相对论重离子碰撞中的强电磁场和自旋极化

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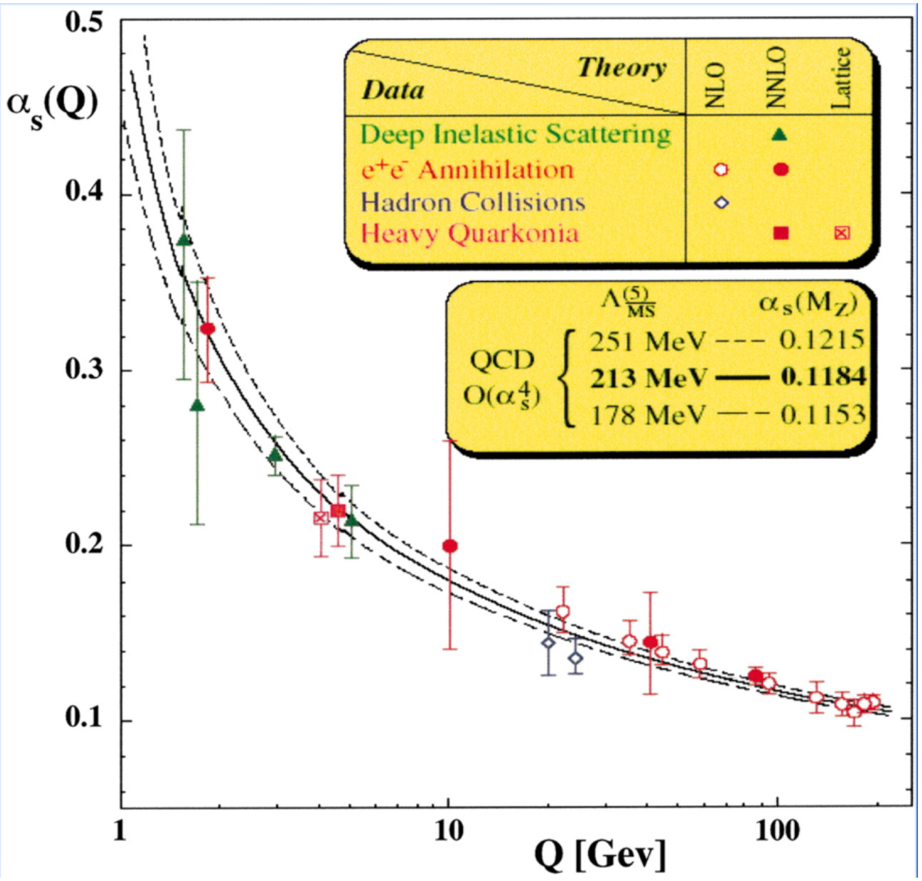
Outline

- **Introduction**
- **Strong magnetic fields, chiral magnetic effects and chiral kinetic theory**
- **Most vortical fluid, Polarization and Spin hydrodynamics**
- **Summary**

Introduction

Asymptotic freedom of QCD



The Nobel Prize in Physics 2004

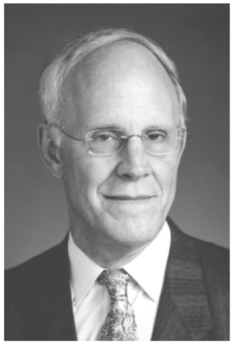


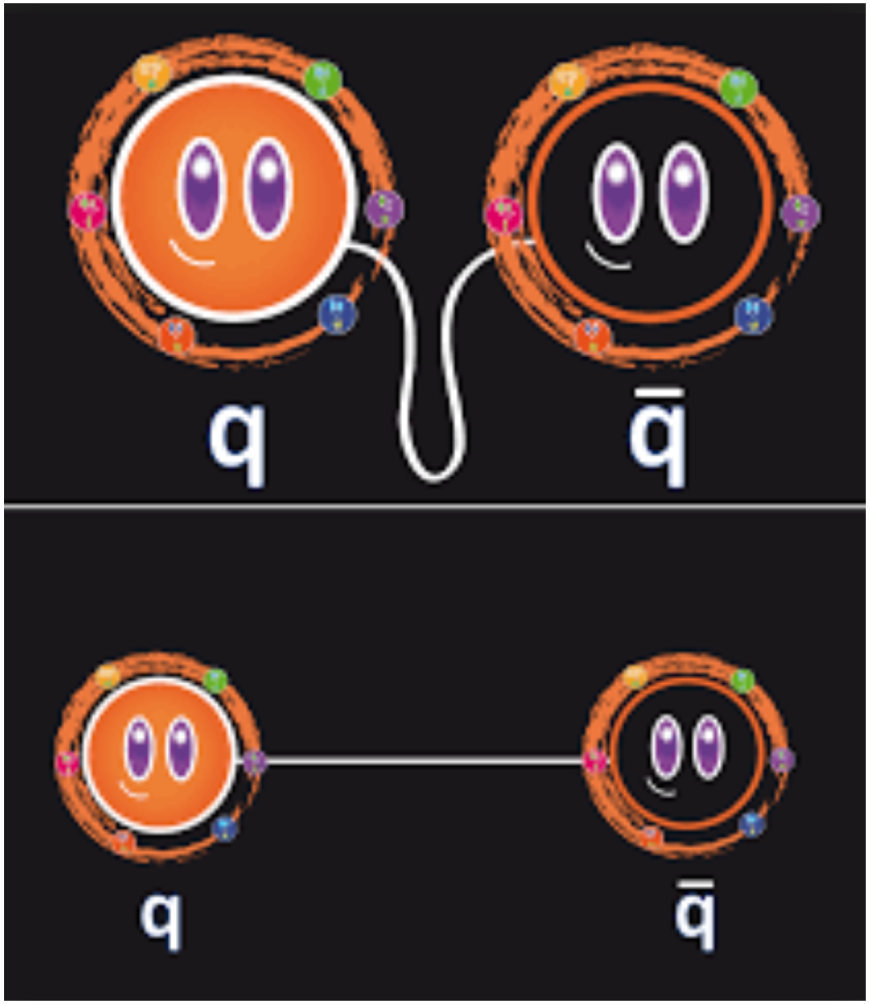
Photo from the Nobel Foundation archive.
David J. Gross
Prize share: 1/3

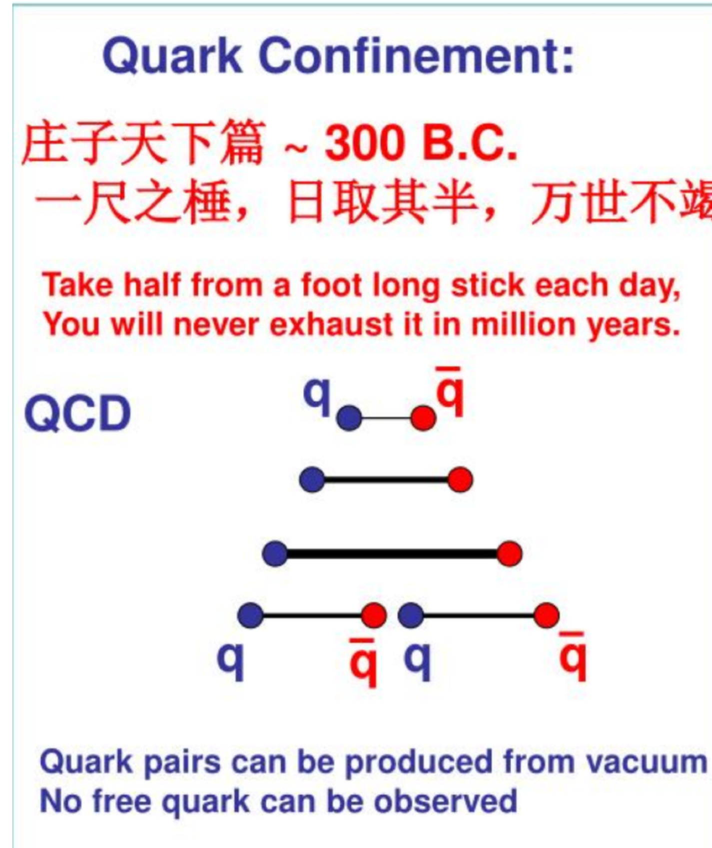
Photo from the Nobel Foundation archive.
H. David Politzer
Prize share: 1/3

Photo from the Nobel Foundation archive.
Frank Wilczek
Prize share: 1/3

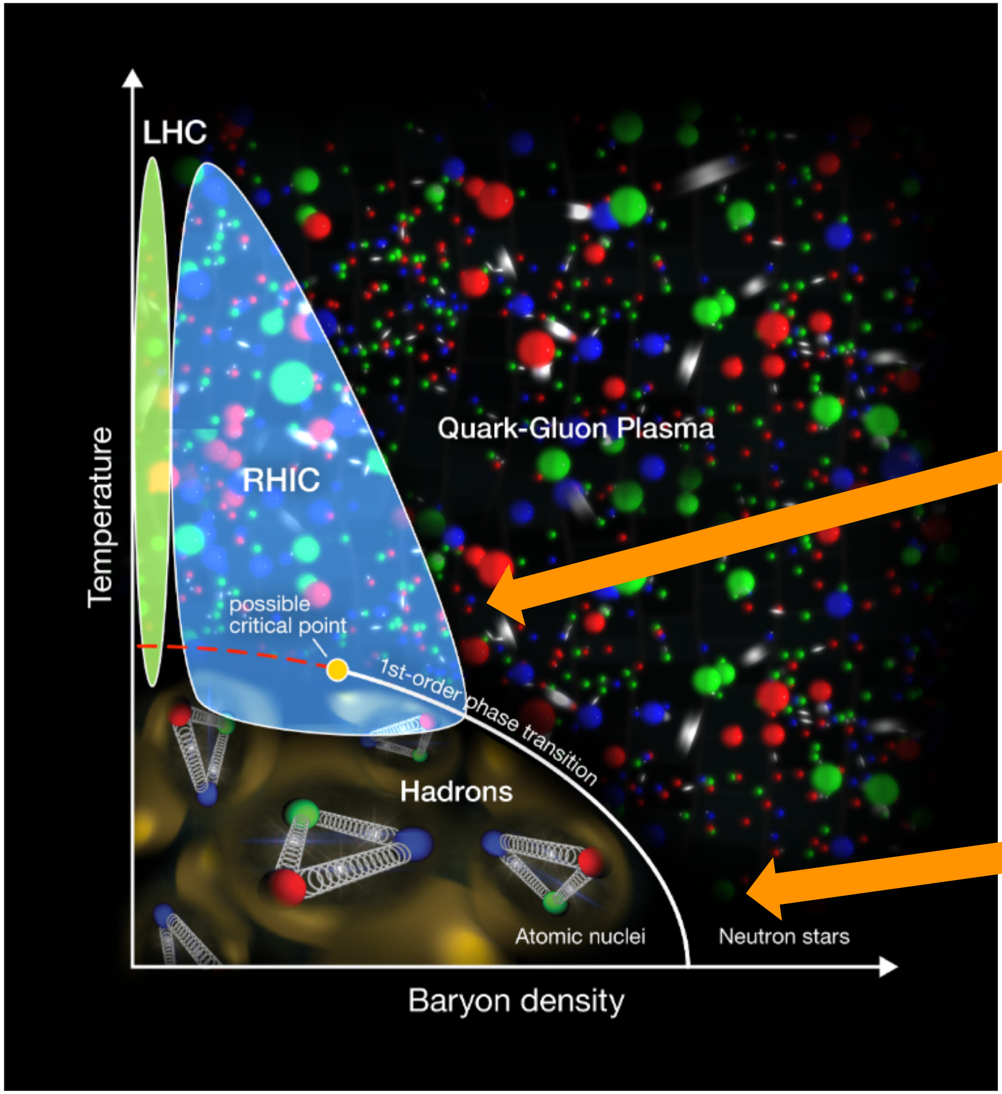
The Nobel Prize in Physics 2004 was awarded jointly to David J. Gross, H. David Politzer and Frank Wilczek "for the discovery of asymptotic freedom in the theory of the strong interaction."

Quark Confinement





Deconfinement phase transition



High temperature

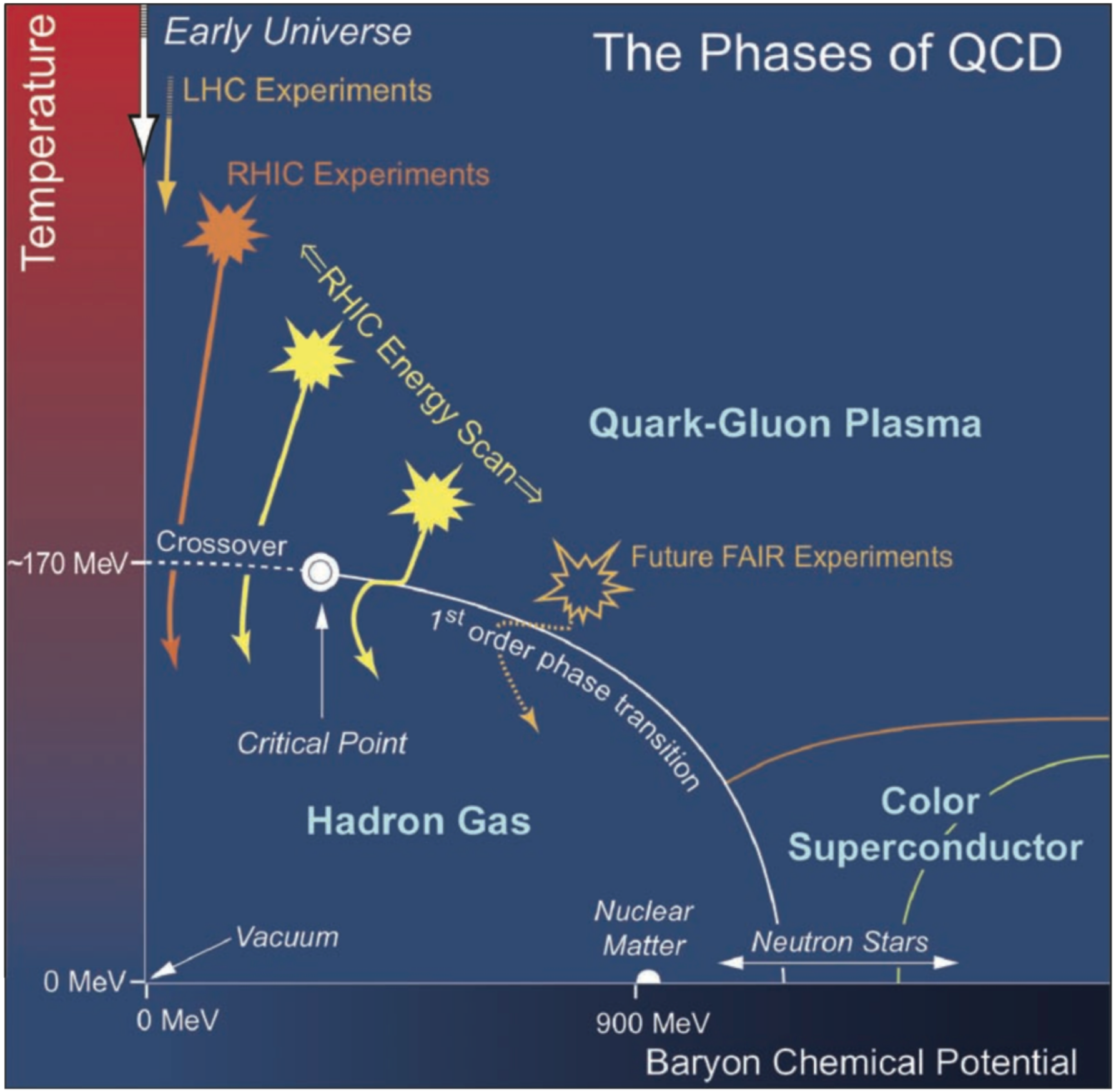
High pressure

核子重如牛 对撞生新态

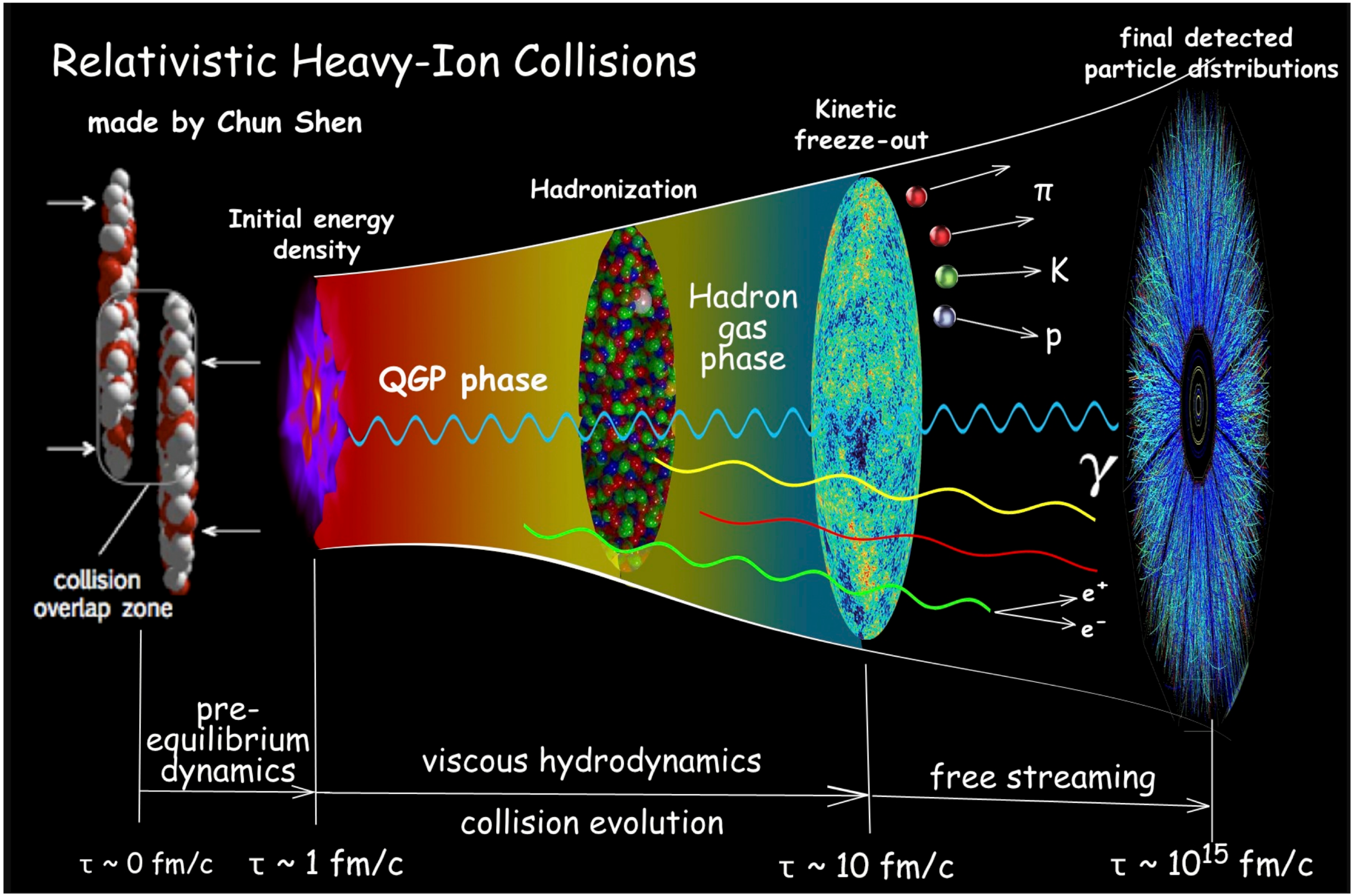
T.D. Lee (1974) and Collins (1975): Heavy ion collision to create a new form of matter!

李可染大师 1986年，
为 李政道先生作。

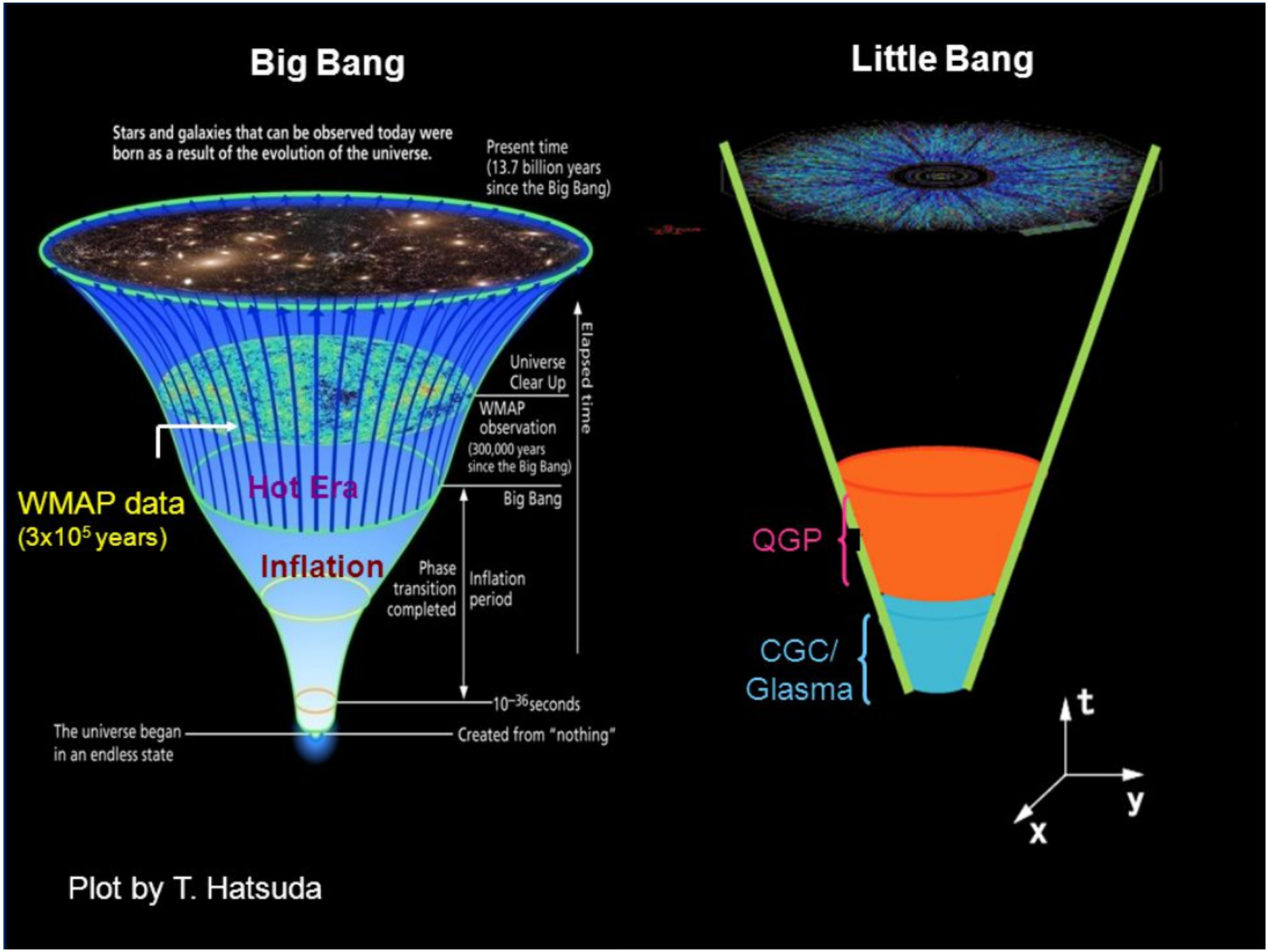
Phases of QCD



Relativistic heavy ion collisions



Little Bang VS Big Bang

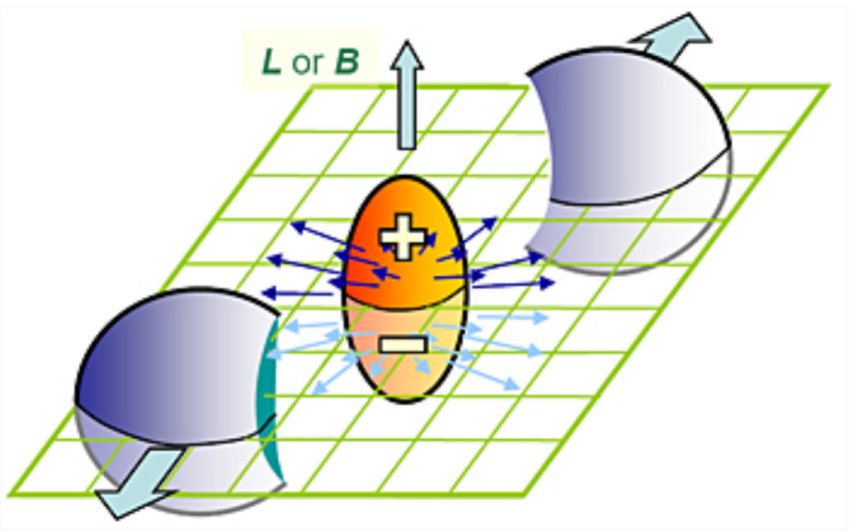


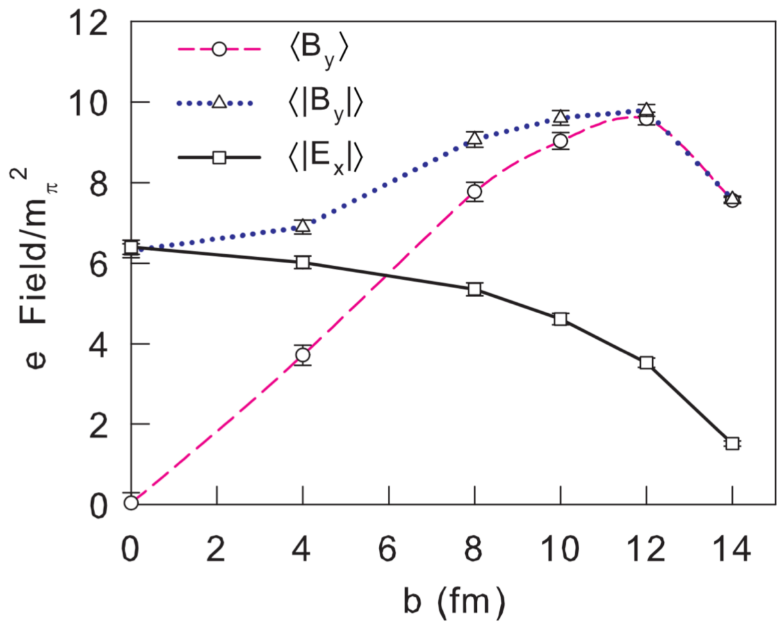
Strong magnetic fields
chiral magnetic effects
chiral kinetic theory

