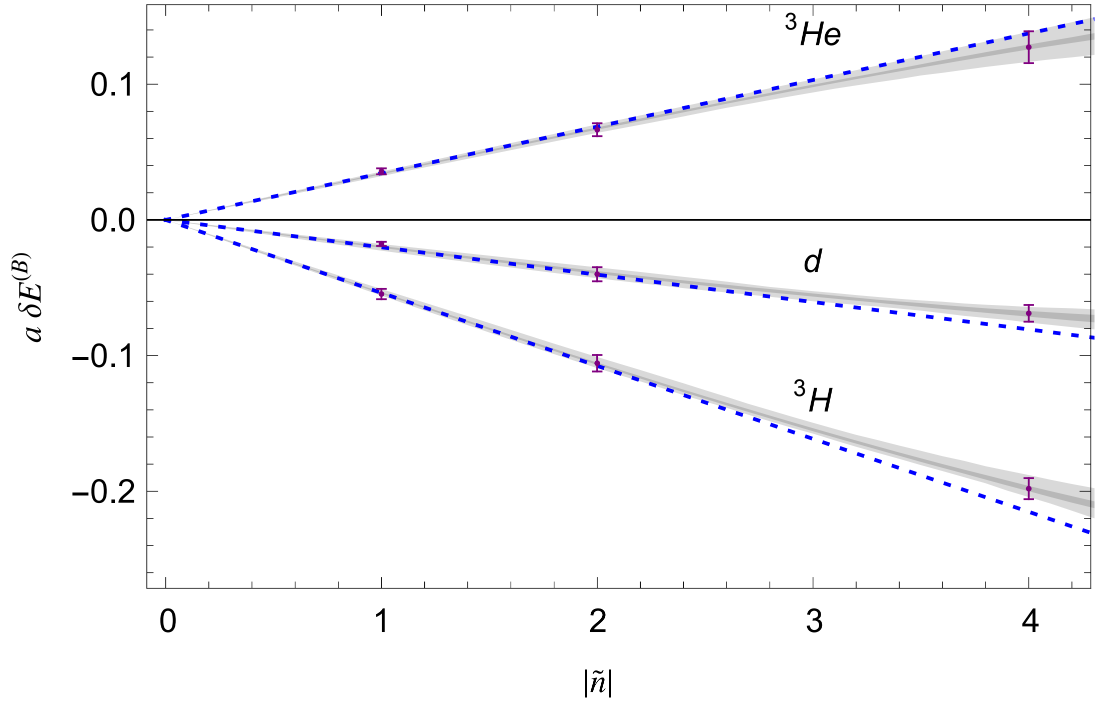


Nucleon



A quanta of magnetic field

Light nuclei



$$N_f = 3,\;\; m_\pi = 0.806\ \text{GeV},\;\; a = 0.145(2)\ \text{fm}$$

Beane et al.(NPLQCD), phys.rev.lett.113 (2014) 25, 252001.
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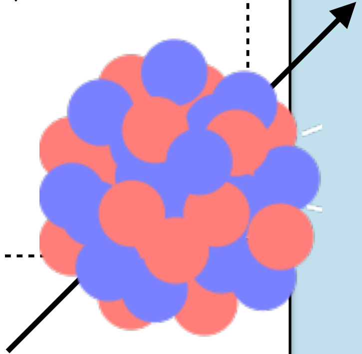