Strong Electromagnetic fields in HIC

- **Theoretical estimation**:

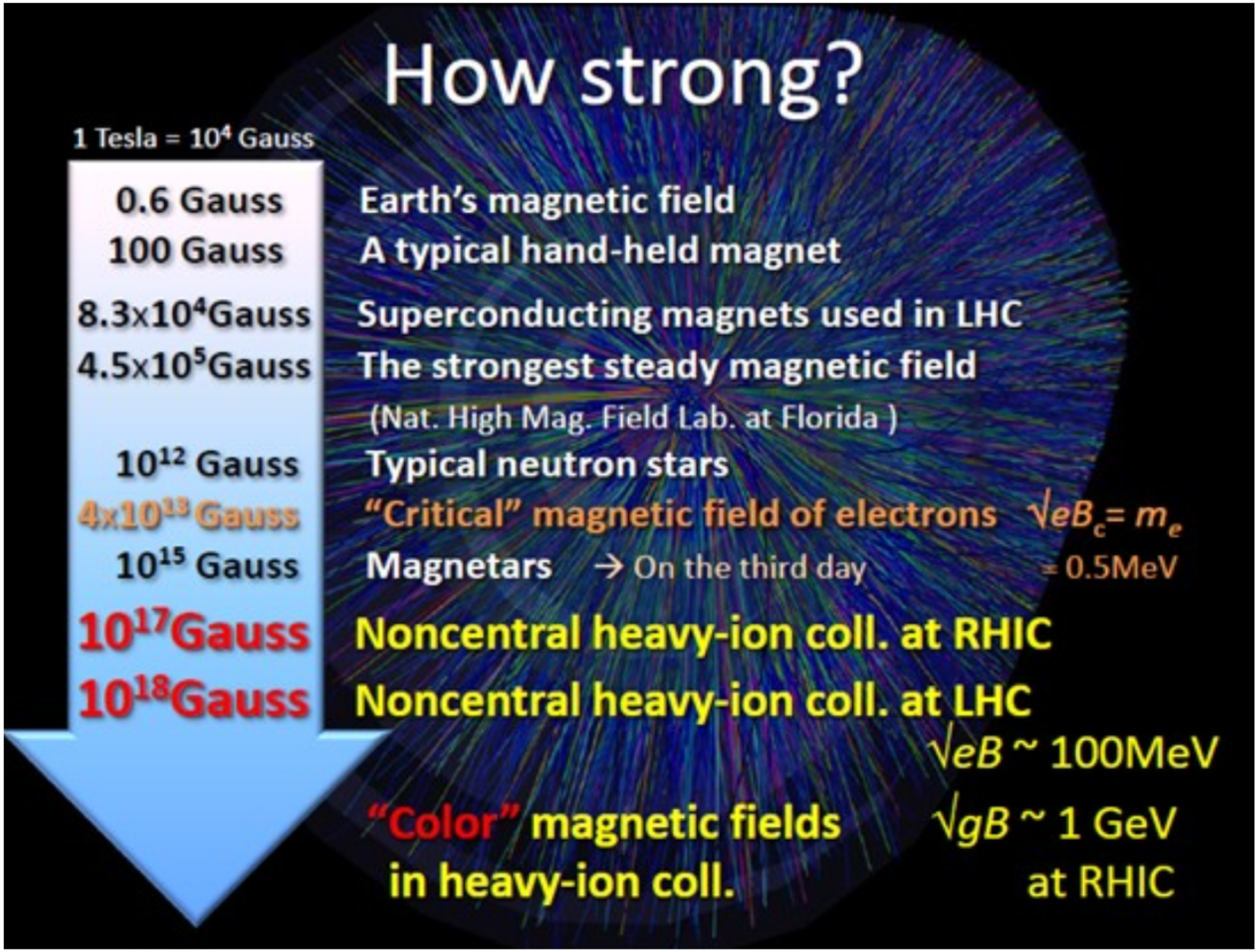
Lienard-Wiechert potential + Event-by-event



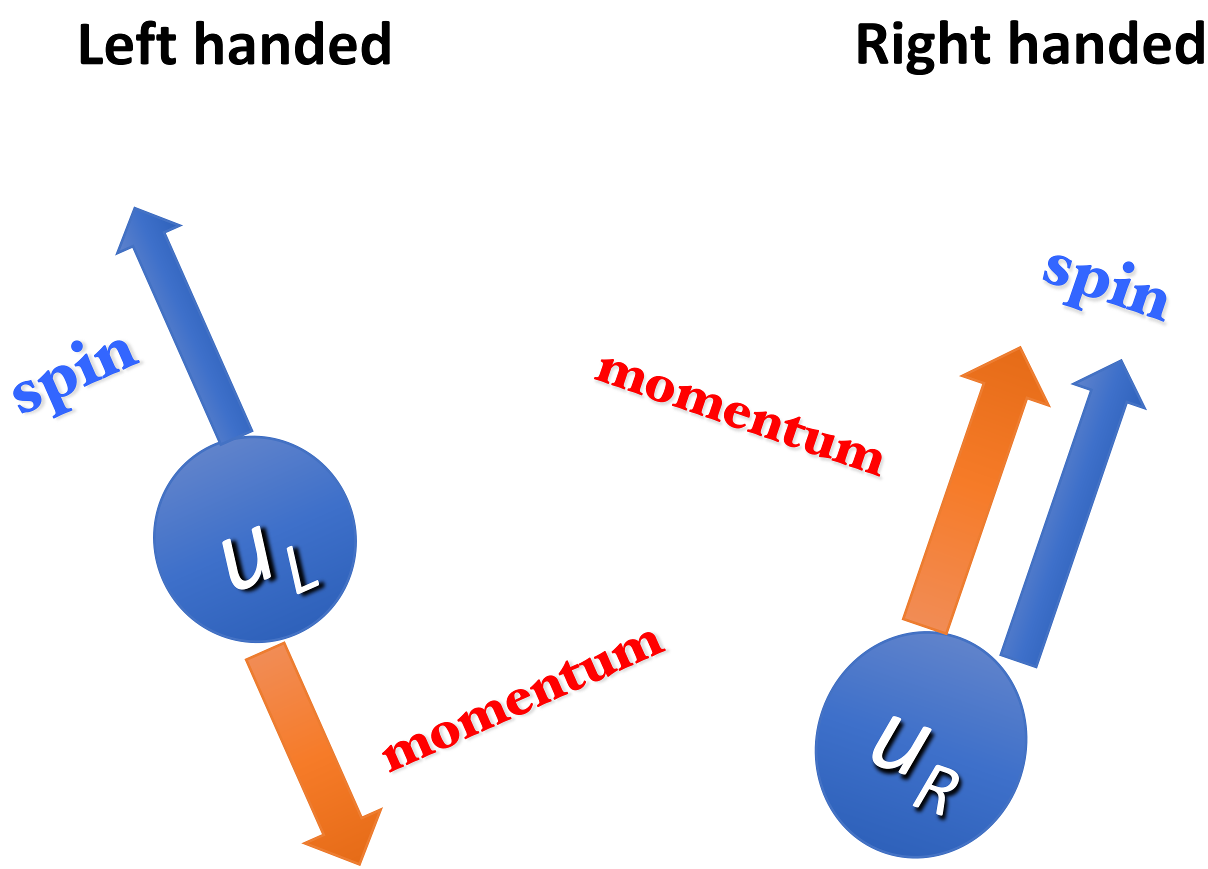


A. Bzdak, V. Skokov PRC 2012 ;W.T. Deng, X.G. Huang PRC 2012;V.Roy, SP, PRC 2015; H. Li, X.l. Sheng, Q.Wang, 2016; etc. / review: K. Tuchin 2013

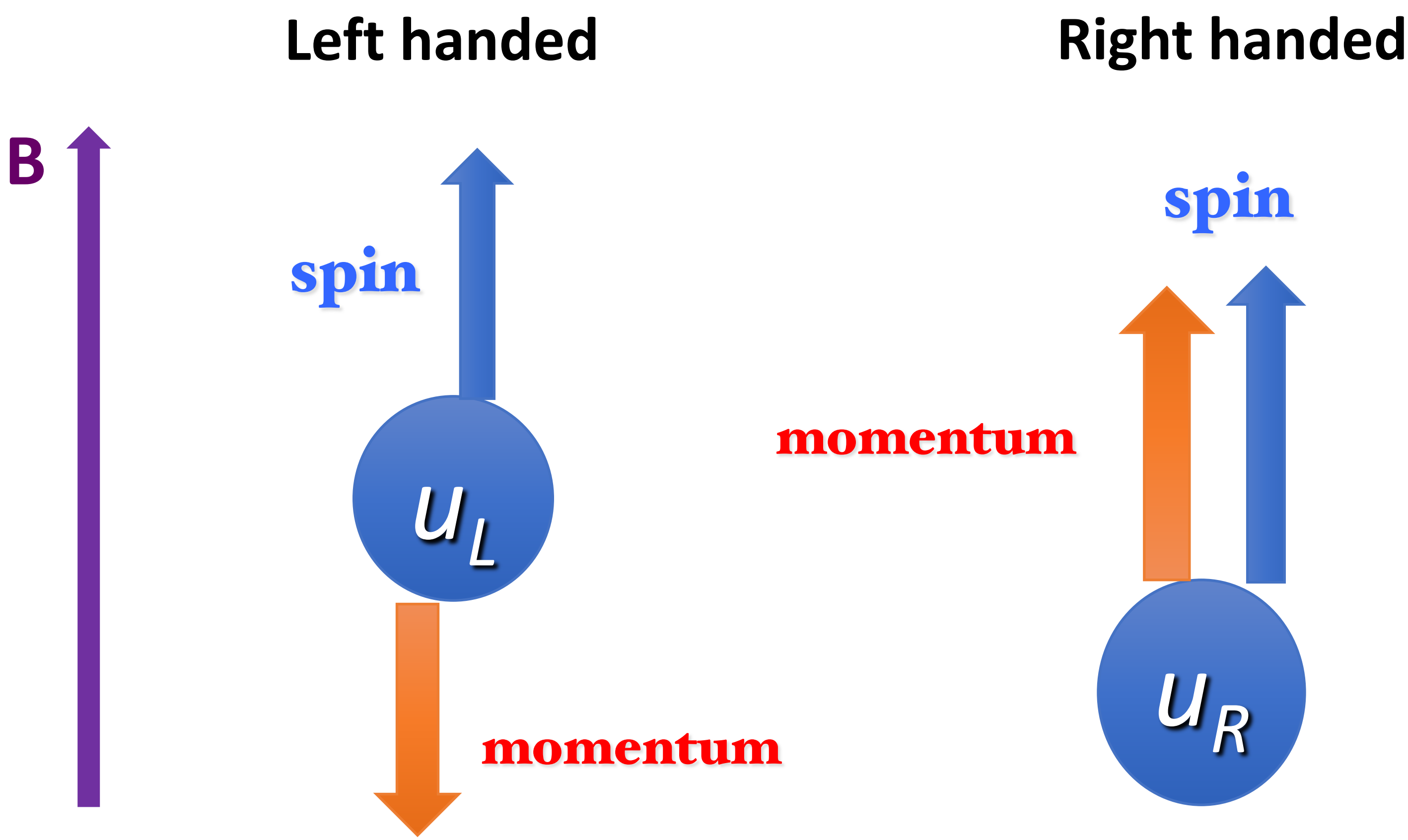
Strong magnetic fields



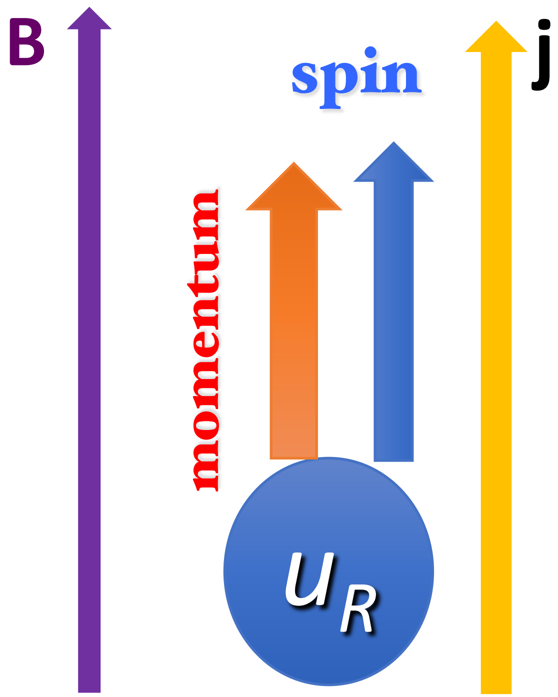
Chirality and massless fermions



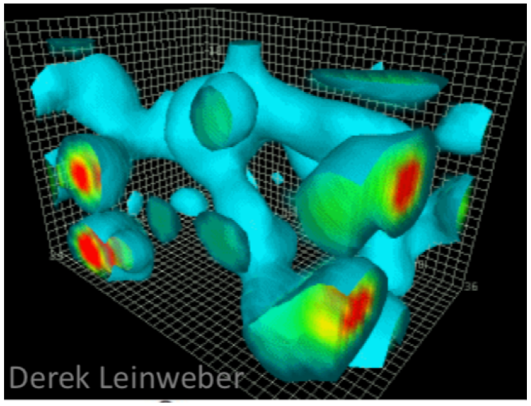
Polarization by magnetic fields

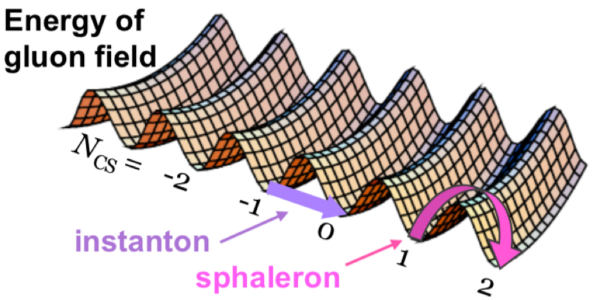


Chiral Magnetic Effect



- Magnetic fields
- Nonzero axial chemical potential
- Number of Left handed fermions ≠ Number of Right handed fermions





- Charge current: charge separation

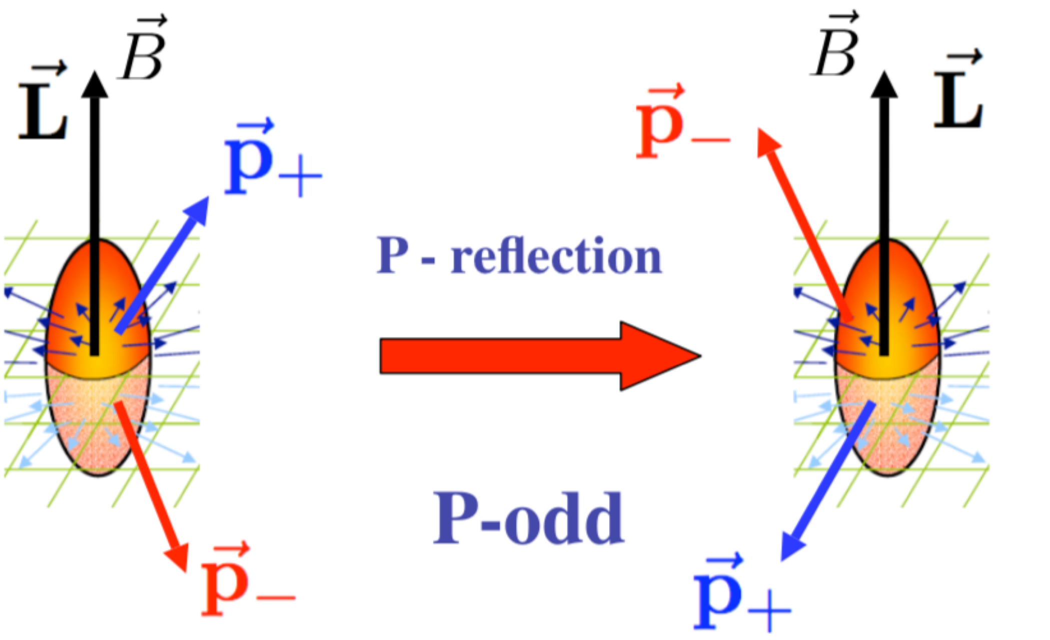
$$\boxed{\mathbf{j} = \frac{e^2}{2\pi^2}\mu_5 \mathbf{B},}$$

Kharzeev, Fukushima, Warrigna, (08,09), etc. ...

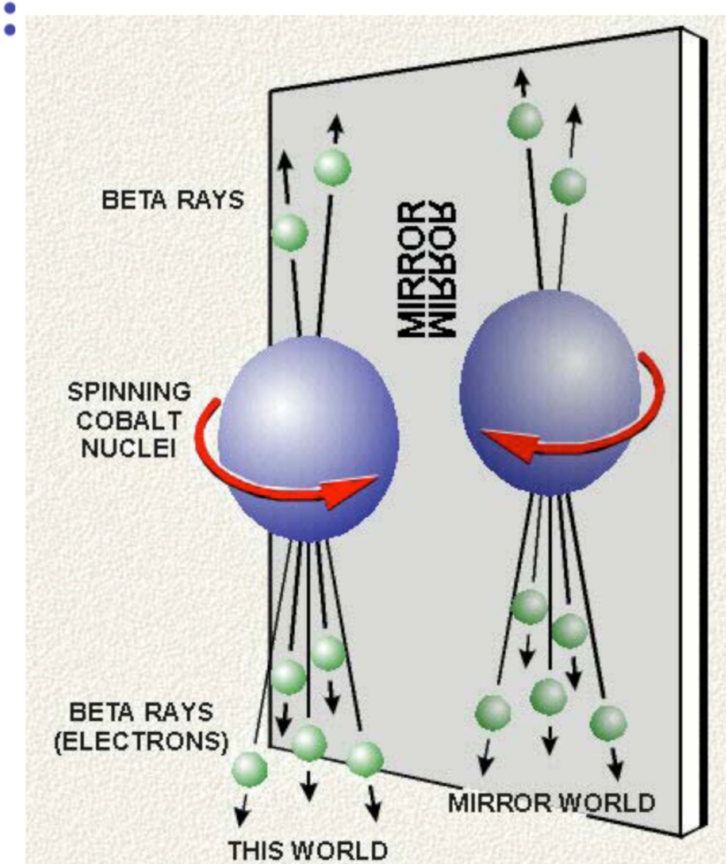
Charge separation ?= Parity Violation

Slides from Kharzeev's talk at 26th Winter Workshop on Nuclear Dynamics (2010)

Charge separation = parity violation:

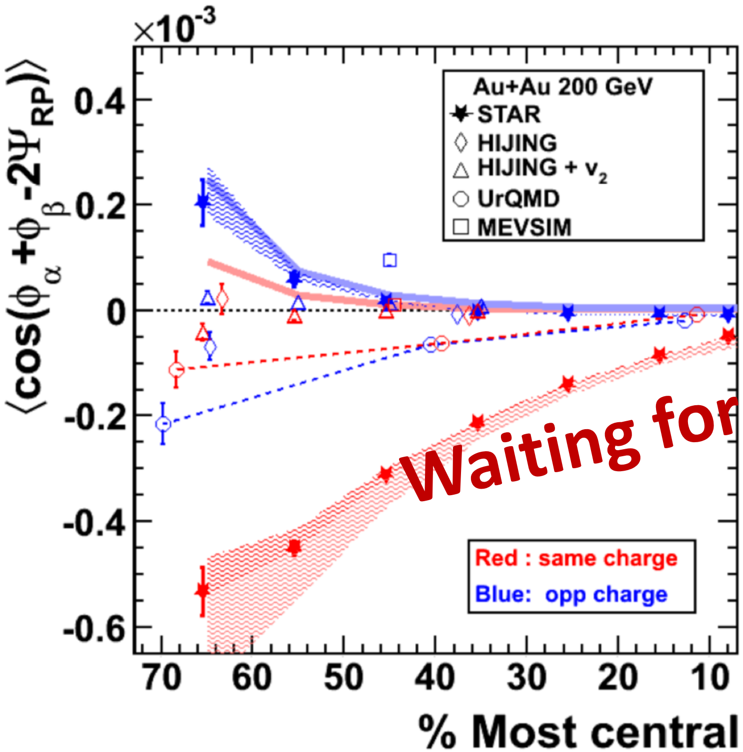


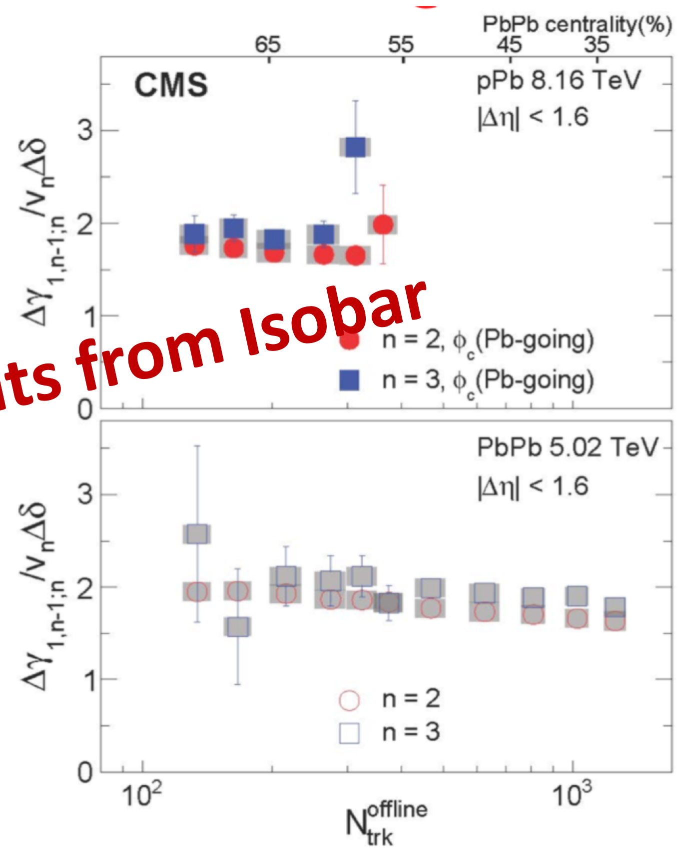
$$\mathcal{P}: \quad \vec{p} \to -\vec{p}; \quad \vec{B} \to \vec{B}; \quad \vec{L} \to \vec{L}$$