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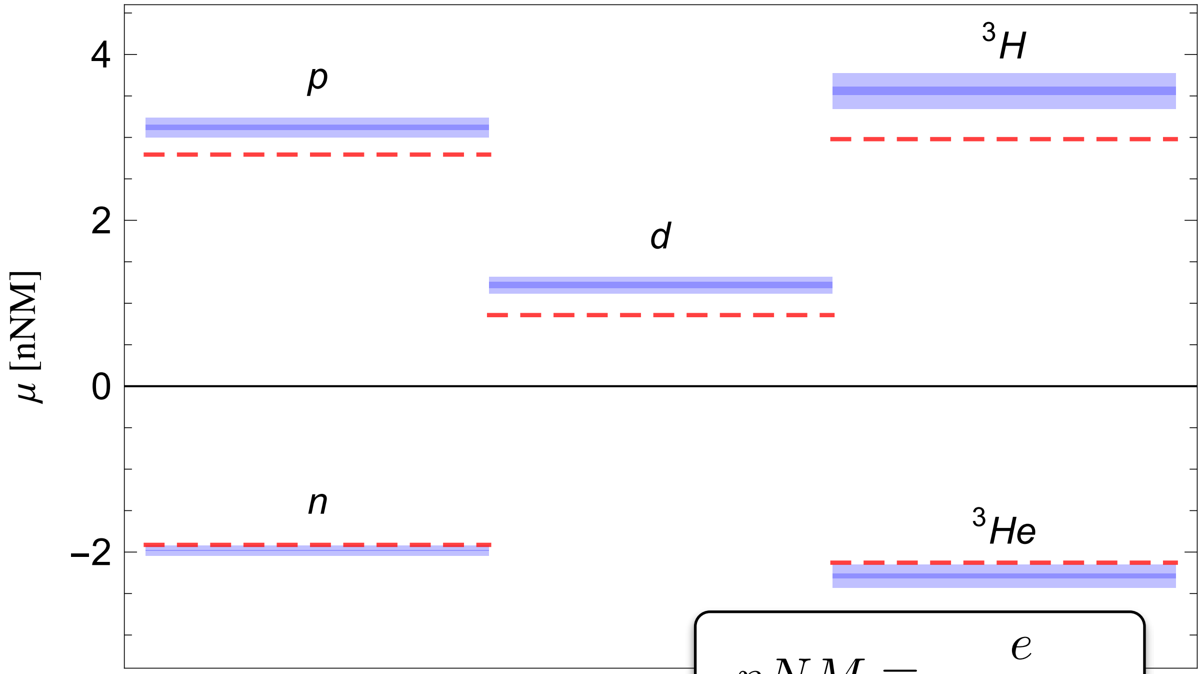

$$nNM \equiv \frac{e}{2M_{@m_q}}$$

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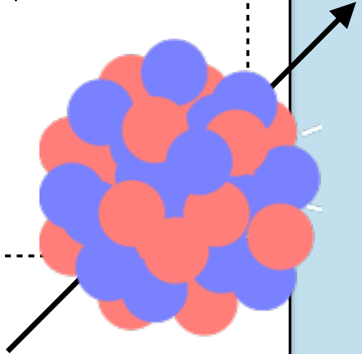

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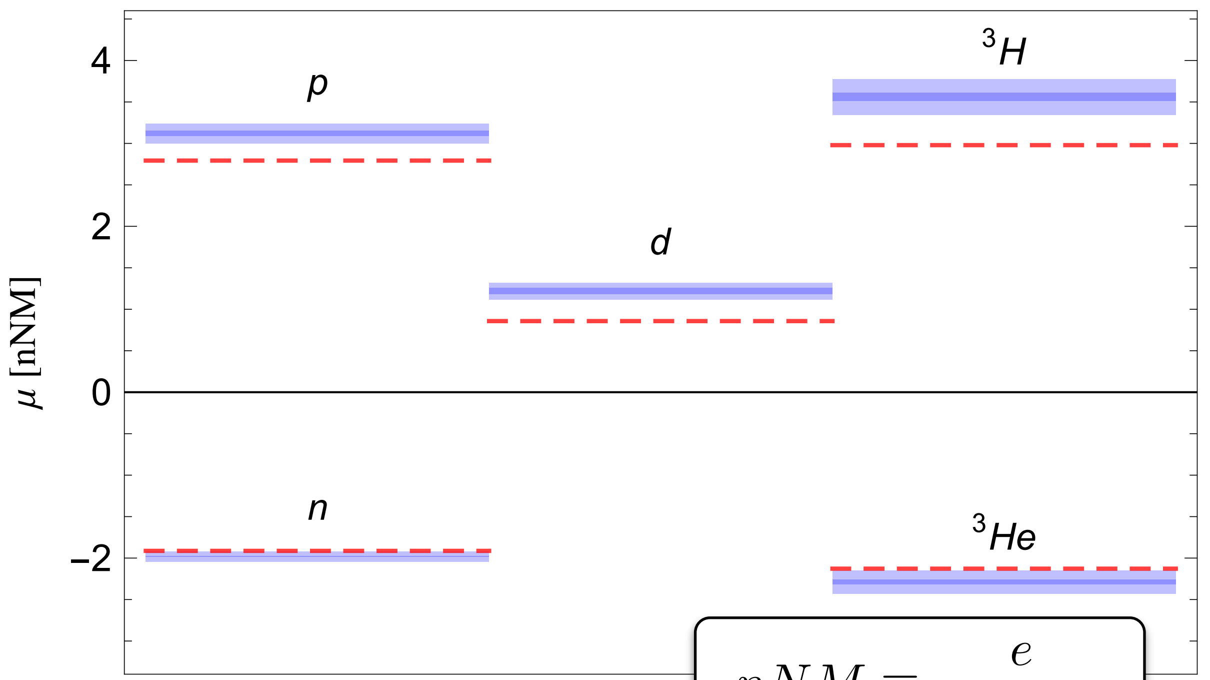

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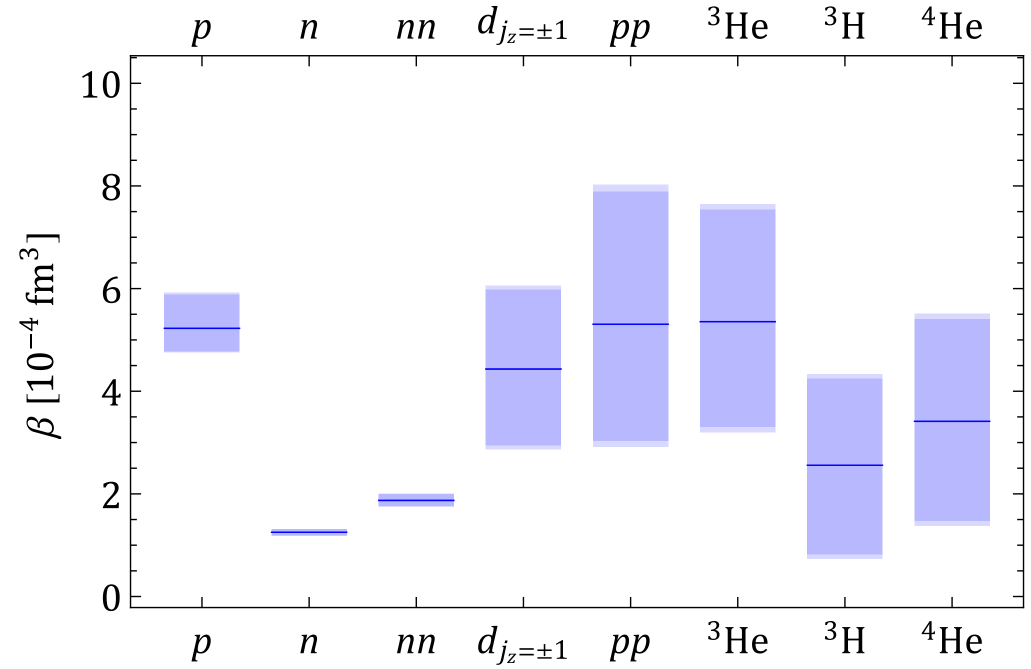
Magnetic polarizability



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