Experiments

- **Experiments: signal VS background**





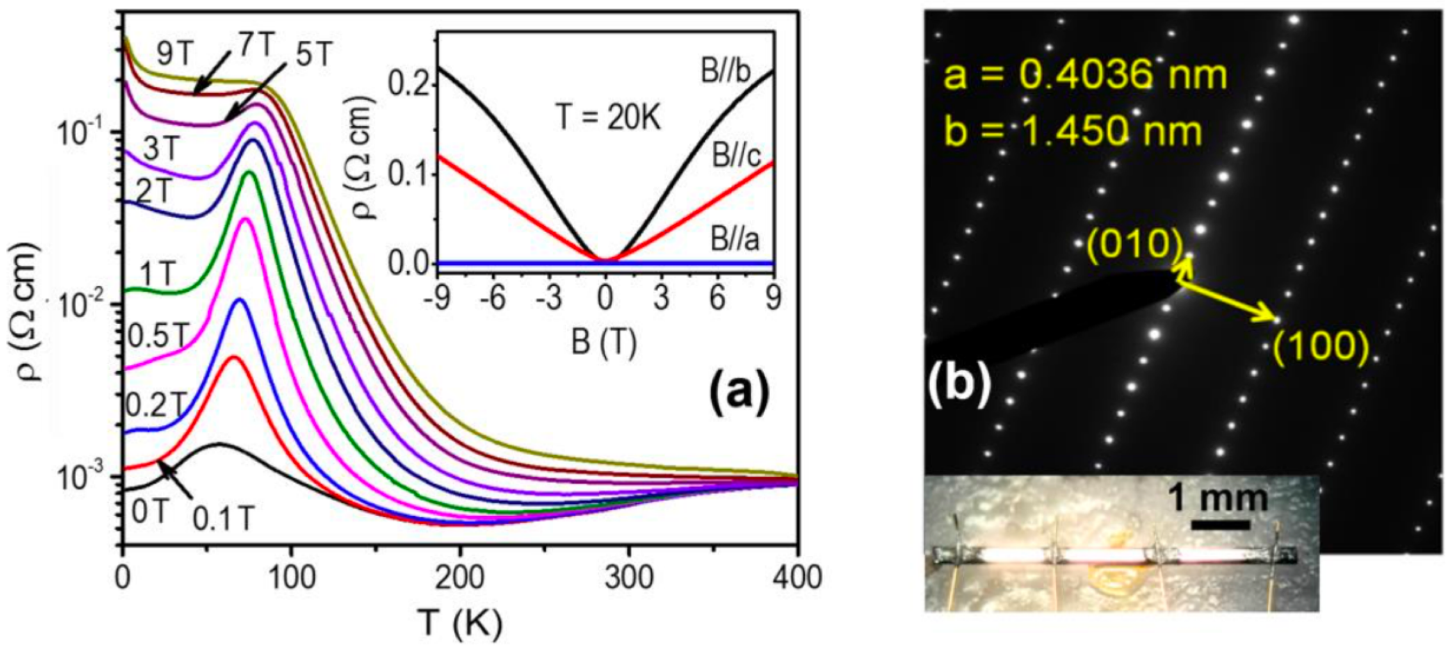
Waiting for the results from Isobar

STAR PRL 103, 251601(2009); PRC 81, 054908

CMS PRL 118, 122301 (2016); PRC 97, 044912

In condense matter

- **Weyl Semi-metal: new transport effects**



ZrTe$_5$: *Nature Physics, 12 , 550–554, (2016)*

Quantum kinetic theory (massless fermion)

- **Hamiltonian formulism, effective theory**

  Son, Yamamoto, PRL, (2012); PRD (2013)

- **Path integration**

  Stephanov, Yin, PRL (2012);

  Chen, Son, Stephanov, Yee, Yin, PRL, (2014);

  J.W. Chen, J.Y. Pang, SP, Q. Wang, PRD (2014)

- **Wigner function (Quantum field theory)**
  - **hydrodynamics, equilibrium**

    J.W. Chen, SP, Q. Wang, X.N. Wang, PRL (2013);
  - **out-of-equilibrium, quantum field theory**

    Y. Hidaka, SP, D.L. Yang, PRD(RC) (2017)
  - **Other studies**

    A.P. Huang, S.Z. Su, Y. Jiang, J.F. Liao, P.F. Zhuang, PRD (2018)

- **World-line formulism**

  N. Muller, R, Venugopalan PRD 2017

Also see Talks at Chirality Workshop in 2018, 2019 and Quark Matter Chirality section 2019

Kinetic theory

- **Assumptions:**

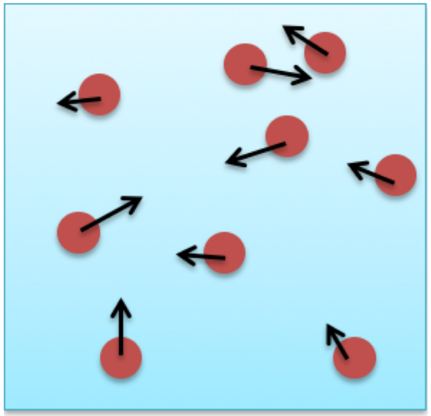
Mean free path >> collision length scaling

- **“distribution function” f(x,p,t)**

how many particles in a small

volume of phase space (x+dx, p+dp)

e.g. Fermi-Dirac distribution function



- **Ordinary kinetic theory: Boltzmann equation**

Dynamical evolution equation for f(x,p,t)

Ordinary Boltzmann equation

Particle's velocity:

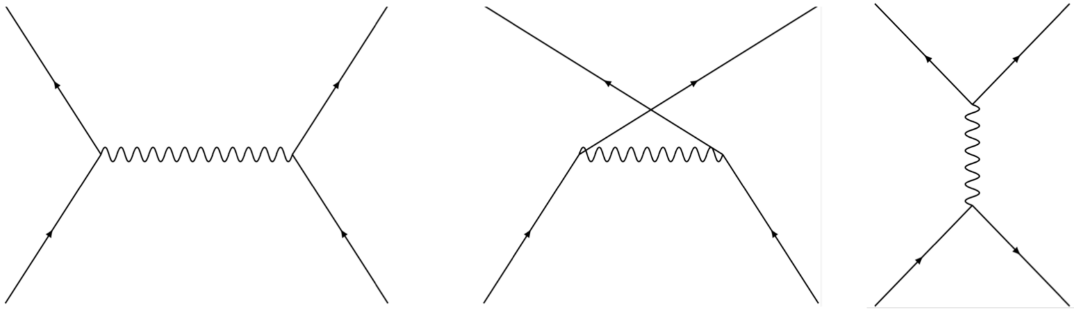
$$\dot{\mathbf{x}} = \frac{\partial \varepsilon}{\partial \mathbf{p}},$$

ε : Particle's energy

Lorentz force:

$$\dot{\mathbf{p}} = \mathbf{E} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{B},$$

$$\partial_t f + \dot{\mathbf{x}} \cdot \nabla_x f + \dot{\mathbf{p}} \cdot \nabla_p f = C[f],$$



Collison term:

What is Chiral kinetic theory?

Chiral Kinetic Theory = **Boltzmann equation** + **Massless fermions** + **Quantum Corrections**

Boltzmann equation ⟷

$$\partial_t f + \dot{\mathbf{x}} \cdot \nabla_x f + \dot{\mathbf{p}} \cdot \nabla_p f = C[f]$$

+

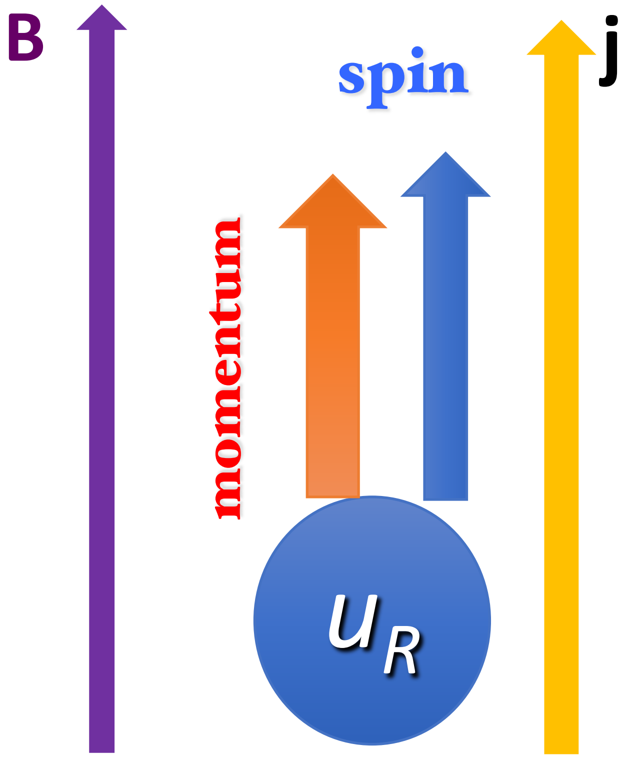
Quantum Corrections ⟷

$$\varepsilon = |\mathbf{p}| + \hbar \ldots$$

$$\dot{\mathbf{x}} = \frac{\partial \varepsilon}{\partial \mathbf{p}} + \hbar \ldots$$

$$\dot{\mathbf{p}} = \mathbf{E} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{B} + \hbar \ldots$$

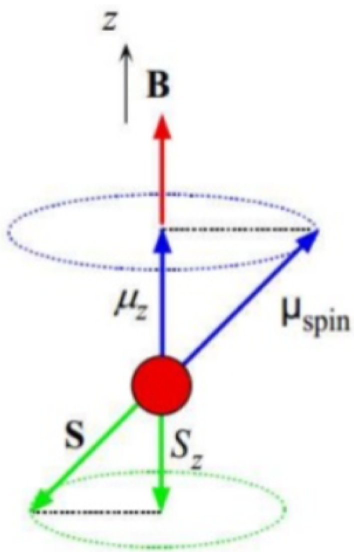
Let us "Guess" what the corrections are



Chiral Magnetic Effect

Spin coupled with magnetic fields

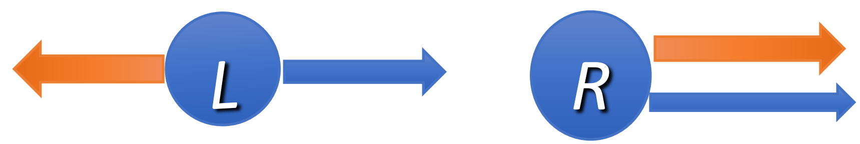
Quantum correction (I)



Spin magnetic moment:

$$\mu_S = -g_S \frac{e}{2m_e}\mathbf{S} \rightarrow -\frac{e}{|\mathbf{p}|}\mathbf{S} \rightarrow \mp \frac{e}{|\mathbf{p}|}\frac{\mathbf{p}}{2|\mathbf{p}|}$$

Massless **Chirality**

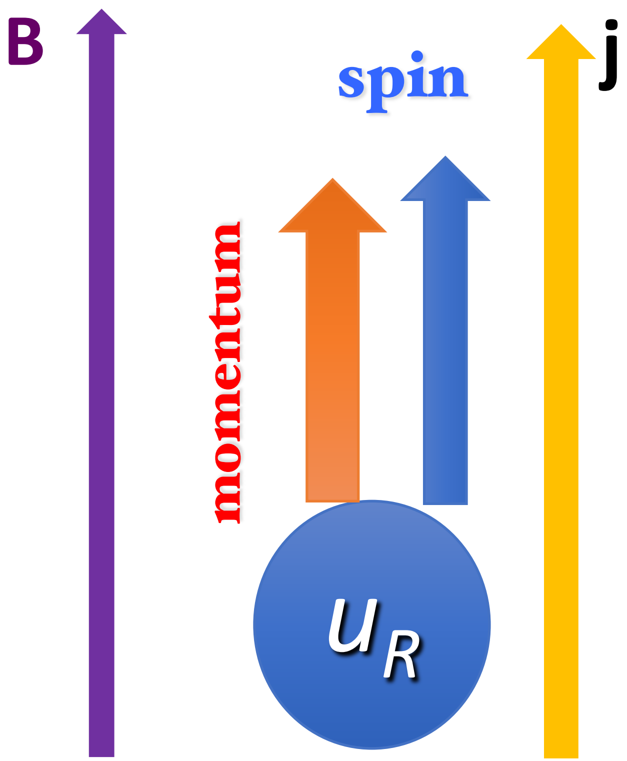


Zeeman effect:

$$\Delta\varepsilon = -\hbar\mu_S \cdot \mathbf{B} = \mp\hbar\frac{|e|}{|\mathbf{p}|}\frac{\mathbf{p}\cdot\mathbf{B}}{2|\mathbf{p}|}$$

Quantum correction (II)

- **Correction to effective velocity/w.o. E fields**



Particles move parallel or anti-parallel to B

$$\Delta\dot{\mathbf{x}} \propto \mathbf{B}$$

Dimension analysis

$$\Delta\dot{\mathbf{x}} \propto \frac{\mathbf{B}}{|\mathbf{p}|^2}$$

Final results:

$$\Delta\dot{\mathbf{x}} = \hbar\frac{\mathbf{B}}{2|\mathbf{p}|^2}$$

Quantum correction (II)

- **Correction to effective velocity with E fields**

$$\Delta\dot{\mathbf{x}} = \hbar \frac{\mathbf{B}}{2|\mathbf{p}|^2}$$

For moving particles, they feel like:

$$\mathbf{B} \to \mathbf{B} + \mathbf{E} \times \mathbf{v}$$

$$\Rightarrow \quad \Delta\dot{\mathbf{x}} = \hbar \frac{\mathbf{B}}{2|\mathbf{p}|^2} + \hbar \frac{1}{2|\mathbf{p}|^2} \mathbf{E} \times \mathbf{v}$$

Quantum correction (III)

- **Are there corrections to effective force?**

$$\dot{\mathbf{p}} = \mathbf{E} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{B} + \hbar \ldots$$

- **History: in condensate matter physics:**

Could be neglected!

D. Xiao, M.C. Chang, Q. Niu, Rev. Mod. Phys. 82, 1959 (2010)

- **QFT: Chiral anomaly!**

Son, Yamamoto, PRL, (2012); PRD (2013)

Stephanov, Yin, PRL (2012);

J.W. Chen, SP, Q. Wang, X.N. Wang, PRL (2013);

Chiral kinetic equation

$$\boxed{\sqrt{G}\partial_t f + \sqrt{G}\dot{\mathbf{x}}\cdot\nabla_x f + \sqrt{G}\dot{\mathbf{p}}\cdot\nabla_p f = C[f].}$$

- **Particle's effective velocity:**

$$\sqrt{G}\dot{\mathbf{x}} = \frac{\partial\varepsilon}{\partial\mathbf{p}} + \hbar\left(\frac{\partial\varepsilon}{\partial\mathbf{p}}\cdot\mathbf{\Omega}\right)\mathbf{B} + \hbar\mathbf{E}\times\mathbf{\Omega},$$

- **Effective force:**

$$\sqrt{G}\dot{\mathbf{p}} = \mathbf{E} + \frac{\partial\varepsilon}{\partial\mathbf{p}}\times\mathbf{B} + \hbar(\mathbf{E}\cdot\mathbf{B})\mathbf{\Omega},$$

- **Berry curvature**

$$\sqrt{G} = 1 + \hbar\mathbf{B}\cdot\mathbf{\Omega}, \qquad \mathbf{\Omega} = \frac{\mathbf{p}}{2|\mathbf{p}|^3},$$

Wigner function (I)

- **Wigner operator**

$$\hat{W}_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \bar{\psi}_\beta(x_+) U(x_+, x_-) \psi_\alpha(x_-),$$

- **Wigner function:**

Gauge link $U(x_+, x_-) \equiv e^{-iQ\int_{x-}^{x+} dz^\mu A_\mu(z)},$

$$W(x,p) = \left\langle : \hat{W}(x,p) : \right\rangle$$

W operator in thermal ensemble average and normal ordering of the operators

- **Physical meaning: QFT version density matrix**

Vasak, Gyulassy, Elze, Ann. Phys. (N.Y.) 173, 462 (1987);

Elze, Heinz,Phys.Rep.183,81(1989).

Wigner function (II)

- **Master equations for Wigner function:**

$$\gamma_\mu\left(p^\mu + \frac{i}{2}\nabla^\mu\right)W(x,p) = 0, \qquad \nabla^\mu \equiv \partial_x^\mu - QF^\mu{}_\nu\partial_p^\nu$$

- **Matrix decomposition**

$$W = \frac{1}{4}\left[\mathscr{F} + i\gamma^5\mathscr{P} + \gamma^\mu\mathscr{V}_\mu + \gamma^5\gamma^\mu\mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right],$$

Charge current

$$\mathscr{V}^\mu = \int \frac{d^4y}{(2\pi)^4}e^{-ip\cdot y} <: \bar{\psi}_\beta\left(x+\frac{1}{2}y\right)\gamma^\mu U\left(x+\frac{1}{2}y, x-\frac{1}{2}y\right)\psi_\alpha\left(x-\frac{1}{2}y\right) :>$$

Chiral current

$$\mathscr{A}_\mu = \int \frac{d^4y}{(2\pi)^4}e^{-ip\cdot y} <: \bar{\psi}_\beta\left(x+\frac{1}{2}y\right)\gamma^\mu\gamma^5 U\left(x+\frac{1}{2}y, x-\frac{1}{2}y\right)\psi_\alpha\left(x-\frac{1}{2}y\right) :>$$

Wigner function (III)

- **Left and right handed currents**

$$\mathscr{J}^s_\mu(x,p) = \frac{1}{2}[\mathscr{V}_\mu(x,p) + s\mathscr{A}_\mu(x,p)]; \qquad s = \pm$$

- **In massless limit**

$$\begin{aligned} p^\mu \mathscr{J}^s_\mu(x,p) &= 0, \\ \nabla^\mu \mathscr{J}^s_\mu(x,p) &= 0, \\ 2s(p^\lambda \mathscr{J}^\rho_s - p^\rho \mathscr{J}^\lambda_s) &= -\epsilon^{\mu\nu\lambda\rho}\nabla_\mu \mathscr{J}^s_\nu . \end{aligned}$$

hbar expansion

- **hbar (gradient) expansion**

$$\mathscr{J}^s_\mu(x,p) = \mathscr{J}^s_{\mu,(0)}(x,p) + \mathscr{J}^s_{\mu,(1)}(x,p) + \cdots ,$$

- **Leading order is the classical currents.**

We introduce the initial distribution function f(x,p) as input.

$$\mathscr{J}^\rho_{(0)s}(x,p) \;=\; p^\rho f_s \delta(p^2),$$

- **Next-to-leading order**

$$\mathscr{J}^\rho_{(1)s}(x,p) \;=\; -\frac{s}{2}\tilde{\Omega}^{\rho\lambda} p_\lambda \frac{df_s}{dp_0}\delta(p^2) - \frac{s}{p^2} e\tilde{F}^{\rho\lambda} p_\lambda f_s \delta(p^2).$$

$$\Omega_{\nu\sigma} = \tfrac{1}{2}(\partial_\nu u_\sigma - \partial_\sigma u_\nu), \text{ and } \Omega^{\mu\nu} = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\tilde{\Omega}_{\rho\sigma}.$$

Currents

- **Integral over momentum**

$$j_s^\mu = \int d^4p\, \mathscr{J}_s^\mu = n_s u^\mu + \xi_{B,s} B^\mu + \xi_s \omega^\mu,$$

Charge current

$$j^\mu = \sum_{s=\pm} j_s^\mu = n u^\mu + \underline{\xi_B B^\mu} + \xi \omega^\mu,$$

Chiral current

$$j_5^\mu = \sum_{s=\pm} s j_s^\mu = n_5 u^\mu + \xi_{B5} B^\mu + \xi_5 \omega^\mu,$$

$$\xi_B = \frac{e}{2\pi^2}\mu_5, \qquad \xi_{B5} = \frac{e}{2\pi^2}\mu,$$

$$\xi = \frac{1}{\pi^2}\mu\mu_5, \qquad \xi_5 = \frac{1}{6}T^2 + \frac{1}{2\pi^2}\left(\mu^2 + \mu_5^2\right)$$

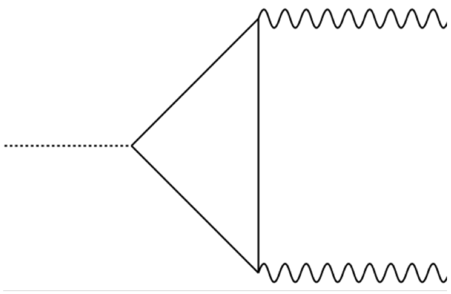
Gao, Liang, Pu, Q. Wang, X.N. Wang, PRL 109, 232301 (2012)

Chiral anomaly

- **Chiral anomaly**

$$\partial_\mu j^\mu = 0,$$

$$\partial_\mu j_5^\mu = -\frac{e^2}{2\pi^2} E\cdot B.$$



We reproduce the chiral anomaly from the kinetic theory!!!

Gao, Liang, Pu, Q. Wang, X.N. Wang, PRL 109, 232301 (2012)

Derivation of chiral kinetic theory

- **Constrain equation.**

$$\nabla^{\mu}\mathscr{J}_{\mu}^{s}(x,p) \;=\; 0,$$

- **We can insert our results into this equation and get the constraint equation for distribution function.**
- **Then, we need to integral over p0 to get 3-dim form.**

J.W. Chen, SP, Q. Wang, X.N. Wang, PRL (2013);

Y. Hidaka, SP, D.L. Yang, PRD(RC) (2017)

Review: *Gao, Liang, Wang, Int.J.Mod.Phys A 36 (2021), 2130001*

Hidaka, SP, D.L. Yang, Q. Wang, invited review, in preparation

Chiral kinetic equation

$$\sqrt{G}\partial_t f + \sqrt{G}\dot{\mathbf{x}} \cdot \nabla_x f + \sqrt{G}\dot{\mathbf{p}} \cdot \nabla_p f = C[f].$$

- **Particle's effective velocity:**

$$\sqrt{G}\dot{\mathbf{x}} = \frac{\partial \varepsilon}{\partial \mathbf{p}} + \hbar \left(\frac{\partial \varepsilon}{\partial \mathbf{p}} \cdot \mathbf{\Omega} \right) \mathbf{B} + \hbar \mathbf{E} \times \mathbf{\Omega},$$

- **Effective force:**

$$\sqrt{G}\dot{\mathbf{p}} = \mathbf{E} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{B} + \hbar (\mathbf{E} \cdot \mathbf{B}) \mathbf{\Omega},$$

- **Berry curvature**

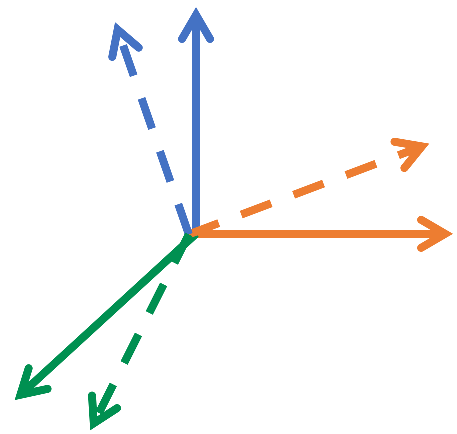
$$\sqrt{G} = 1 + \hbar \mathbf{B} \cdot \mathbf{\Omega}, \qquad \mathbf{\Omega} = \frac{\mathbf{p}}{2|\mathbf{p}|^3},$$

Subgroup of Lorentz symmetry

- **Massive particles: Rest frame**

$$p^\mu = (m, 0, 0, 0)$$

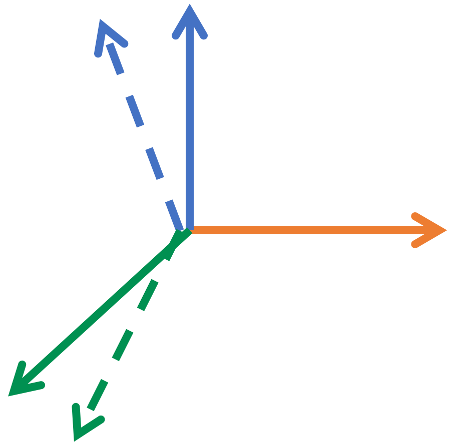
Subgroup: SO(3)



- **Massless particles: No rest frame**

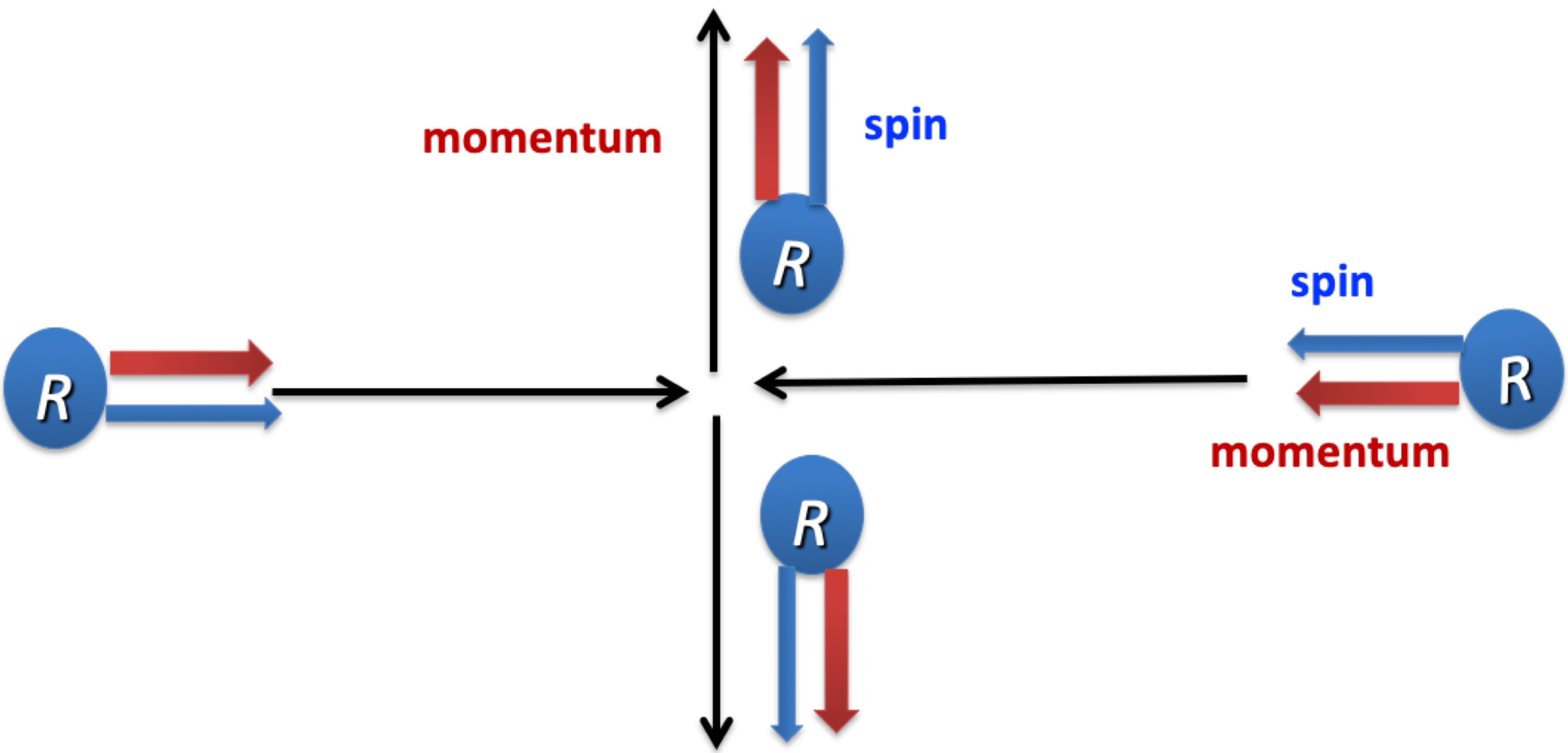
$$p^\mu = (|p_z|, 0, 0, p_z)$$

Subgroup: ISO(2)



Side-jump (I)

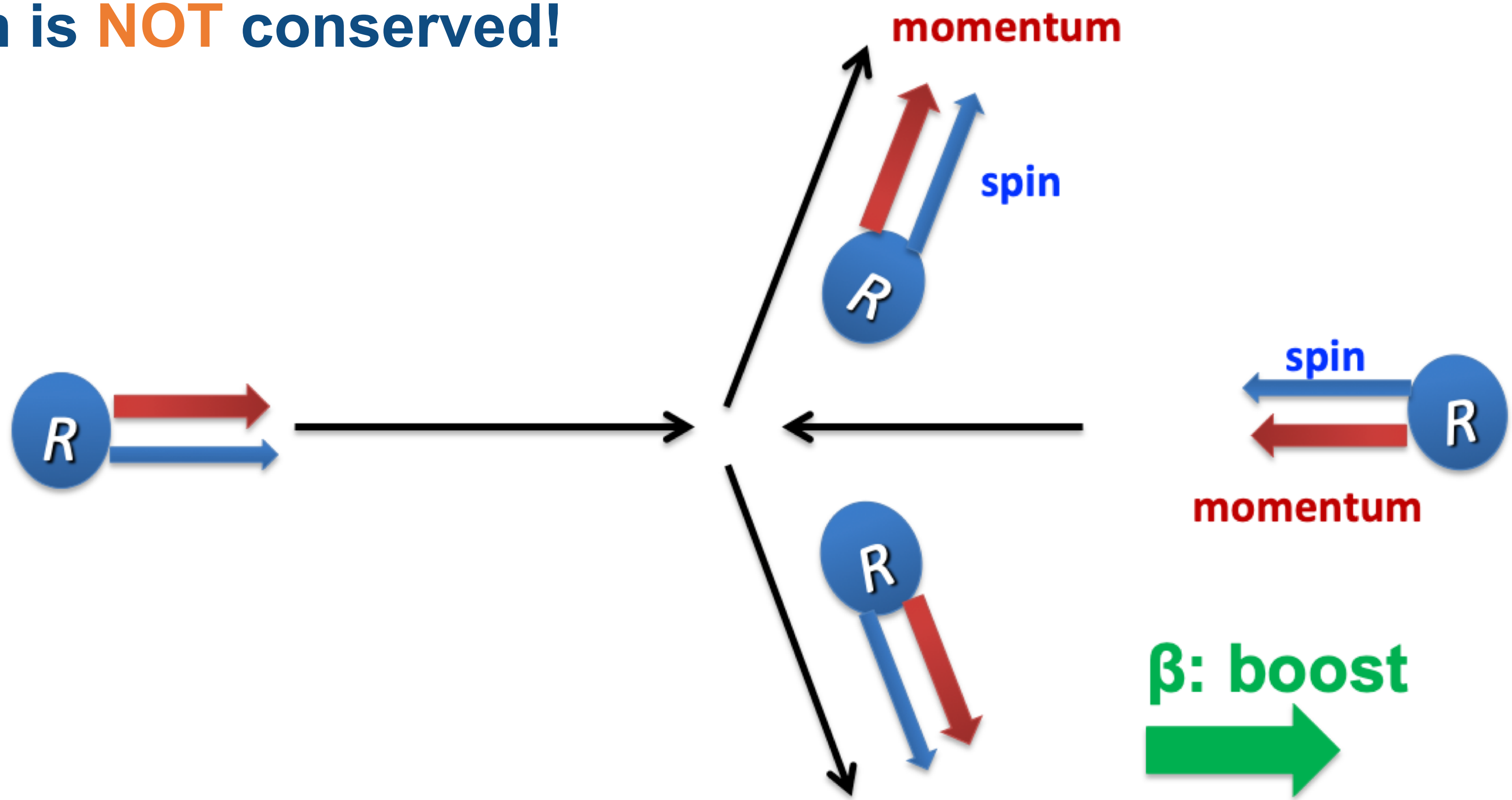
Orbital angular momentum and spin are conserved separately



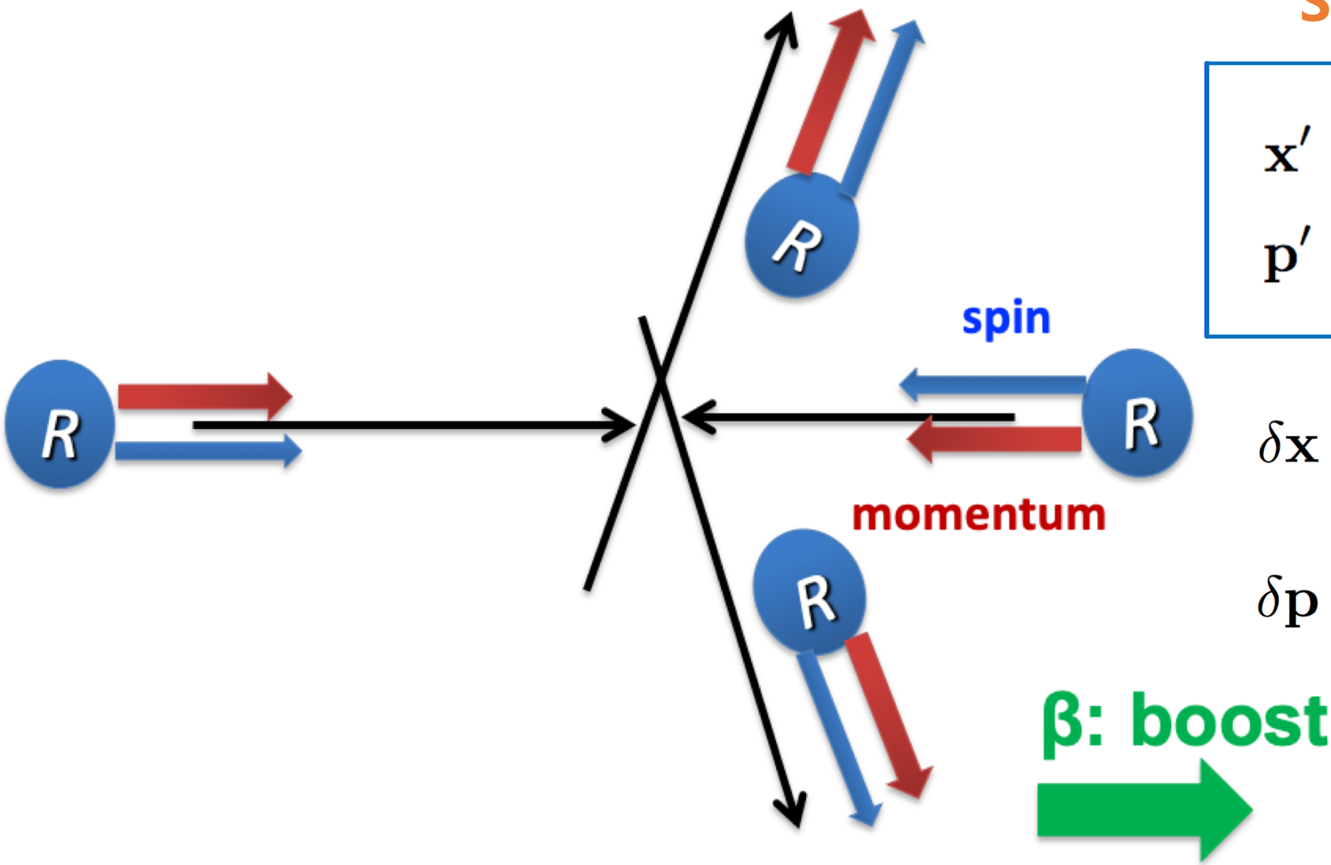
Chen, Son, Stephanov, Yee, Yin, PRL, (2014)

Side-jump (II)

Orbital angular momentum ?
Spin is NOT conserved!



Side-jump (III)



x has a shift!!!

"Side-jump":

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} + \boldsymbol{\beta} t + \delta \mathbf{x}, \\ \mathbf{p}' &= \mathbf{p} + \boldsymbol{\beta} \varepsilon + \delta \mathbf{p}, \end{aligned}$$

$$\begin{aligned} \delta \mathbf{x} &= \hbar \frac{\boldsymbol{\beta} \times \hat{\mathbf{p}}}{2|\mathbf{p}|}, \\ \delta \mathbf{p} &= \hbar \frac{\boldsymbol{\beta} \times \hat{\mathbf{p}}}{2|\mathbf{p}|} \times \mathbf{B} \end{aligned}$$

Chen, Son, Stephanov, PRL, (2015);
Y. Hidaka, SP, D.L. Yang, PRD (2016)

Non-trivial Lorentz symmetry

- **Quantum field theory**

$$j^\mu = \overline{\psi}\sigma^\mu\psi \to \Lambda^\mu_\nu j^\nu$$

- **Lorentz transformation**

$$x^{\mu\prime} = \Lambda^\mu_\nu x^\nu,\ p^{\mu\prime} = \Lambda^\mu_\nu p^\nu,$$

$$\boxed{f'(x',p',t') = f(x',p',t') + \hbar N^\mu(\partial^x_\mu + F_{\nu\mu}\partial^\nu_p)f,}$$

Infinitesimal Lorentz Transform

$$\delta \mathbf{x} = \hbar\frac{\boldsymbol{\beta}\times\hat{\mathbf{p}}}{2|\mathbf{p}|},$$

$$\delta \mathbf{p} = \hbar\frac{\boldsymbol{\beta}\times\hat{\mathbf{p}}}{2|\mathbf{p}|}\times \mathrm{B}$$

Chen, Son, Stephanov, PRL, (2015); Y. Hidaka, SP, D.L. Yang, PRD (2016)

Summary for CKT

- $\boxed{\text{Chiral Kinetic Theory}} = \boxed{\text{Boltzmann equation}} + \boxed{\text{Massless fermions}}\ \boxed{\text{Quantum Corrections}}$

- **Lorentz symmetry**

  side-jump

Some other related topics

- **Chirality production and Schwinger Mechanism**
- **Anomalous magnetohydrodynamics**
- **Lepton pair production in Ultra-Peripheral Collisions**

Some other related topics

- **Chirality production and Schwinger Mechanism**

P. Copinger, K. Fukshima, SP, Phys.Rev.Lett. 121 (2018), 261602

P. Copinger, SP, IJMA 35(2020)28, 2030015 (invited review)

- **Anomalous magnetohydrodynamics**

- **Lepton pair production in Ultra-Peripheral Collisions**

Axial Ward identity

- **Axial Ward identity**

$$\partial_\mu j_5^\mu = 2im\overline{\psi}\gamma^5\psi - \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

Chiral current **Pseudo-scalar** **~ E.B, Chiral anomaly**

- **Volume integral**

$$\frac{d}{dt}N_5 = \int d^3x \left(2im\overline{\psi}\gamma^5\psi - \frac{e^2}{2\pi^2}E\cdot B\right)$$

Pseudo-scalar

$$\partial_\mu j_5^\mu = \underline{2im\overline{\psi}\gamma^5\psi} - \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

- **Massless limit:** Pseudo-scalar term->0
- **Method:**
  - **Perturbative:**

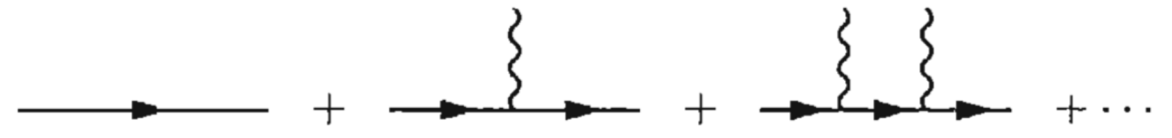
Gao, Liang, Q. Wang, X.N. Wang, arXiv:1802.06216; Weickgenannt, Sheng, Speranza,Q. Wang,1902.06513; Hattori, Hidaka, Yang, 1903.01653; Wang, Guo, Shi, Zhuang, 1903.03461

  - **Non-perturbative: World-line formulism**

World-line Formulism: IN-OUT Propagator

- **Spinor Feynman propagator at background fields:**

$$S_A(x,y) =$$



$$= \left(i\not{D}_x + m\right) \underline{\Delta(x,y)}$$

- **Path integral: (Homogenous, Constant E,B)**

$$\Delta(x,y) = \int_0^\infty ds e^{-im^2 s} \frac{e^2 EB}{(4\pi)^2} \frac{\exp\left[-\frac{i}{2} eF\sigma s + \frac{i}{2} xeFy - \frac{i}{4} z \coth(eFs) eFz\right]}{\sinh(eEs)\sin(eBs)}$$

s: Schwinger proper time

M. D. Schwartz, Quantum Field theory and the standard model;
Christian Schubert: lecture note on the Worldline Formalism

Puzzle: chirality production vanishes?

- **Homogenous Constant E,B at z direction**

- **Using world-line formulism (or original Schwinger's methods):**

$$< \partial_\mu j_5^\mu > = -2im < \overline{\psi}\gamma^5\psi > - \frac{e^2}{2\pi^2}\mathbf{E}\cdot\mathbf{B} = 0$$

?

$$< \overline{\psi}\gamma^5\psi > = i\frac{1}{4\pi^2}\frac{EB}{m}$$

J. Schwinger, Phys. Rev. 82,5 (1951); M. D. Schwartz, Quantum Field theory and the standard model;

Puzzle: All vanishing?

$$< \partial_\mu j_5^\mu > = -2im < \overline{\psi}\gamma^5\psi > -\frac{e^2}{2\pi^2}\mathbf{E}\cdot\mathbf{B} = 0$$

- **Taking m->0 at the very beginning: Weyl fermions**

$$\partial_\mu j_5^\mu = -\frac{e^2}{2\pi^2}\mathbf{E}\cdot\mathbf{B}$$

- **After all the calculations, taking m->0.**

$$< \partial_\mu j_5^\mu > = -2im < \overline{\psi}\gamma^5\psi > -\frac{e^2}{2\pi^2}\mathbf{E}\cdot\mathbf{B} = 0$$

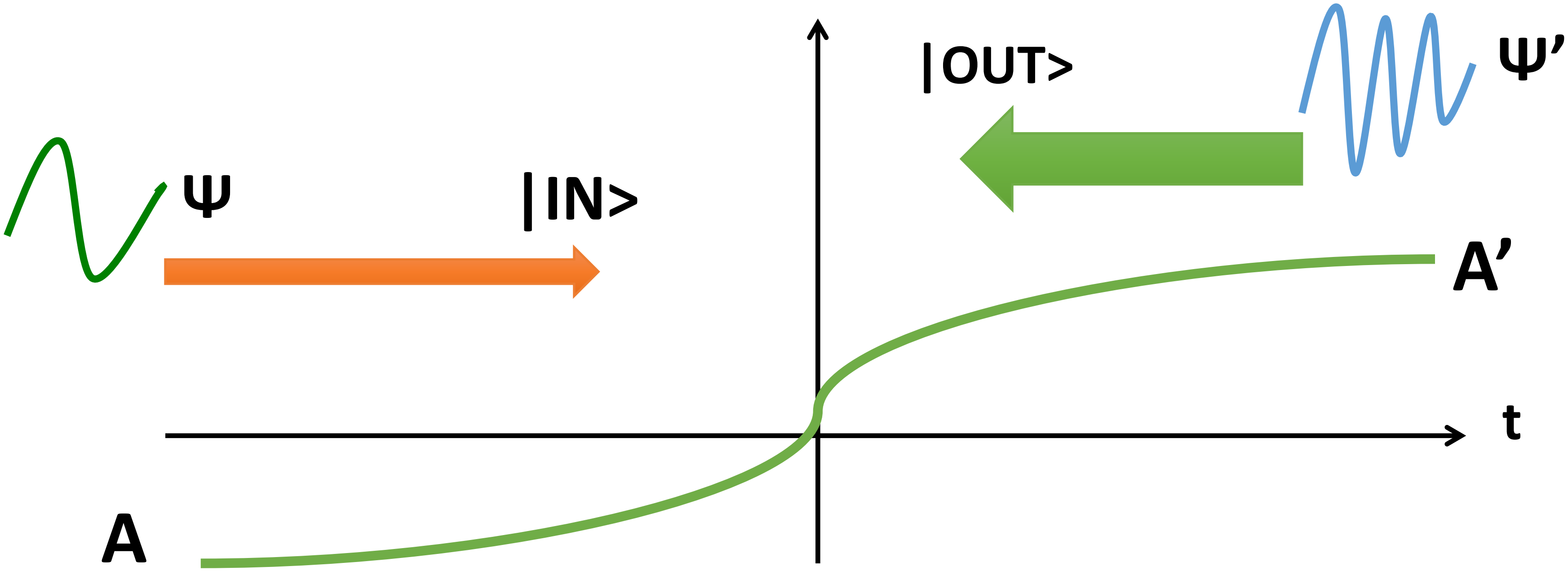
- **We can take the smooth massless limits (m->0) in world-line formulism.**

IN and OUT States

- **Homogenous Constant Ez,Bz field: (Schwinger Fock gauge)**

$$A^z(t) = eE_z t, \quad H = H(A(t)),$$

- **Vacuum at In states is different with it at OUT states:**



Unstable vacuum

- **|0, IN> is NOT equal to |0, OUT>**

$$|<0, out \mid 0, in>|^2 \neq 1$$

- **Schwinger Pair Production Rate:**

$$P_0 = 1 - |<0, out \mid 0, in>|^2 = \frac{e^2 E_z B_z}{4\pi^2} \coth\left(\frac{B_z}{E_z}\pi\right) \exp\left(-\frac{m^2\pi}{|eE_z|}\right)$$

(n=1 world-line instanton)

Expectation Value: IN-IN states

- **Transition amplitude: IN-OUT**

$$< 0, out|\partial_\mu j_5^\mu|0, in >$$

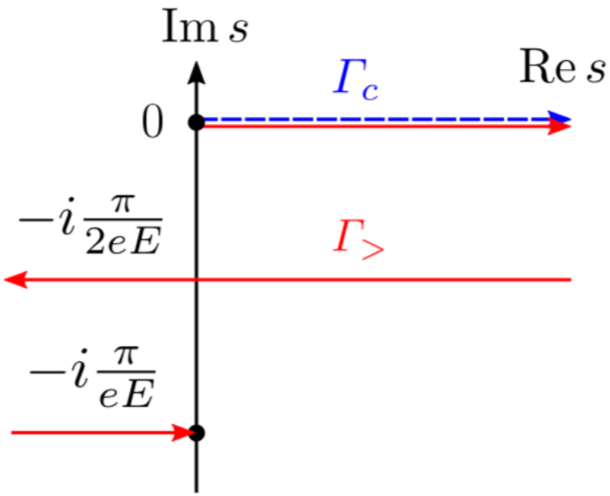
- **Expectation value: IN-IN**

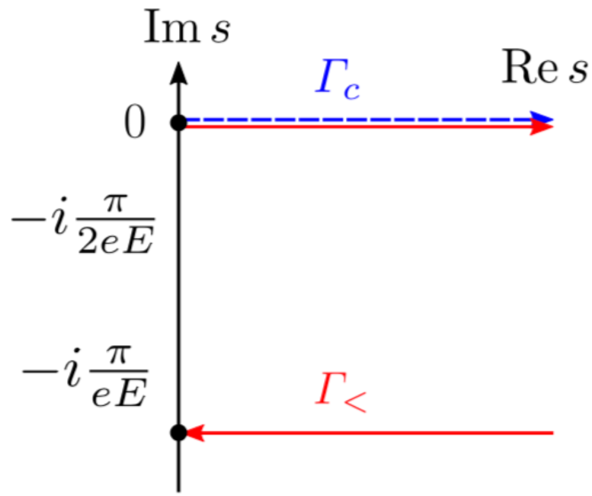
$$< 0, in|\partial_\mu j_5^\mu|0, in >$$

Review: F. Gelis, N. Tanji 2015; N. Tanji 2009

Textbook: E.S. Fradkin, D.M. Gitman, Sh.M. Shvartsman: Quantum Electrodynamic with Unstable Vacuum, 1991

Feynman Propagator for IN-IN





To compute a quantity at IN-IN states, we can use all the expression in ordinary world-line formulism but with new integral paths (red one in figures).

- **IN-OUT Propagator: Path in Blue**

$$S_A(x,y) = \left(i\not{D}_x + m\right)\Delta(x,y)$$

$$\Delta(x,y) = \left[\theta(x_3-y_3)\int_{\Gamma^C} + \theta(y_3-x_2)\int_{\Gamma^C}\right] ds \times e^{-im^2 s} g(x,y,s),$$

- **IN-IN Propagator: Path in Red**

$$S_{in}(x,y) = \left(i\not{D}_x + m\right)\Delta_{in}(x,y)$$

$$\Delta_{in}(x,y) = \left[\theta(x_3-y_3)\int_{\Gamma^>} + \theta(y_3-x_2)\int_{\Gamma^<}\right] ds \times e^{-im^2 s} g(x,y,s),$$

Textbook: E.S. Fradkin, D.M. Gitman, Sh.M. Shvartsman: Quantum Electrodynamic with Unstable Vacuum, 1991

Chirality Production

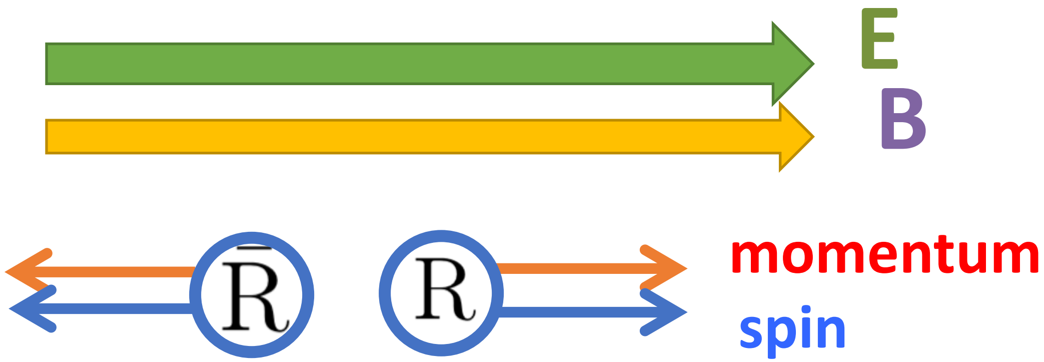
- **We obtained the chirality production rate:**

$$\partial_\mu j_5^\mu = \frac{e^2 EB}{2\pi^2} \exp\left(-\frac{\pi m^2}{eE}\right)$$

smooth massless limit:

$m \to 0$,Chiral anomaly

- **Consistent with physical picture**



$$\frac{1}{2}\partial_t n_5 = \text{Schwinger Pair Production rate}$$

K. Fukushima, D.Kharzeev, H. Warringa PRL 2010

Mass correction to CME

- **Assuming E,B at z direction, we obtain the current**

$$j^3 = \frac{e^2 EB}{2\pi^2} \coth\left(\frac{B}{E}\pi\right) \exp\left(-\frac{\pi m^2}{eE}\right) t$$

- **Non-perturbative:** $\sim \frac{1}{eE}$
- **Sum over all Landau levels:** $Coth(\frac{B}{E}\pi)$

Some other related topics

- **Chirality production and Schwinger Mechanism**

- **Anomalous magnetohydrodynamics**

  Siddique, R.-j. Wang, Pu, Q. Wang, PRD 2019

- **Lepton pair production in Ultra-Peripheral Collisions**

Anomalous MagentoHydroDynamics

- **Conservation equations：**
  - **Energy-momentum conservation**

$$\partial_\mu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = T_F^{\mu\nu} + T_{EM}^{\mu\nu}.$$

Fluid part **Electromagnetic part**

  - **(anomalous) currents conservation**

$$\partial_\mu j_e^\mu = 0, \qquad \partial_\mu j_5^\mu = -e^2 C E \cdot B,$$

Electric Charge current **Chiral current**

- **Maxwell's equation：**

$$\partial_\mu F^{\mu\nu} = j_e^\nu, \qquad \partial_\mu(\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}) = 0.$$

Previous Studies: ideal MHD without CME

- **Preview studies:**

➢ **1+1 D ideal MHD Bjorken flow**

V.Roy, SP, L. Rezzolla, D. Rischke, Phys.Lett. B750, 45

SP, V. Roy, L. Rezzolla, D. Rischke,Phys.Rev. D93, 074022

➢ **2+1 D ideal MHD Bjorken flow (perturbative)**

SP, Di-Lun Yang, Phys.Rev. D93, 054042

➢ **Background Magnetic field: contribution to v2**

V.Roy, SP, L. Rezzolla, D. Rischke, Phy.Rev. C96, 054909

- **Problem:**

How to add the CME to Magentohydrodynamics?

Beyond ideal limit of MHD

- **Ideal limit of MHD:**
  - **Electric conductivity is infinite**

$$\sigma \to \infty$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \Rightarrow \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$$

Maxwell's equation

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E} = -\nabla \times (\mathbf{v} \times \mathbf{B})$$

- **No space for the CME**

$$\nabla \times \mathbf{B} = \mathbf{j} + \partial_t \mathbf{E}$$

- **Anomalous MHD needs finite conductivity**

Constitution Eqs. for Anomalous MHD

$$\partial_\mu T^{\mu\nu} = 0,$$

$$\begin{aligned} T^{\mu\nu} &= (\varepsilon + p + E^2 + B^2)u^\mu u^\nu - (p + \frac{1}{2}E^2 + \frac{1}{2}B^2)g^{\mu\nu} \\ &\quad -E^\mu E^\nu - B^\mu B^\nu - u^\mu \epsilon^{\nu\lambda\alpha\beta}E_\lambda B_\alpha u_\beta - u^\nu \epsilon^{\mu\lambda\alpha\beta}E_\lambda B_\alpha u_\beta, \end{aligned}$$

$$\partial_\mu j_e^\mu = 0,$$

Electric Conducting flow

CME

$$\partial_\mu j_5^\mu = -e^2 C E \cdot B,$$

$$\begin{aligned} j_e^\mu &= n_e u^\mu + \sigma E^\mu + \xi B^\mu, \\ j_5^\mu &= n_5 u^\mu + \sigma_5 E^\mu + \xi_5 B^\mu, \end{aligned}$$

CSE

$$\partial_\mu F^{\mu\nu} = j_e^\nu,$$

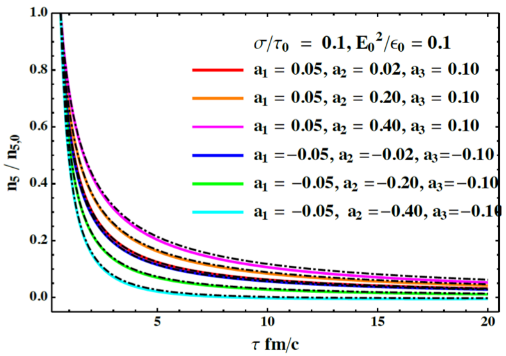
$$\partial_\mu(\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}) = 0.$$

$$F^{\mu\nu} = E^\mu u^\nu - E^\nu u^\mu + \epsilon^{\mu\nu\alpha\beta}u_\alpha B_\beta,$$

Analytic solution for Anomalous magneto-hydrodynamics

- *Siddique, R.-j. Wang, Pu, Q. Wang, PRD 2019*
- Anomalous MHD:

  Hydrodynamic eq. + Maxwell's eq. + Chiral currents
- We obtained the analytic solutions of anomalous magneto-hydrodynamics in Bjorken flow with transverse EB fields

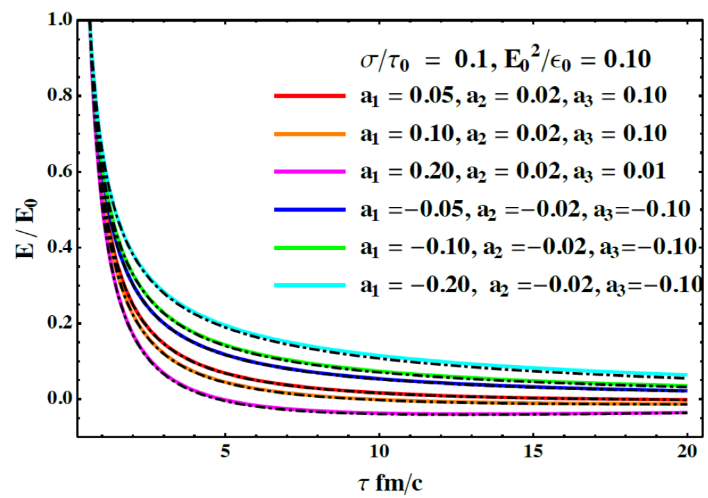


$$n_5(\tau) = n_{5,0}\left(\frac{\tau_0}{\tau}\right)\{1 + a_2 e^{\sigma\tau_0}[\mathrm{E}_1(\sigma\tau_0) - \mathrm{E}_1(\sigma\tau)]\},$$

$$\varepsilon(\tau) = \epsilon_0\left(\frac{\tau_0}{\tau}\right)^{1+c_s^2}\left\{1 + \sigma\frac{E_0^2}{\varepsilon_0}e^{2\sigma\tau_0}\left[\tau_0\mathrm{E}_{1-c_s^2}(2\sigma\tau_0) - \tau\left(\frac{\tau}{\tau_0}\right)^{c_s^2-1}\mathrm{E}_{1-c_s^2}(2\sigma\tau')\right]\right.$$
$$\left.+\frac{a_3}{\tau_0}e^{\sigma\tau_0}\left[\tau_0\mathrm{E}_{2-3c_s^2}(\sigma\tau_0) - \tau\left(\frac{\tau_0}{\tau}\right)^{2-3c_s^2}\mathrm{E}_{2-3c_s^2}(\sigma\tau)\right]\right\}.$$

- **Time evolution of chirality and energy density, EB fields with quantum corrections from CME, chiral anomaly.**

- Time evolution of EB fields
  - **In lab frame, B field decays much slower than in the vacuum**
  - **By decays like ~ $1/\tau$,**
  - **Bx decays like~ $\exp(-\sigma\tau)/\tau$**



Some other related topics

- **Chirality production and Schwinger Mechanism**
- **Anomalous magnetohydrodynamics**
- **Lepton pair production in Ultra-Peripheral Collisions**

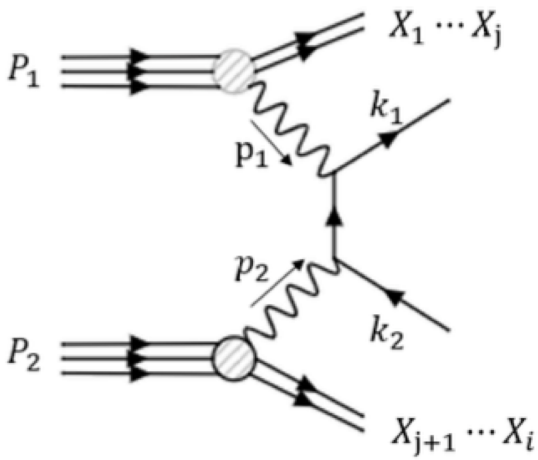
  R.J. Wang, SP, Q. Wang, arXiv: 2106.05462

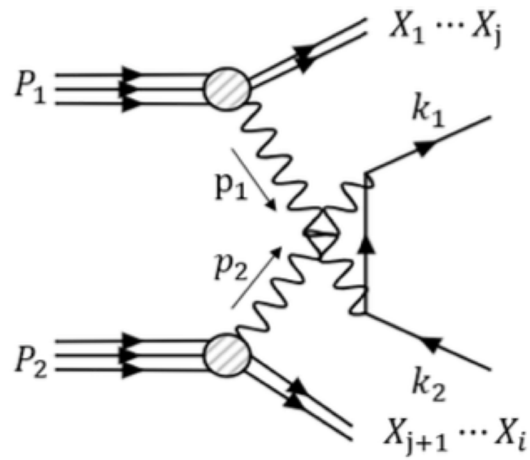
Ultra-Peripheral Collisions

- **Ultra-Peripheral Collisions (UPC): the impact parameter is larger than 2 times the radius of a nucleus**

- **Since the QCD effects are higher orders and QED effects are enhanced by the Ze, UPC provides a nice platform to study the strong EB effects.**

Dilepton production





Equivalent photon approximation

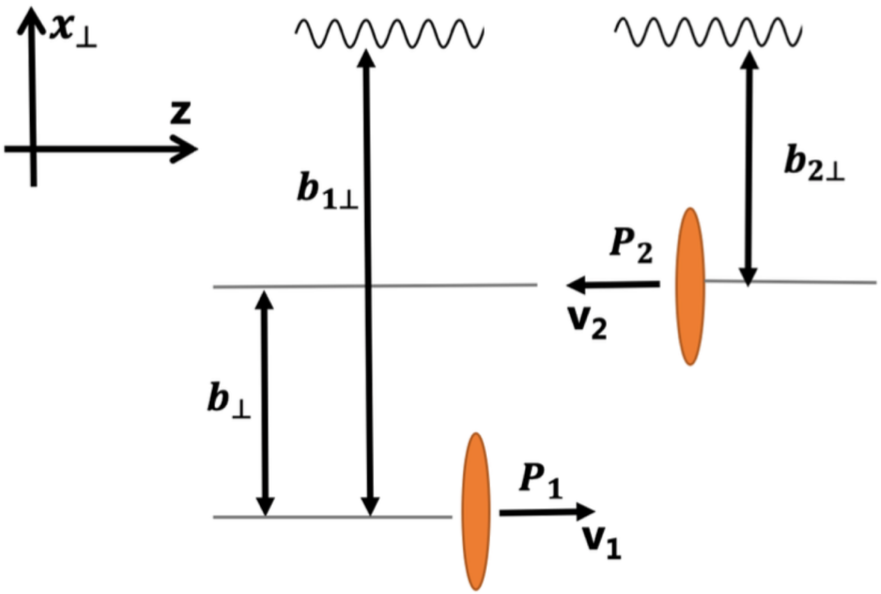
- *A. J. Baltz, Y. Gorbunov, S. R. Klein and J. Nystrand, PRC 80, 044902 (2009)*
- *W. Zha, L. Ruan, Z. Tang, Z. Xu and S. Yang, PLB 781, 182 (2018)*
- *W. Zha, J. D. Brandenburg, Z. Tang and Z. Xu, PLB 800 (2020) 135089*

Based on QED calculations

- *C. Li, J. Zhou and Y. J. Zhou, Phys. Lett. B 795, 576 (2019) ; arXiv:1911.00237 [hep-ph]].*
- *Klein, Muller, Xiao, Yuan, PRL 122 (2019) 13, 132301; PRD 102 (2020) 9, 094013*
- *Xiao, Yuan, Zhou, PRL 125 (2020) 23, 232301*
- *R.J. Wang, SP, Q. Wang, arXiv: 2106.05462*

Our theoretical framework

- **We derive the general expression of the cross section including several impact parameters.**



R.J. Wang, SP, Q. Wang, arXiv: 2106.05462

Our theoretical framework

$$
\begin{aligned}
\frac{d\sigma}{d^3k_1 d^3k_2} \approx & \frac{1}{32(2\pi)^6}\frac{1}{E_{k1}E_{k2}}\int d^2\mathbf{b}_T d^2\mathbf{b}_{1T} d^2\mathbf{b}_{2T}\int d^4p_1 d^4p_2 \\
& \times\delta^{(2)}\left(\mathbf{b}_T-\mathbf{b}_{1T}+\mathbf{b}_{2T}\right)(2\pi)^4\delta^{(4)}\left(p_1+p_2-k_1-k_2\right) \\
& \times\int\frac{d^2\mathbf{P}_{(1+1')T}}{(2\pi)^2}\frac{d^2\mathbf{P}_{(2+2')T}}{(2\pi)^2}\frac{1}{v\sqrt{E_{P1}E_{P2}E_{P1'}E_{P2'}}} \\
& \times G^2\left[(P_1'^z-P_{A1}^z)^2\right]\phi_T(\mathbf{P}_{1T})\phi_T(\mathbf{P}_{2T})\phi_T^*(\mathbf{P}'_{1T})\phi_T^*(\mathbf{P}'_{2T}) \\
& \times\mathcal{S}_{\sigma\mu}(p_1,\mathbf{b}_{1T})\mathcal{S}_{\rho\nu}(p_2,\mathbf{b}_{2T}) \\
& \times L^{\mu\nu;\sigma\rho}\left(p_1,p_2;p_1-P_1+P_1',p_2-P_2+P_2';k_1,k_2\right),
\end{aligned}
$$

- **If we consider A as background fields and take the virtuality expansion, we can reproduce the Equivalent photon approximation**

$$
\sigma_0(A_1A_2\to l\bar{l}) = \int d\omega_1 d\omega_2 n_{A1}(\omega_1)n_{A2}(\omega_2)\sigma_{\gamma\gamma\to l\bar{l}}(\omega_1,\omega_2),
$$

Our theoretical framework

$$\frac{d\sigma}{d^3k_1 d^3k_2} = \frac{1}{32(2\pi)^6}\frac{1}{E_{k1}E_{k2}}\int d^4p_1 d^4p_2$$
$$\times \int \frac{d^2\mathbf{P}_{1T}}{(2\pi)^2}\frac{d^2\mathbf{P}_{2T}}{(2\pi)^2}\frac{1}{vE_{P1}E_{P2}}\left|\phi_T(\mathbf{P}_{1T})\right|^2\left|\phi_T(\mathbf{P}_{2T})\right|^2$$
$$\times \underline{S_{\sigma\mu}(P_1,p_1)S_{\rho\nu}(P_2,p_2)}L^{\mu\nu;\sigma\rho}(p_1,p_2;p_1,p_2;k_1,k_2)$$
$$\times(2\pi)^4\delta^4(p_1+p_2-k_1-k_2),$$

$$S_{\sigma\mu}(P,p) \equiv \int \frac{d^4y}{(2\pi)^4}e^{ip\cdot y}\left\langle P\right| A_\sigma^\dagger(0)A_\mu(y)\left|P\right\rangle,$$

→ **Transverse Momentum Dependent (TMD) Photon Distribution**

Klein, Muller, Xiao, Yuan, PRL 122 (2019) 13, 132301; 2003.02947

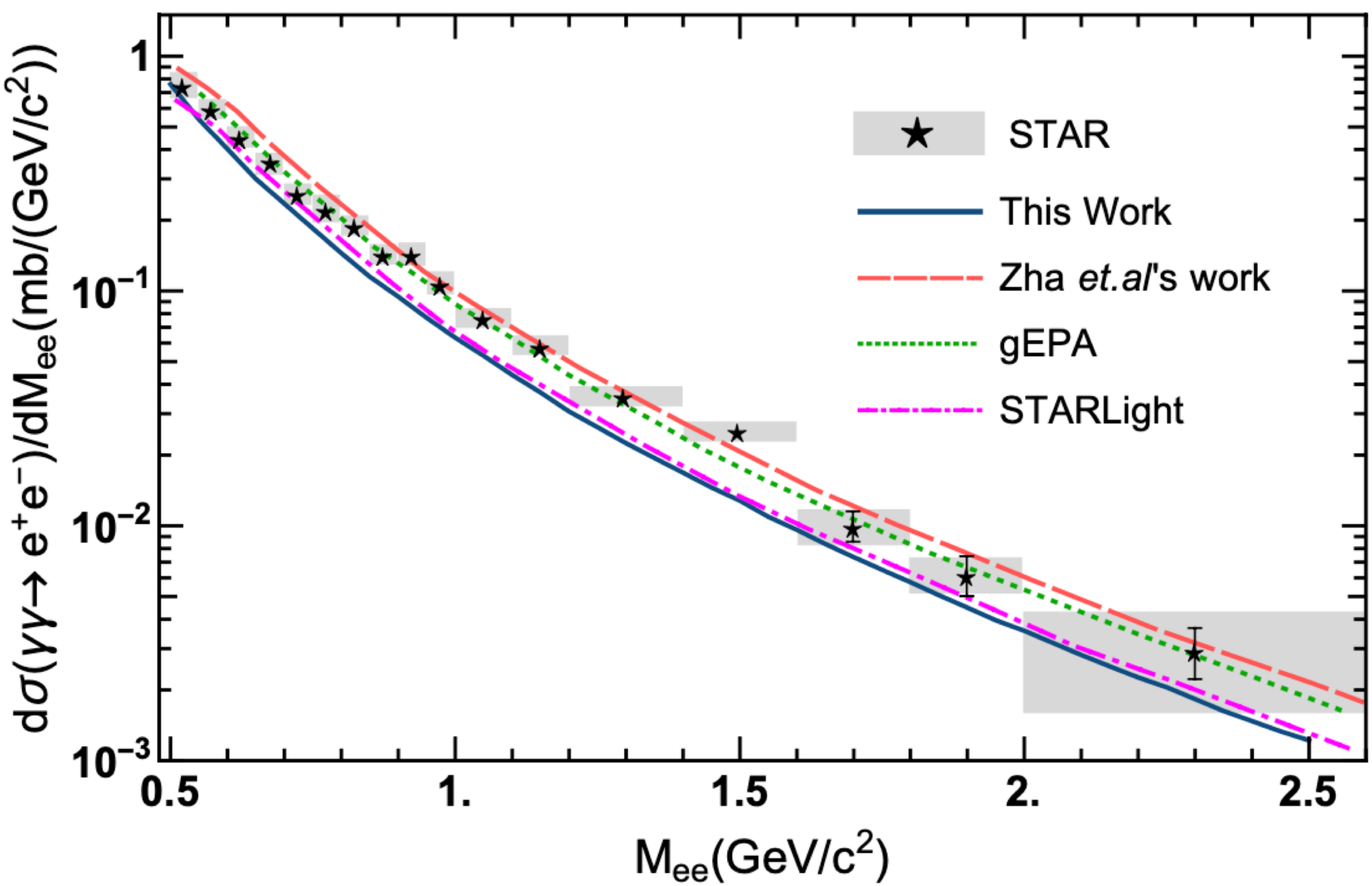
Our primary result (I)

- **Total cross section**

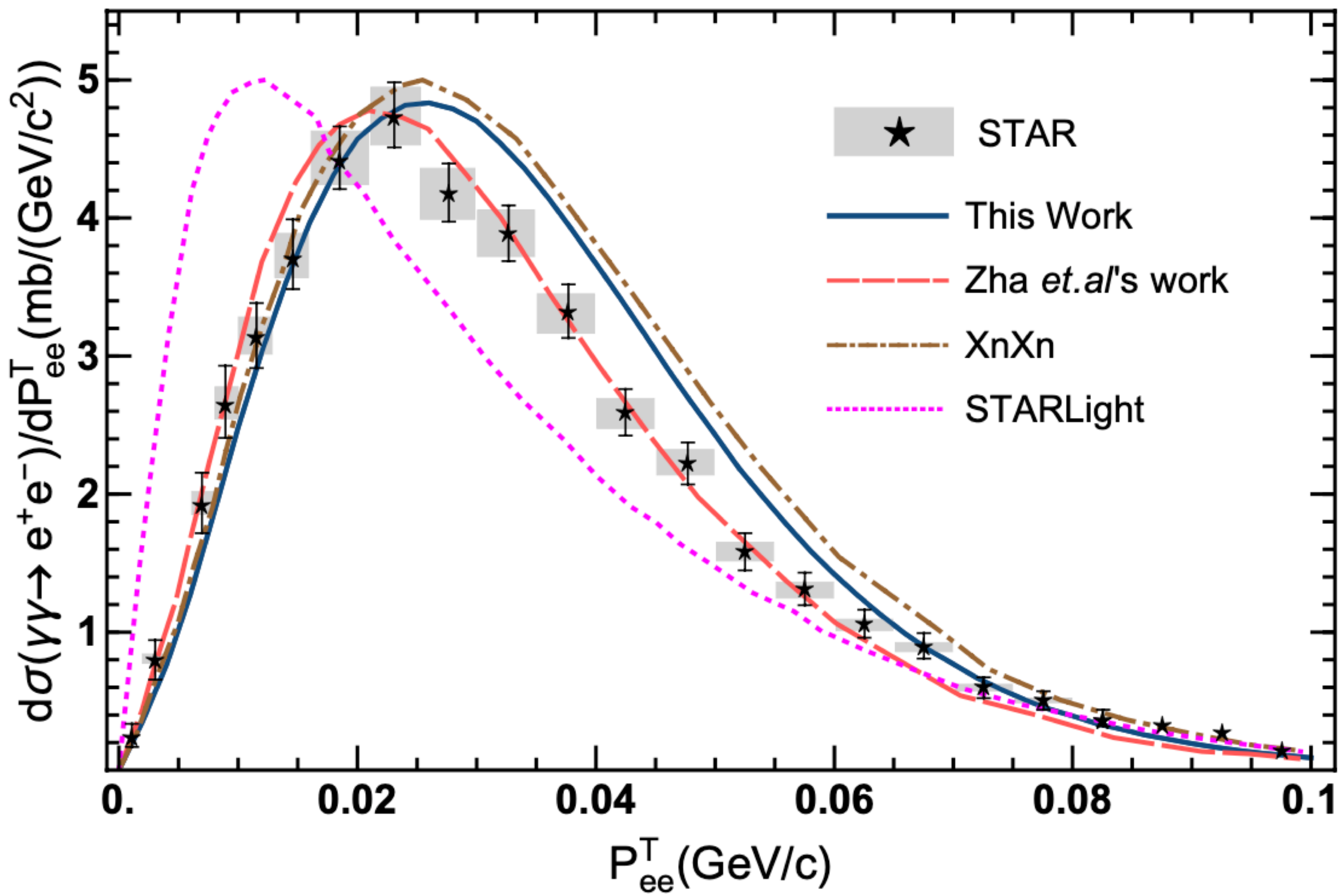
Table I: The total cross sections from STAR measurements and theoretical models. The numerical integration errors are labeled as "int.".

Data or models	Total cross sections
STAR data [67]	0.261±0.004 (stat.)±0.013 (sys.)±0.034 (scale) mb
STARLight [73]	0.22 mb
Zha et al's gEPA [77]	0.26 mb
Zha et al's work [77]	0.26 mb
This work Eq. (20)	0.252±0.0016 (int.) mb
gEPA Eq. (30)	0.256±0.0030 (int.) mb

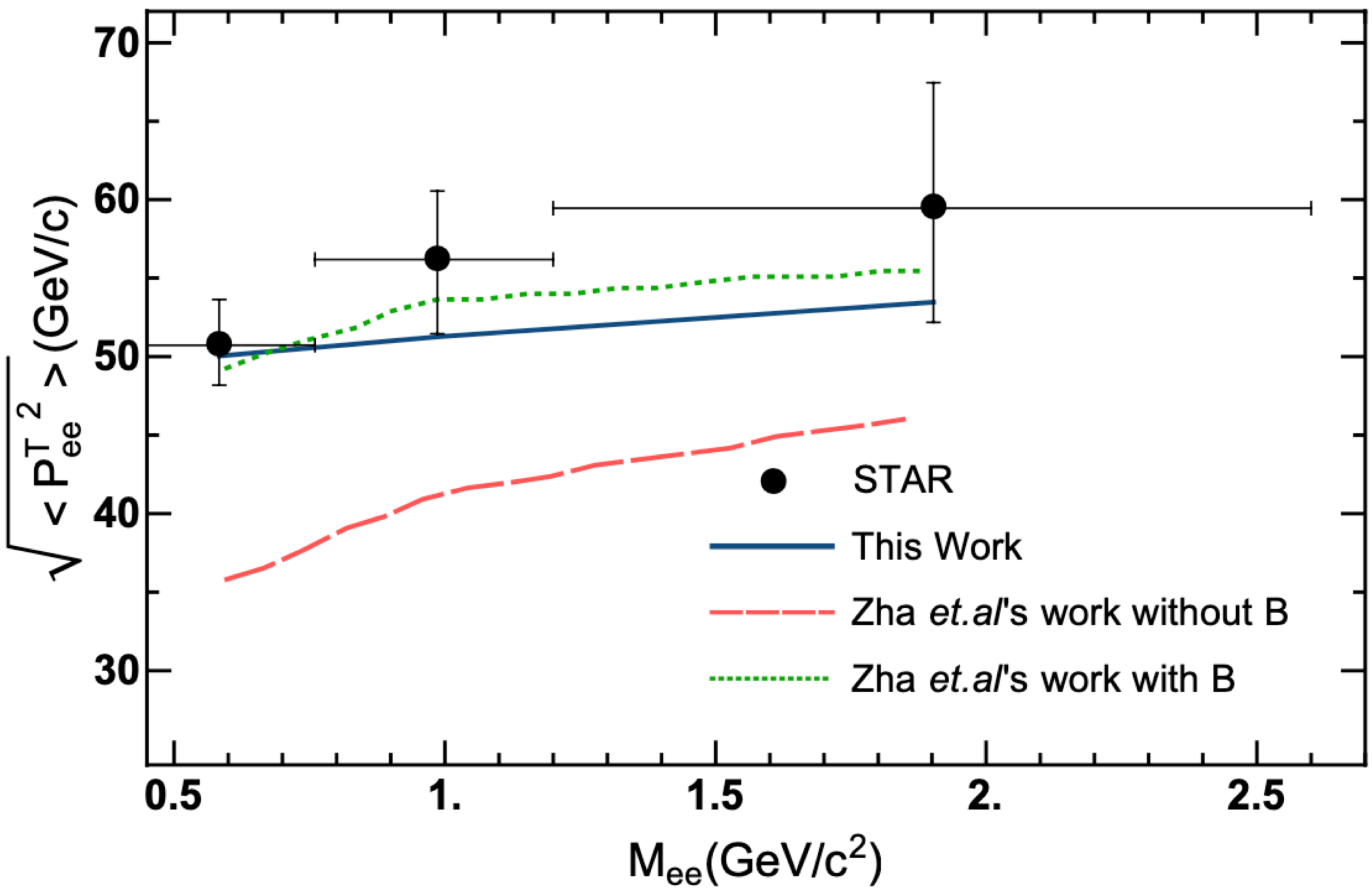
Our primary result (II)



Our primary result (III)

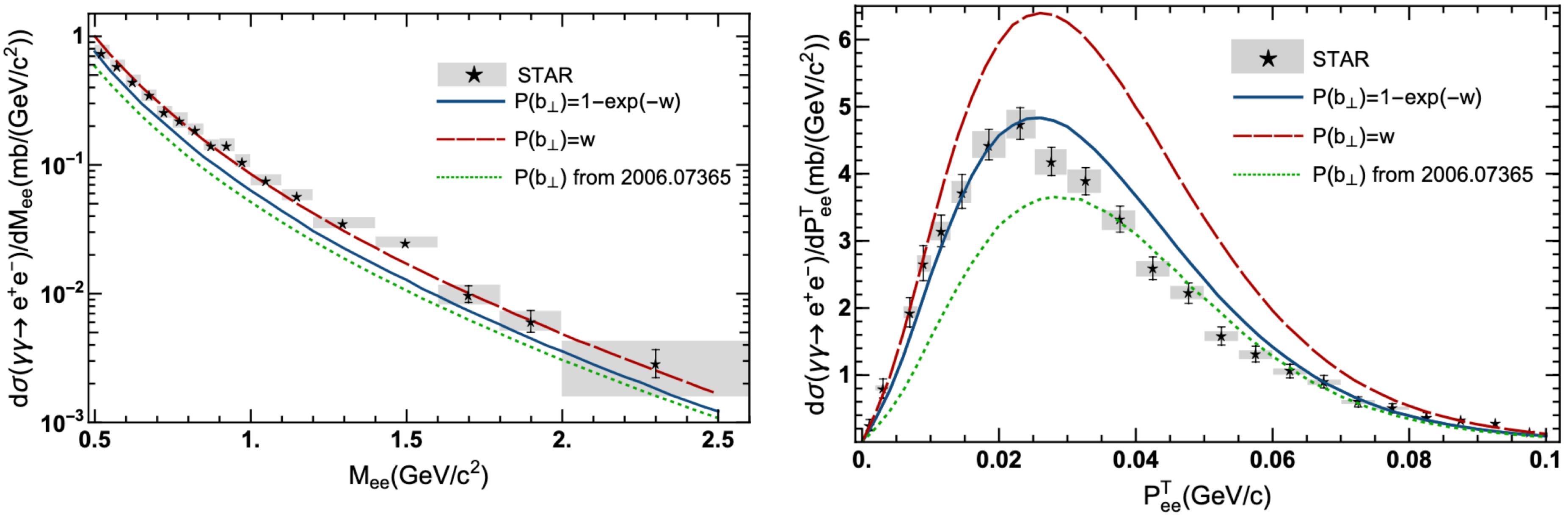


Our primary result (IV)



Our primary result (V)

We find that the differential cross section strongly depends on the choice of $P(b_\perp)$ and the radius of nuclei.

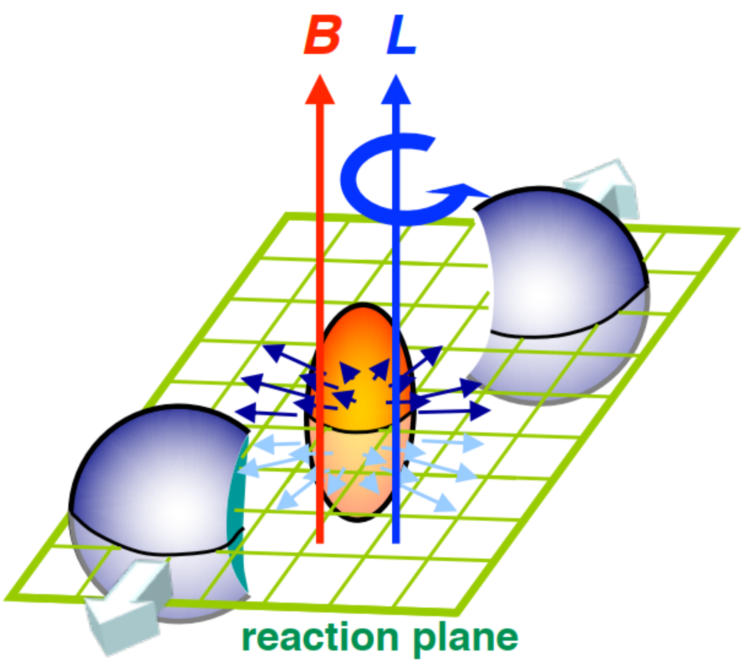


Most vortical fluid

Global and local Polarization

Spin hydrodynamics

Huge angular momentum

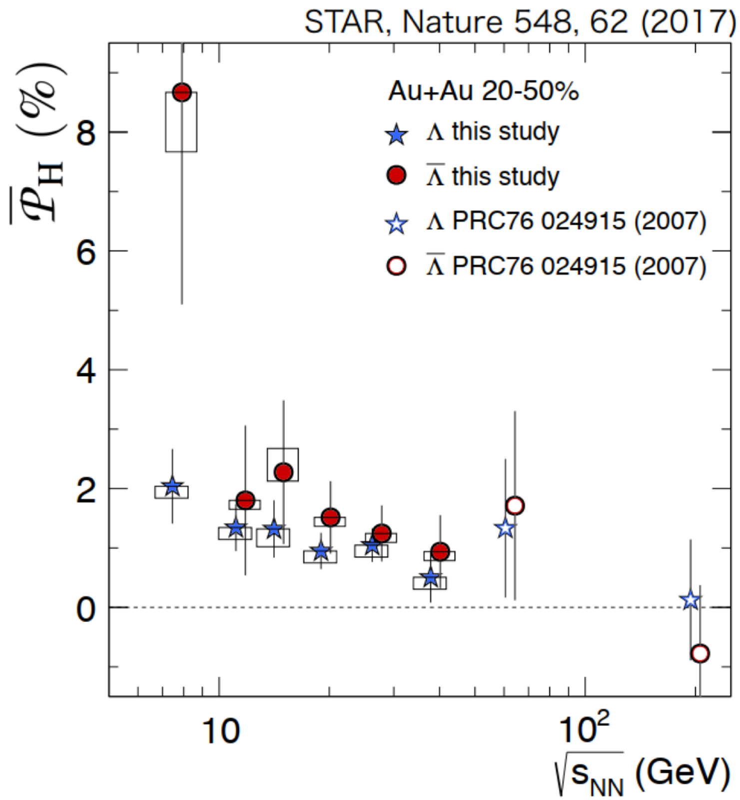


- **Huge global orbital angular momenta are produced**

$$L \sim 10^5 \hbar$$

- **How do orbital angular momenta be transferred to the matter created?**

Global Polarization of Λ and $\bar{\Lambda}$



- $\sqrt{s_{NN}}$ < 62.4GeV , we observe the signal for polarization of Λ and $\bar{\Lambda}$
- The lower energy, the stronger polarization effects
- $P_{\bar{\Lambda}} > P_{\Lambda}$

$$P_{\Lambda} \approx \frac{1}{2}\frac{\omega}{T} + \frac{\mu_{\Lambda}B}{T}$$
$$P_{\bar{\Lambda}} \approx \frac{1}{2}\frac{\omega}{T} - \frac{\mu_{\Lambda}B}{T}$$

ω = (9 ± 1)x10^{21}/s, greater than previously observed in any system.

Liang, Wang, PRL (2005)
Betz, Gyulassy, Torrieri, PRC (2007)
Becattini, Piccinini, Rizzo, PRC (2008)
Becattini, Karpenko, Lisa, Upsal, Voloshin, PRC (2017)
Fang, Pang, Q. Wang, X. Wang, PRC (2016)

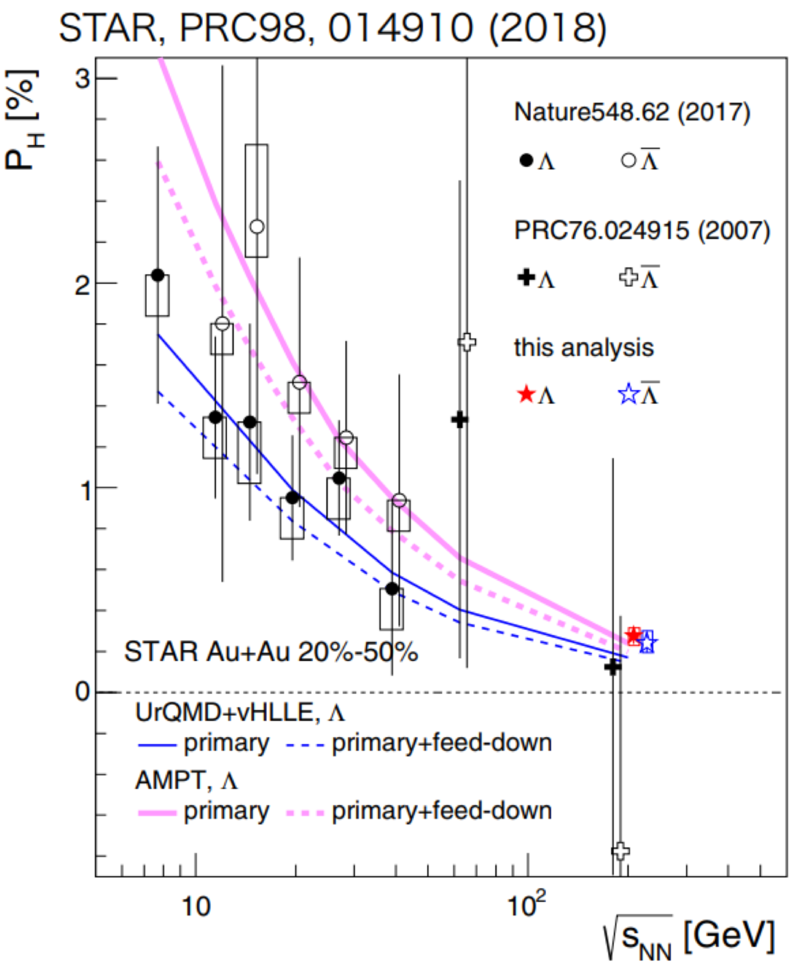
The fastest fluid

The Fastest Fluid

by Sylvia Morrow

Superhot material spins at an incredible rate.

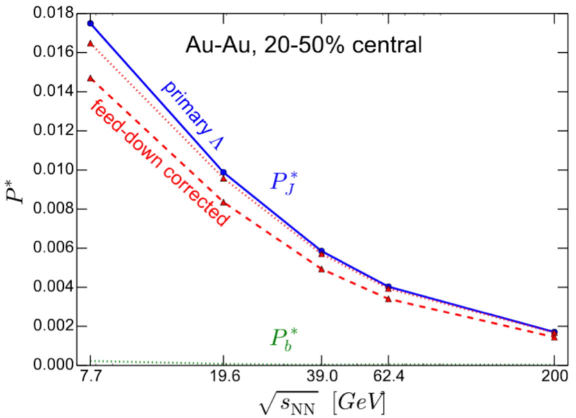
Global Polarization



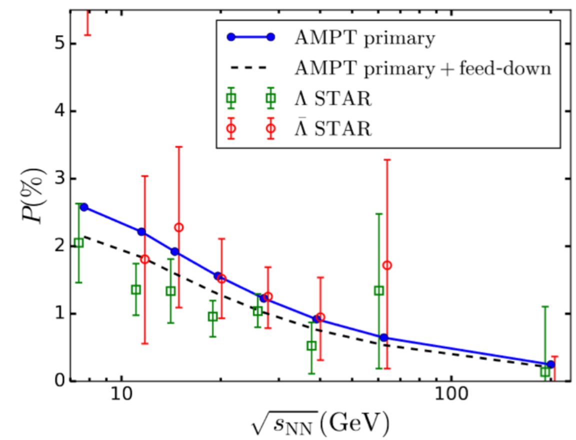
The results from both UrQMD+hydro and AMPT are consistent with the experimental data.

- ***UrQMD+vHLLE: Karpenko, Becattini, EPJC(2017)***
- ***AMPT: Li, Pang, Wang, Xia, PRC (2017)***

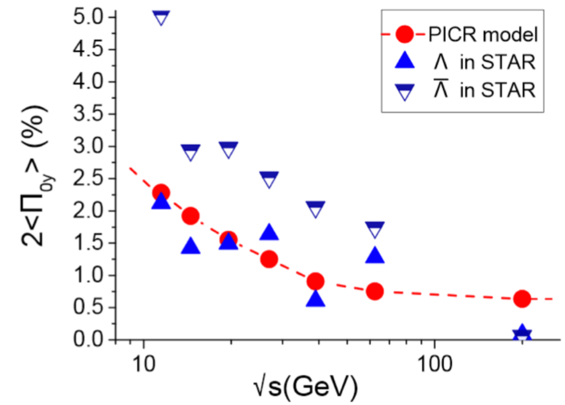
Global Polarization from different models



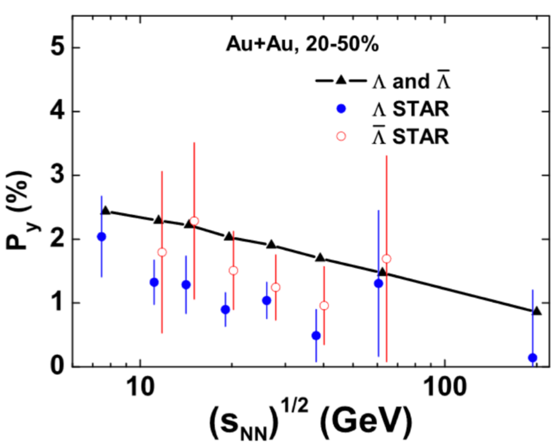
Karpenko, Becattini, EPJC(2017)



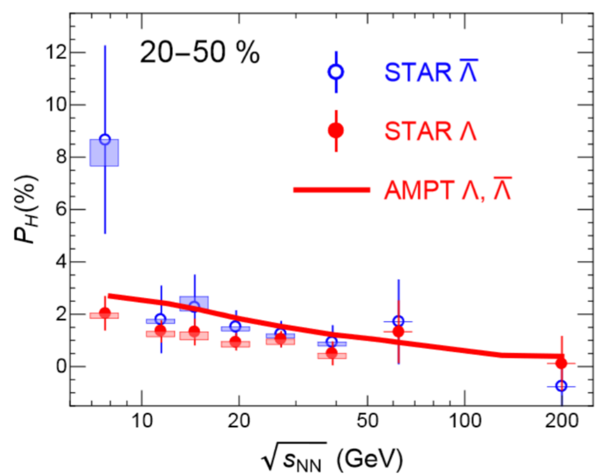
Li, Pang, Wang, Xia PRC(2017)



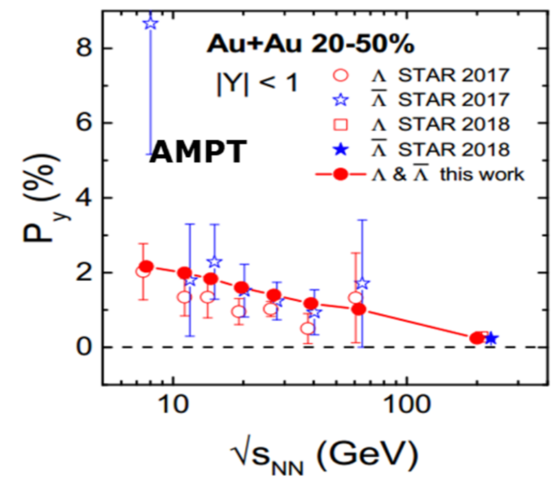
Xie, Wang, Csernai, PRC(2017)



Sun, Ko, PRC(2017)

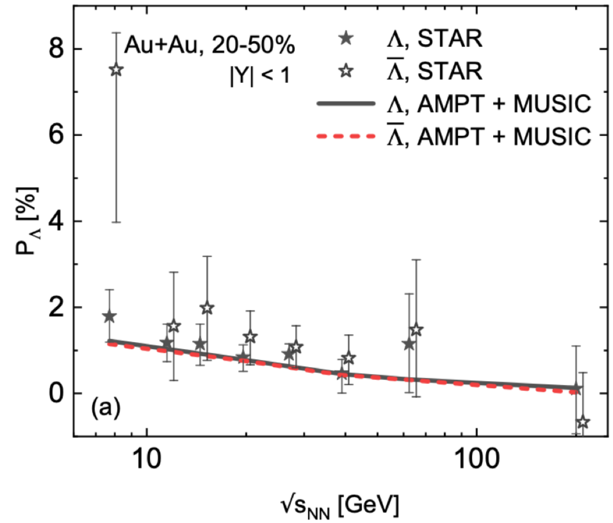


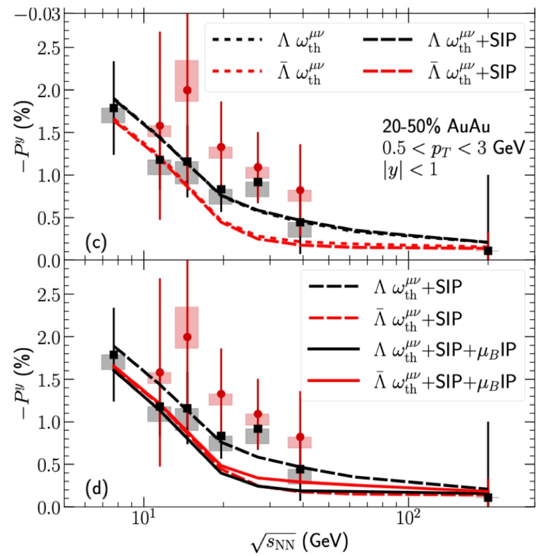
Shi, Li, Liao, PLB(2018)



Wei, Deng, Huang, PRC(2019)

Global Polarization from different models

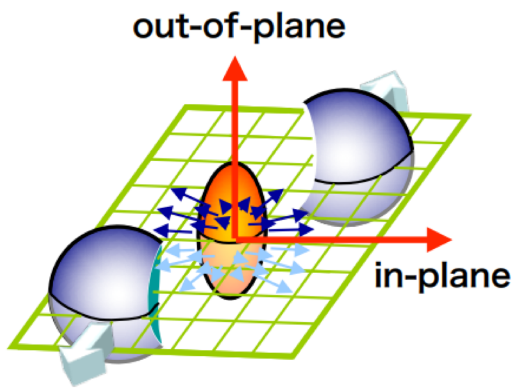


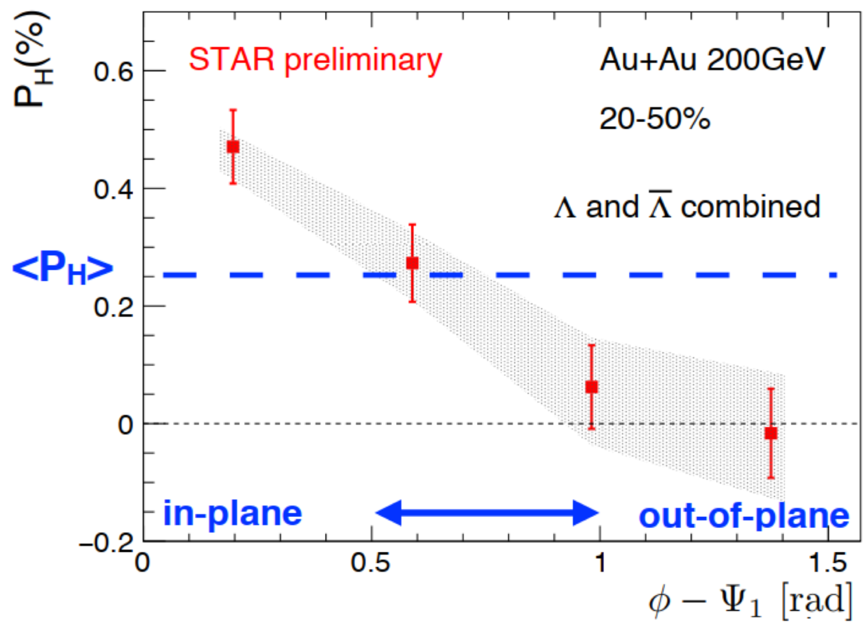


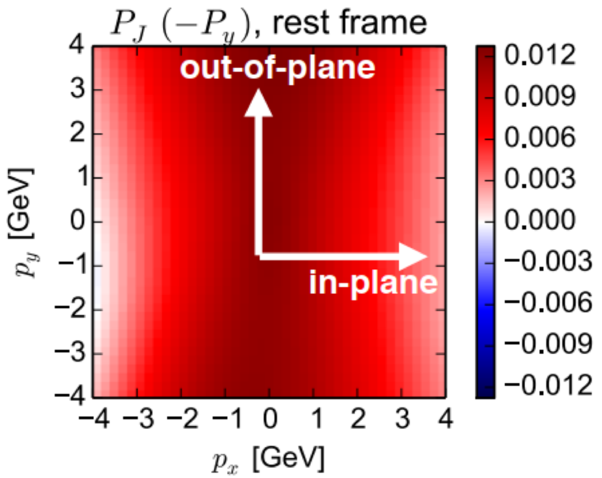
B.C. Fu, K. Xu, X.G. Huang, H.C. Song, Phys. Rev. C 103, 024903 (2021)

S. Ryu, V. Jupic, C. Shen, arXiv:2106.08125

Local Polarization







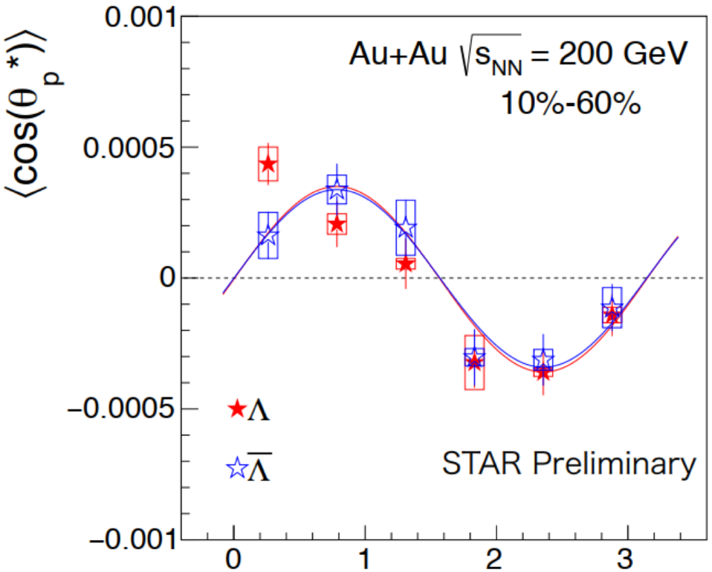
- **Exp data:**

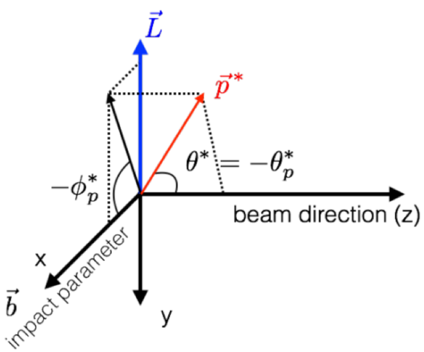
  P_H **in-plane >** P_H **out-of-plane**

- **Simulations:**

  <u>Sign is opposite of expected!</u>

Local Polarization alone beam direction





$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha_H \mathbf{P_H} \cdot \mathbf{p}_p^*)$$

$$\langle \cos\theta_p^* \rangle = \int \frac{dN}{d\Omega^*} \cos\theta_p^* d\Omega^*$$

$$= \alpha_H P_z \langle (\cos\theta_p^*)^2 \rangle$$

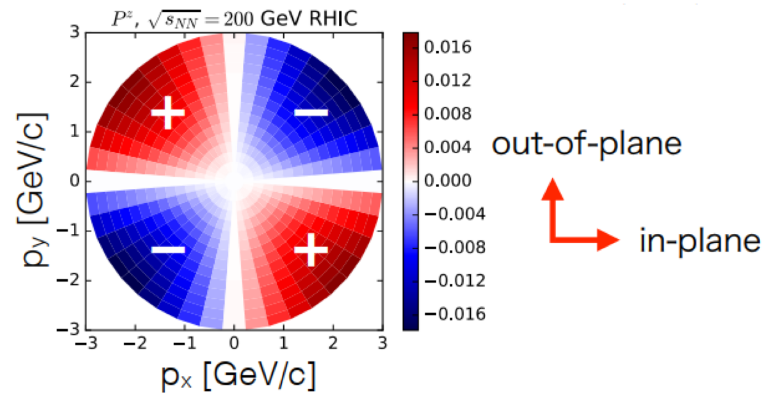
$$\therefore \; P_z = \frac{\langle \cos\theta_p^* \rangle}{\alpha_H \langle (\cos\theta_p^*)^2 \rangle}$$

$$= \frac{3\langle \cos\theta_p^* \rangle}{\alpha_H} \quad \text{(if perfect detector)}$$

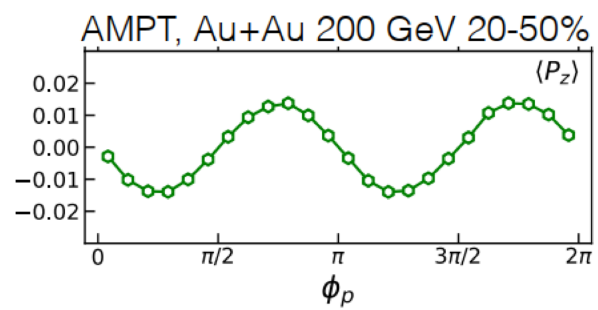
α_H: hyperon decay parameter

θ_p^*: θ of daughter proton in Λ rest frame

Again, sign is opposite of expected!



UrQMD : ***Becattini, Karpenko, PRL (2018)***

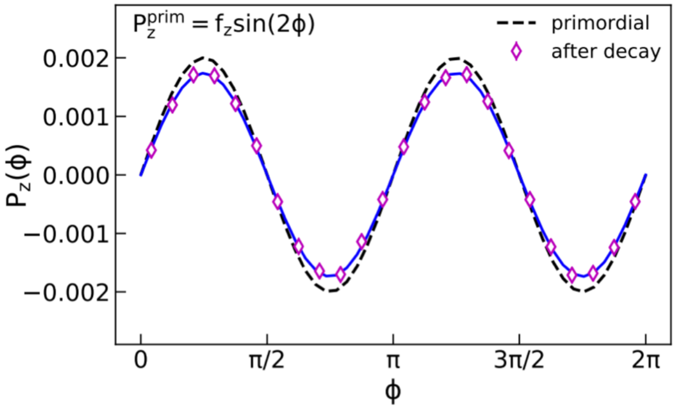


AMPT: ***Xia, Li, Tang, Wang, PRC (2018)***

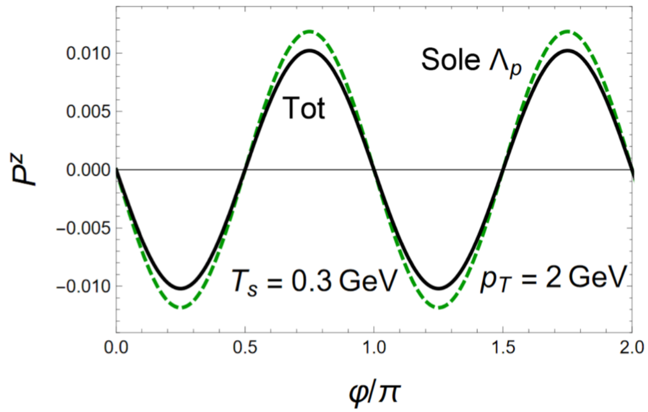
Feed-down effects: NO!

- **Feed-down effects**

Lambda may come from decays of heavier particles



Xia, Li, Huang, Huang PRC(2019)



Becattini, Cao, Speranza, EPJC(2019)

Different approaches

- **Spin hydrodynamics**

  Florkowski, Friman, Jaiswal, Ryblewski, Speranza (2017-2018);

  Montenegro, Tinti, Torrieri (2017-2019);

  Hattori, Hongo, Huang, Matsuo, Taya PLB(2019)

  Fukushima, SP, Lecture Note (2020); PLB(2021)

  S.Y. Li, M.A Stephanov, H.U Yee, arXiv:2011.12318

  D. She, A. Huang, D.F. Hou, J.F Liao, arXiv: 2105.04060

  ....

- **Quantum kinetic theory for massive fermions**

- **Other approaches:**

Different approaches

- **Spin hydrodynamics**
- **Quantum kinetic theory for massive fermions**
- **Other approaches:**

Quantum kinetic theory (massive fermions)

- **Quantum kinetic theory (for massive fermions)**
- **Collision term with quantum corrections**

  Weickgenannt, Sheng, Speranza, Wang, Rischke, PRD 100, 056018 (2019)

  Hattori, Hidaka, Yang, PRD100, 096011 (2019); Yang, Hattori, Hidaka, arXiv: 2002.02612.

  Liu, Mameda, Huang, arXiv:2002.03753.

  Wang, Guo, Shi, Zhuang, PRD100, 014015 (2019),

  Li ,Yee, PRD100, 056022 (2019)

Recent reviews:

Gao, Ma, SP, Wang, NST 31 (2020) 9, 90

Gao, Liang, Wang, Int.J.Mod.Phys A 36 (2021), 2130001

Hidaka, SP, Yang, Wang, in preparation

Different approaches

- **Spin hydrodynamics**

- **Quantum kinetic theory for massive fermions**

- **Other approaches:**
  - **Side-jump effect** ***Liu, Sun, Ko PRL(2020)***
  - **Mesonic mean-field** ***Csernai, Kapusta, Welle, PRC(2019)***
  - **Using different vorticity** ***Wu, Pang, Huang, Wang, PRR (2019)***

- **Also see recent review**

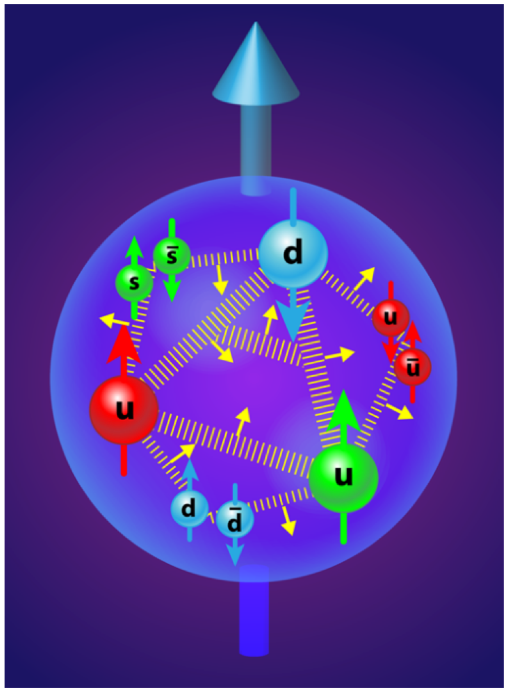
  J.H. Gao, G.L. Ma, SP, Q. Wang, NST 31(2020)9, 90

Spin hydrodynamics

- **Relativistic hydrodynamics + spin degree of freedom**

- **Problem: how to introduce spin of a massive fermionic fluid in a relativistic theory?**

Connection to "spin physics" (QCD)



- **Proton spin problem:**

  (slides from Hatta-son's talk)

$$\boxed{\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q + L^g}$$

Quarks' helicity — **Gluons' helicity** — **Orbital angular Momentum (OAM)**

Total angular momentum conservation

- **Nöther's theorem ：**

$$x^\mu \;\rightarrow\; x'^\mu = \Lambda^\mu_{\ \nu} x^\nu = (\delta^\mu_\nu + \epsilon^\mu_{\ \nu}) x^\nu \,, \qquad A^\mu(x) \;\rightarrow\; A'^\mu(x) = \Lambda^\mu_{\ \nu} A^\nu(\Lambda^{-1} x),$$

$$\psi(x) \;\rightarrow\; \psi'(x) = \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1} x),$$

$$\Rightarrow \qquad \boxed{\partial_\lambda (J_A^{\lambda\mu\nu} + J_\psi^{\lambda\mu\nu}) = 0}$$

- **Nöther current**

Gauge part

$$J_A^{\lambda\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A^\alpha)} \Delta A^{\mu\nu\alpha} = -F^\lambda_{\ \alpha}(x^\mu \partial^\nu - x^\nu \partial^\mu) A^\alpha - F^{\lambda\mu} A^\nu + F^{\lambda\nu} A^\mu \,.$$

Fermionic part

$$J_\psi^{\lambda\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \psi)} \Delta\psi^{\mu\nu} = \bar{\psi} i \gamma^\lambda \left(x^\mu \partial^\nu - x^\nu \partial^\mu - i\Sigma^{\mu\nu} \right) \psi$$

- **How to define the orbital and spin parts?**

Jaffe-Manohar (canonical) decomposition

- **Canonical energy momentum tensor** — Non-symmetric

Gauge part

$$T^{\mu\nu}_{\mathrm{A,can}} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\alpha)}\partial^\nu A^\alpha - g^{\mu\nu}\mathcal{L}_A = -F^\mu_{\ \alpha}\partial^\nu A^\alpha + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$$

Fermionic part

$$T^{\mu\nu}_{\psi,\mathrm{can}} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial^\nu \psi - g^{\mu\nu}\mathcal{L}_\psi = \bar{\psi}i\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\bar{\psi}(i\gamma^\alpha D_\alpha - m)\psi$$

- **Canonical decomposition**

$$J^{\mu\nu\lambda} = \underbrace{x^\nu T^{\mu\lambda}_{can} - x^\lambda T^{\mu\nu}_{can}}_{\text{Orbital angular momentum}} \underbrace{- \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\overline{\psi}\gamma_5\gamma_\rho\psi}_{\text{Quark helicity (spin)}} + \underbrace{F^{\mu\lambda}A^\nu - F^{\mu\nu}A^\lambda}_{\text{Gluon helicity (Spin)}}$$

- **Jaffe-Manohar decomposition (1990)**

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q_{can} + L^g_{can}$$

Operators are not gauge invariant.

Pseudo-gauge transformation

- **The variation of (canonical) spin is the anti-symmetric part of energy momentum tensor**

$$0 = \partial_\lambda J^{\lambda\mu\nu} = \partial_\lambda\left(x^\mu T^{\lambda\nu}_{\text{can}} - x^\nu T^{\lambda\mu}_{\text{can}} + S^{\lambda\mu\nu}_{\text{can}}\right) \;\Rightarrow\; T^{\mu\nu}_{\text{can}} - T^{\nu\mu}_{\text{can}} = -\partial_\lambda S^{\lambda\mu\nu}_{\text{can}}\,,$$

- **Belinfante energy momentum tensor**

$$\boxed{T^{\mu\nu}_{\text{Bel}} \equiv T^{\mu\nu}_{\text{can}} + \partial_\lambda K^{\lambda\mu\nu}_{\text{Bel}}}$$

$$K^{\lambda\mu\nu} = -K^{\mu\lambda\nu}$$

$$K^{\lambda\mu\nu}_{\text{Bel}} = \frac{1}{2}\left(S^{\lambda\mu\nu}_{\text{can}} - S^{\mu\lambda\nu}_{\text{can}} + S^{\nu\mu\lambda}_{\text{can}}\right)$$

Conserved $\boxed{\partial_\mu T^{\mu\nu}_{\text{Bel}} = 0}$

$\boxed{T^{\mu\nu}_{\text{Bel}} - T^{\nu\mu}_{\text{Bel}} = 0}$ Symmetric

$$T^{\mu\nu}_{A,\text{Bel}} \equiv -F^{\mu}_{\ \alpha}F^{\nu\alpha} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\,,$$

$$T^{\mu\nu}_{\psi,\text{Bel}} \equiv \bar{\psi}i\gamma^\mu \overleftrightarrow{D}^\nu\psi + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}\partial_\lambda(\bar{\psi}\gamma_5\gamma_\rho\psi)\,.$$

Gauge invariant

Ji (Belinfante) decomposition

- **Belinfante improved total angular momentum**

$$J_{\mathrm{Bel}}^{\lambda\mu\nu} \equiv J^{\lambda\mu\nu} + \partial_\rho\left(x^\mu K_{\mathrm{Bel}}^{\rho\lambda\nu} - x^\nu K_{\mathrm{Bel}}^{\rho\lambda\mu}\right)$$

$$\partial_\lambda J^{\lambda\mu\nu} = 0 \quad \Longleftrightarrow \quad \partial_\lambda J_{\mathrm{Bel}}^{\lambda\mu\nu} = 0$$

$$\boxed{J_{A/\psi,\mathrm{Bel}}^{\lambda\mu\nu} = x^\mu \tilde{T}_{A/\psi,\mathrm{Bel}}^{\lambda\nu} - x^\nu \tilde{T}_{A/\psi,\mathrm{Bel}}^{\lambda\mu}}$$

- **Ji decomposition (1997)**

$$\frac{1}{2} = J_q + J_g$$

Table of two different forms

- **Canonical (Jaffe-Manohar) decomposition**

$$\boldsymbol{J} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5\boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} + \underbrace{\boldsymbol{E}\times\boldsymbol{A}}_{\Delta G} \underbrace{-i\psi^\dagger(\boldsymbol{x}\times\boldsymbol{\nabla})\psi}_{L^q_{\text{can}}} + \underbrace{\boldsymbol{E}(\boldsymbol{x}\times\boldsymbol{\nabla})\boldsymbol{A}}_{L^g_{\text{can}}}$$

$$\boxed{T^{\mu\nu}_{\psi,\text{can}} = \bar{\psi}i\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\bar{\psi}(i\gamma^\alpha D_\alpha - m)\psi}$$

non-symmetric
Not gauge invariant

- **Belinfante (Ji) decomposition**

$$\boldsymbol{J} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5\boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} \underbrace{-i\psi^\dagger(\boldsymbol{x}\times\boldsymbol{D})\psi}_{L^q_{\text{Ji}}} + \underbrace{\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{B})}_{J^g_{\text{Ji}}}$$

Connected by pseudo gauge transformation

$$\boxed{T^{\mu\nu}_{\psi,\text{Bel}} \equiv \bar{\psi}i\gamma^\mu\overleftrightarrow{D}^\nu\psi + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}\partial_\lambda(\bar{\psi}\gamma_5\gamma_\rho\psi)}$$

Symmetric
Gauge invariant

Two spin communities divided

measured by PHENIX, STAR, COMPASS, HERMES

Nonexistent in Ji's scheme...

Jaffe-Manohar

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q_{can} + L^g_{can}$$

common and well-known

not measured yet
not even well-defined?

Ji

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + L^q_{\mathrm{Ji}} + J^g$$

Define rigorously.
Must be related to GPD!

accessible from GPD measured at JLab, COMPASS, HERMES, J-PARC
also calculable in lattice QCD

(slides from Hatta-san's talk)

E. Leader, C. Lorce, Phys. Rept. 541 (2014) 163-248

GLW decomposition

- **Another Pseudo gauge transformation**

$$T^{\mu\nu}_{\rm can} = T^{\mu\nu}_{\rm GLW} + \frac{1}{2}\partial_\lambda\left(\Phi^{\lambda,\mu\nu}_{\rm can} + \Phi^{\mu,\nu\lambda}_{\rm can} + \Phi^{\nu,\mu\lambda}_{\rm can}\right)$$

$$S^{\lambda,\mu\nu}_{\rm can} = S^{\lambda,\mu\nu}_{\rm GLW} - \Phi^{\lambda,\mu\nu}_{\rm can}$$

$$\Phi^{\lambda,\mu\nu}_{\rm can} \equiv S^{\mu,\lambda\nu}_{\rm GLW} - S^{\nu,\lambda\mu}_{\rm GLW}$$

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm can}(x) = T^{\nu\mu}_{\rm can} - T^{\mu\nu}_{\rm can} = -\partial_\lambda S^{\mu,\lambda\nu}_{\rm GLW}(x) + \partial_\lambda S^{\nu,\lambda\mu}_{\rm GLW}(x).$$

$$T^{\mu\nu}_{\rm Bel} = T^{\mu\nu}_{\rm GLW} - \frac{1}{2}\partial_\lambda\left(S^{\nu,\lambda\mu}_{\rm GLW} + S^{\mu,\lambda\nu}_{\rm GLW}\right)$$

***Textbook** written by de Groot, van Leeuven, and van Weert*

Review: ***W. Florkowski, R. Ryblewski and Avdhesh Kumar, Prog. Part. Nucl. Phys. 108 (2019) 103709***

- **Microscopic kinetic theory: GLW is the classical one.**

$$T^{\mu\nu}_{\rm GLW}(x) = \frac{1}{m}{\rm tr}_4\int d^4k\, k^\mu\, k^\nu \mathcal{W}(x,k) = \frac{1}{m}\int d^4k\, k^\mu\, k^\nu \mathcal{F}(x,k).$$

Question

- **Which kinds of energy momentum tensor are measured or preferred by the experiments?**

- **Hints:**
  - ✓ Ordinary relativistic hydrodynamics formulism are symmetric. (Relatively easy to be extended to spin hydro ?)
  - ✓ Anomalous (magneto-) hydrodynamics (including the spin current for massless fermions) are symmetric. (Relatively easy to be checked in massless limit)
  - ✓ Maybe, we eventually need to add the gluons' contributions. (A gauge invariant macroscopic theory may be more acceptable.)

Canonical form of spin hydrodynamics

Ref:

K. Hattori, M. Hongo, X.-G.Huang, M. Matsuo, H. Taya,
"Fate of spin polarization in a relativistic fluid: An entropy-current analysis,"
Phys. Lett. B795 (2019) 100–106, arXiv:1901.06615 [hep-th].

Also see recent work:

S.Y. Li, M.A Stephanov, H.U Yee, "Non-dissipative second-order transport, spin, and pseudo-gauge transformations in hydrodynamics", arXiv:2011.12318

D. She, A. Huang, D.F. Hou, J.F Liao, "Relativistic Viscous Hydrodynamics with Angular Momentum", arXiv: 2105.04060

Basic conservation equations

- **Total angular momentum conservation**

$$\partial_\alpha J^{\alpha\mu\nu}_{\mathrm{can}} = 0 \qquad J^{\alpha\mu\nu}_{\mathrm{can}} = \underbrace{x^\mu T^{\alpha\nu}_{can} - x^\nu T^{\alpha\mu}_{can}}_{\text{Orbital part}} + \underbrace{\Sigma^{\alpha\mu\nu}}_{\text{Spin tensor}}$$

$$\Rightarrow \boxed{\partial_\alpha \Sigma^{\alpha\mu\nu} = T^{\nu\mu}_{can} - T^{\mu\nu}_{can}} \quad ✓$$

- **Energy-momentum conservation**

$$\boxed{\partial_\mu T^{\mu\nu}_{can} = 0} \quad ✓$$

- **Currents conservation**

$$\boxed{\partial_\mu j^\mu = 0} \quad ✓$$

Common strategy for derivation of fluid equations

- **Tensor decomposition:**
  - Parallel or perpendicular to fluid velocity u^μ
  - Traceless part and other part

- **Gradient (∂) expansion: $\partial X \ll X$**

- **Entropy principle:**

  to derive the general expression for all components of tensors

An example: charge currents

- **Charge currents:**

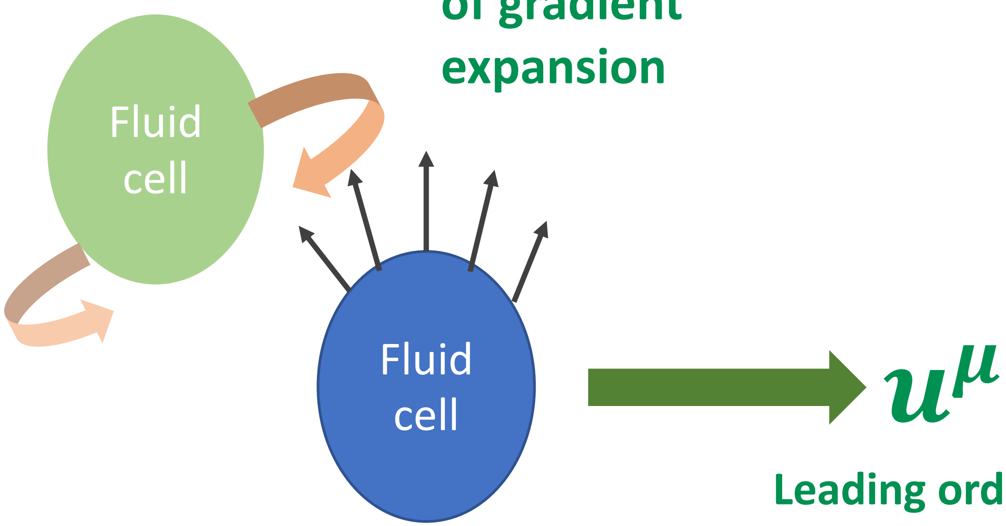
n: charge density

$$j^\mu = n u^\mu + j^\mu_{(1)}$$

Parallel to fluid velocity u^μ;
Leading order of gradient expansion

Perpendicular to fluid velocity u^μ;
Higher orders of gradient expansion

Higher orders: exchange the heat and particles with other cells.



Leading order:
moving alone the u^μ in average

Spin tensor decomposition

- **Analogy to the decomposition for currents:**

$$j^\mu = nu^\mu + j^\mu_{(1)}$$

Parallel to fluid velocity u^μ; Leading order

Perpendicular to fluid velocity u^μ; Higher order

- **One can assume that**

Spin density

$$\Sigma^{\alpha\mu\nu} = u^\alpha S^{\mu\nu} + \Sigma^{\alpha\mu\nu}_{(1)}$$

spin tensor

Parallel to fluid velocity u^μ; Leading order

Perpendicular to fluid velocity u^μ; Higher order

Modified thermodynamic relations

- **Density vs Chemical potential**

Charge density: n ↔ **Charge chemical potential: μ**

Spin density: $S^{\mu\nu}$ ↔ **Spin chemical potential: $\omega^{\mu\nu}$**

Physical interpretation:
Variation of total energy when we add one particle with spin $S^{\mu\nu}$

- **Thermodynamic relations**

$$e + p = Ts + \mu n + \omega_{\mu\nu} S^{\mu\nu}$$

energy density; pressure; temperature X entropy density; spin chemical potential X spin density

- **Gibbs relations**

$$de = Tds + \mu dn + \omega_{\mu\nu} dS^{\mu\nu} \qquad dp = sdT + nd\mu + S^{\mu\nu} d\omega_{\mu\nu}$$

Orders of $S^{\mu\nu}$ and $\omega^{\mu\nu}$

- In Ref. *K. Hattori, M. Hongo, X.-G.Huang, M. Matsuo, H. Taya, Phys. Lett. B795 (2019) 100–106.*

$$\boxed{S^{\mu\nu}, \omega^{\mu\nu} \sim O\left(\partial^1\right);\ \omega_{\mu\nu}S^{\mu\nu} \sim O\left(\partial^2\right)}$$

- In our recent work, *K. Fukushima, SP, PLB 2010.01608*

$$\boxed{S^{\mu\nu} \sim O(1), \omega^{\mu\nu} \sim O\left(\partial^1\right);\ \omega_{\mu\nu}S^{\mu\nu} \sim O\left(\partial^1\right)}$$

Density is classic $O(1)$, but the variation of energy is quantum $O\left(\partial^1\right)$!

- We only consider the spin hydro up to the first order in gradient expansion.

Entropy production rate

- **Two ways to derive the entropy flow:**
  - Directly using

$$u_\nu \partial_\mu T^{\mu\nu}_{can} + \mu \partial_\mu j^\mu = 0$$

+ Gibbs relation

K. Hattori, M. Hongo, X.-G.Huang, M. Matsuo, H. Taya, PLB795 (2019) 100–106.

$$\longrightarrow \quad \partial_\mu \mathcal{S}^\mu_{can} \geq 0$$

  - Using the extended entropy flow

W. Israel, J. Stewart, Annals Phys. 118, 341 (1979)

Relativistic fluid generation of Gibbs relation

$$\begin{aligned}\mathcal{S}^\mu_{\rm can} &= \frac{u_\nu}{T}\Theta^{\mu\nu} + \frac{p}{T}u^\mu - \frac{\mu}{T}j^\mu - \frac{1}{T}\omega_{\rho\sigma}S^{\rho\sigma}u^\mu + \mathcal{O}(\partial^2) \\ &= su^\mu + \frac{u_\nu}{T}\Theta^{\mu\nu}_{(1)} - \frac{\mu}{T}j^\mu_{(1)} + \mathcal{O}(\partial^2)\,,\end{aligned}$$

K. Fukushima, SP, PLB 2010.01608

$$\longrightarrow \quad \partial_\mu \mathcal{S}^\mu_{can} \geq 0$$

Constrains from entropy principle

$$T^{\mu\nu}_{can} = \underbrace{T^{\mu\nu}_{(0)}}_{\text{Leading order}} + \underbrace{T^{\mu\nu}_{(1)}}_{\text{Next-to-Leading order}}$$

$$\boxed{\begin{aligned} T^{\mu\nu}_{(1s)} &= T^{\mu\nu}_{(1)} + T^{\nu\mu}_{(1)} \\ T^{\mu\nu}_{(1a)} &= T^{\mu\nu}_{(1)} - T^{\nu\mu}_{(1)} \end{aligned}}$$

symmetric

anti-symmetric

$$\partial_\mu S^\mu_{can} = \underbrace{T^{\mu\nu}_{(1s)}\partial_\mu \frac{u_\nu}{T} - j^\mu_{(1)}\partial_\mu \frac{\mu}{T}}_{\text{Ordinary terms not related to spin}} + \frac{2T^{\mu\nu}_{(1a)}}{T}\left(\omega_{\mu\nu} + \frac{1}{2}T\omega^{th}_{\mu\nu}\right) \geq 0$$

Thermal vorticity

$$\begin{aligned} \omega^{\mu\nu}_{th} &= (g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta) \\ &\quad \times[\partial_\alpha(u_\beta/T) - \partial_\beta(u_\alpha/T)] \end{aligned}$$

Non-relativistic limit

$$\epsilon^{ijk}\omega^{ij}_{th} \sim (\nabla \times \frac{\mathbf{v}}{T})^k$$

Global equilibrium

$$\omega^{\mu\nu}_{th} = (g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta) \times[\partial_\alpha(u_\beta/T) - \partial_\beta(u_\alpha/T)]$$

Ordinary terms not related to spin

Thermal vorticity

$$\partial_\mu \mathcal{S}^\mu_{can} = \overbrace{T^{\mu\nu}_{(1s)}\partial_\mu \frac{u_\nu}{T} - j^\mu_{(1)}\partial_\mu \frac{\mu}{T}} + \frac{2T^{\mu\nu}_{(1a)}}{T}\underline{\left(\omega_{\mu\nu} + \frac{1}{2}T\omega^{th}_{\mu\nu}\right)} = 0$$

Must vanish!

Widely proved by many approaches:

F. Becattini, L. Bucciantini, E. Grossi, and L. Tinti, Eur. Phys. J. C 75, 191 (2015)
F. Becattini, W. Florkowski, and E. Speranza, Physics Letters B 789, 419 (2019)
K. Hattori, M. Hongo, X.-G.Huang, M. Matsuo, H. Taya, Phys. Lett. B795 (2019) 100–106.
…

Also see recent reviews:

Y. C. Liu and X. G. Huang, Nucl. Sci. Tech. 31,56 (2020)
J.H. Gao, G.L. Ma, SP, Q. Wang, Nucl. Sci. Tech 31 (2020) 9, 90

Spin chemical potential $\omega^{\mu\nu}$ must be related to thermal vorticity as $-T\omega^{th}_{\mu\nu}/2$ in global equilibrium!

Local equilibrium

$$\partial_\mu \mathcal{S}^\mu_{can} = T^{\mu\nu}_{(1s)} \partial_\mu \frac{u_\nu}{T} - j^\mu_{(1)} \partial_\mu \frac{\mu}{T} + \frac{2T^{\mu\nu}_{(1a)}}{T} \left(\omega_{\mu\nu} + \frac{1}{2} T \omega^{th}_{\mu\nu} \right) \geq 0$$

- **Tensor decomposition of energy momentum tensor**

symmetric

$$T^{\mu\nu}_{(1s)} = \underbrace{h^\mu}_{\text{heat flow}} u^\nu + h^\nu u^\mu + \underbrace{\pi^{\mu\nu}}_{\text{viscous tensor}}$$

anti-symmetric

$$T^{\mu\nu}_{(1a)} = q^\mu u^\nu - q^\nu u^\mu + \phi^{\mu\nu}$$

$$\begin{aligned}
\partial_\mu \mathcal{S}^\mu_{can} = & \left.\begin{aligned} &(h^\mu - \frac{e+p}{n} j^\mu_{(1)}) \frac{n}{e+p} (g_{\nu\alpha} - u_\nu u_\alpha) \partial^\nu \frac{\mu}{T} \\ &+ \frac{\pi^{\mu\nu}}{T} \partial_{<\mu} u_{\nu>} + \frac{1}{3} \frac{1}{T} \pi^\mu_\mu (\partial \cdot u) \end{aligned}\right\} \text{dissipative terms in ordinary fluids} \\
& \left.\begin{aligned} &+ q^\mu [-\frac{1}{T}(u\cdot\partial) u_\mu + \partial_\mu \frac{1}{T} + 4\frac{\omega_{\mu\nu} u^\nu}{T}] \\ &+ \phi^{\mu\nu} [\omega^{th}_{\mu\nu} + 2\beta \omega_{\mu\nu}] \geq 0 \end{aligned}\right\} \text{new terms related to spin}
\end{aligned}$$

Entropy principle

$$
\begin{aligned}
\partial_\mu \mathcal{S}^\mu_{can} \;=\;& (h^\mu - \frac{e+p}{n} j^\mu_{(1)}) \frac{n}{e+p} (g_{\nu\alpha} - u_\nu u_\alpha) \partial^\nu \frac{\mu}{T} \\
&+ \frac{\pi^{\mu\nu}}{T} \partial_{<\mu} u_{\nu>} + \frac{1}{3}\frac{1}{T} \pi^\mu_\mu (\partial \cdot u) \\
&+ q^\mu [-\frac{1}{T}(u\cdot\partial) u_\mu + \partial_\mu \frac{1}{T} + 4\frac{\omega_{\mu\nu} u^\nu}{T}] \\
&+ \phi^{\mu\nu} [\omega^{th}_{\mu\nu} + 2\beta\omega_{\mu\nu}] \geq 0
\end{aligned}
$$

dissipative terms in ordinary fluids

new terms related to spin

- **To ensure the entropy production rate be always positive, the only possible way is**

$$
\begin{aligned}
q^\mu &= \lambda[(u\cdot\partial)u^\mu + \frac{1}{T}\Delta^{\mu\nu}\partial_\nu T - 4\omega^{\mu\nu}u_\nu], \\
\phi^{\mu\nu} &= 2\gamma[T\omega^{\mu\nu}_{th} + 2(g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta)\omega_{\alpha\beta}]/T.
\end{aligned}
$$

$\lambda, \gamma \geq 0$ **are new transport coefficients**

Brief summary of canonical form

- **Energy momentum tensor has anti-symmetric part**

$$T^{\mu\nu}_{can} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)}$$

Leading order | Next-to-Leading order

$$T^{\mu\nu}_{(1s)} = T^{\mu\nu}_{(1)} + T^{\nu\mu}_{(1)} \quad \text{symmetric}$$

$$T^{\mu\nu}_{(1a)} = T^{\mu\nu}_{(1)} - T^{\nu\mu}_{(1)} \quad \text{anti-symmetric}$$

- **In global equilibrium, spin chemical potential is related to thermal vorticity**

$$\omega^{\mu\nu} = -\frac{1}{2}T\omega^{\mu\nu}_{th}$$

Power counting: $\omega^{th}_{\mu\nu} \sim O(\partial^1)$

$\Rightarrow \omega_{\mu\nu} \sim O(\partial^1)$

- **Symmetric part is as the same as the ordinary fluid. The expression for anti-symmetric part can be derived by entropy principle.**

$$T^{\mu\nu}_{(1a)} = q^\mu u^\nu - q^\nu u^\mu + \phi^{\mu\nu}$$

$$q^\mu = \lambda[(u\cdot\partial)u^\mu + \frac{1}{T}\Delta^{\mu\nu}\partial_\nu T - 4\omega^{\mu\nu}u_\nu],$$

$$\phi^{\mu\nu} = 2\gamma[T\omega^{\mu\nu}_{th} + 2(g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta)\omega_{\alpha\beta}]/T.$$

Power counting: $T^{\mu\nu}_{(1a)} \sim O(\partial^1)$

$\partial_\rho(u^\rho S^{\mu\nu}) = -2T^{\mu\nu}_{(1a)} \sim O(\partial^1)$

$\Rightarrow S^{\mu\nu} \sim O(1)$

$\lambda, \gamma \geq 0$ **are new transport coefficients**

Belinfante form of spin hydrodynamics

Ref.
K. Fukushima, SP,
"Spin Hydrodynamics and Symmetric Energy - Momentum Tensors
– A current induced by the spin vorticity", PLB 817 (2021) 136346

"Relativistic decomposition of the orbital and the spin angular momentum in chiral physics and Feynman's angular momentum paradox", invited lecture, 2001.00359, Lecture Notes in Physics volume on "Strongly Interacting Matter under Rotation"

Basic conservation equations

- **Total angular momentum conservation**

$$J^{\mu\nu\alpha}_{\text{Bel}} = J^{\mu\nu\alpha} + \partial_\rho(x^\nu K^{\rho\mu\alpha}_{\text{Bel}} - x^\alpha K^{\rho\mu\nu}_{\text{Bel}}),$$
$$= x^\nu T^{\mu\alpha}_{\text{Bel}} - x^\alpha T^{\mu\nu}_{\text{Bel}}.$$

$$\boxed{\partial_\mu J^{\mu\nu\alpha}_{\text{Bel}} \quad = \quad 0,}$$

- **Energy momentum conservation**

✓ $$\boxed{\partial_\mu T^{\mu\nu}_{\text{Bel}} \quad = \quad 0.}$$

These two equations are equivalent!

- **Current conservation**

✓ $$\boxed{\partial_\mu j^\mu = 0}$$

No spin in Belinfante form?

$$\boxed{\partial_\mu J^{\mu\nu\alpha}_{\mathrm{Bel}} = 0,} \quad \overset{\text{equivalent}}{\longleftrightarrow} \quad \boxed{\partial_\mu T^{\mu\nu}_{\mathrm{Bel}} = 0.}$$

<u>The common argument:</u>

- There is no equation for spin.
- There is no degree of freedom for spin.
- We cannot observe spin effect in Belinfante form.

Recalling what we discussed in introduction.

- Belinfante energy momentum tensor is connected to canonical one by pseudo gauge transformation.

If a physical (spin) effect disappears after a physical transformation, then, did that mean this “physical effect” is unphysical?
Or, should it be that this physical effect will appear somewhere?

Belinfante energy momentum tensor

- **We take the pseudo gauge transformation**

$$T^{\mu\nu}_{Bel} = T^{\mu\nu}_{can} + \partial_\lambda K^{\lambda\mu\nu}_{Bel} = T^{\mu\nu}_{0} + T^{\mu\nu}_{(1)}$$

Leading order

Next-to-Leading order

$$T^{\mu\nu}_{(1)} = h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu} + \frac{1}{2}\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})$$

heat flow

viscous tensor

spin density tensor

spin corrections to the energy momentum tensor

Spin corrections

$$T^{\mu\nu}_{(1)} = h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu} + \frac{1}{2}\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})$$

heat flow

viscous tensor

spin density tensor

spin corrections to the energy momentum tensor

Using standard tensor decomposition, we have

$$T^{\mu\nu}_{(1)} = (\delta e_s)u^\mu u^\nu + (h^\mu + \delta h^\mu_s)u^\nu + (h^\nu + \delta h^\nu_s)u^\mu + \pi^{\mu\nu} + \delta\pi^{\mu\nu}_s$$

$$\delta e_s = u_\rho \partial_\sigma S^{\rho\sigma},$$
↔ Spin correction to energy density

$$\delta h^\mu_s = \frac{1}{2}\Delta^\mu_\sigma \partial_\lambda S^{\sigma\lambda} + \frac{1}{2} u_\rho S^{\rho\lambda}\partial_\lambda u^\mu$$
↔ Spin correction to heat flow

$$\delta\pi^{\mu\nu}_s = \partial_\lambda(u^{<\mu}S^{\nu>\lambda}) + \frac{1}{3}\partial_\lambda(u^\sigma S^{\rho\lambda})\Delta_{\rho\sigma}.$$
↔ Spin correction to viscous tensor

Spin will appear as corrections to the ordinary dissipative terms!

Frame dependence

- **In ordinary relativistic fluid, we have Landau (energy) frame and Particle frame.**
- **The heat flow depends on frame. In Landau frame, there is no heat flow,**

$$u_L^\mu = u^\mu + \frac{1}{e+p}(h^\mu + \delta h^\mu),$$

but the dissipative current will be modified by spin correction!

$$j^\mu_{\mathrm{L}(1)} = \left(j^\mu_{(1)} - \frac{n}{e+p} h^\mu \right) + \delta j^\mu_{(1)} \qquad \delta j^\mu_{(1)} = -\frac{n}{e+p}\delta h^\mu \,.$$

Non-relativistic limit →

$$\delta \boldsymbol{j}_{(1)} = -\frac{n}{2(e+p)}\Big[\boldsymbol{\nabla}\times\boldsymbol{S} + \dot{\boldsymbol{v}}\times\boldsymbol{S} + (\boldsymbol{\nabla}\cdot\boldsymbol{v})\mathfrak{s} - 2(\mathfrak{s}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \dot{\mathfrak{s}}\Big] \,.$$

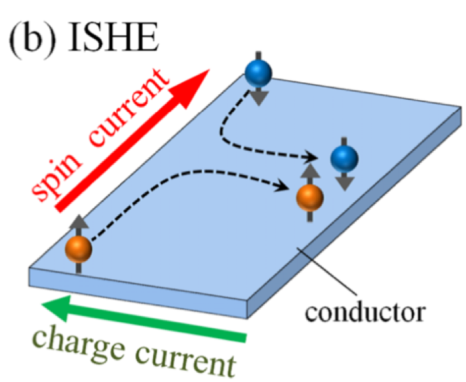
Quantum spin vorticity

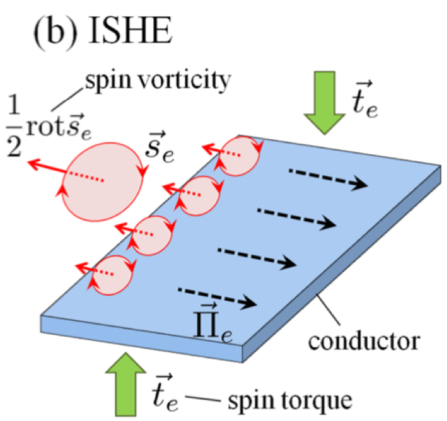
- **We have derived**

$$\delta \mathbf{j}_{(1)} \propto -(\nabla \times \mathbf{S})$$

$S^i = \epsilon^{ijk} S^{ij}$
spin density alone
i-th direction

Curl of spin will induce a current





Standard Inverse Spin Hall Effect (ISHE)

Inverse Spin Hall Effect (SHE) understood by quantum spin vorticity

M. Fukuda, K. Ichikawa, M. Senami, and A. Tachibana, AIP Advances 6, 025108 (2016).

Entropy principle (1)

- Using the same method, we get the entropy production rate

$$\partial_\mu \mathcal{S}^\mu = \left(\frac{n}{e+p} h^\mu - j^\mu_{(1)}\right)\Delta_{\mu\nu}\partial^\nu \frac{\mu}{T} + \frac{1}{T}\pi^{\mu\nu}\partial_\mu u_\nu + \Delta$$

Spin corrections

$$\Delta \equiv \frac{1}{2}\left[\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\right]\partial_\mu \frac{u_\nu}{T} - \frac{\omega_{\rho\sigma}}{T}\partial_\lambda(u^\lambda S^{\rho\sigma}) .$$

It is not in a squared form at all!

- The simplest way to ensure entropy principle is to let

$$S^{\mu\nu} = 0, \Delta = 0 \qquad ???$$

That is the way to "get" the common argument "No degree of freedom for spin in Belinfante form".

Of course, it is a (trivial) solution. But, is it the only solution?

Entropy principle (2)

- **Using the same method, we get the entropy production rate**

$$\partial_\mu \mathcal{S}^\mu = \left(\frac{n}{e+p}h^\mu - j^\mu_{(1)}\right)\Delta_{\mu\nu}\partial^\nu \frac{\mu}{T} + \frac{1}{T}\pi^{\mu\nu}\partial_\mu u_\nu + \Delta$$

Spin corrections

$$\Delta \equiv \frac{1}{2}\left[\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\right]\partial_\mu \frac{u_\nu}{T} - \frac{\omega_{\rho\sigma}}{T}\partial_\lambda(u^\lambda S^{\rho\sigma})\,.$$

It is not in a squared form at all!

$$\Delta = \boxed{\frac{1}{2}\partial_\mu\left[\partial_\lambda(u^\lambda S^{\mu\nu} + u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\frac{u_\nu}{T}\right]} \rightarrow \partial_\mu \delta \mathcal{S}^\mu$$
$$- \frac{1}{2}\left[\partial_\lambda(u^\lambda S^{\mu\nu})\right]\partial_\mu \frac{u_\nu}{T} - \frac{\omega_{\rho\sigma}}{T}\partial_\lambda(u^\lambda S^{\rho\sigma})\,.$$

- **We move the total derivatives to the entropy flow (redefine the entropy flow)**

$$\partial_\mu(\mathcal{S}^\mu + \delta\mathcal{S}^\mu) = \ldots + \Delta' \geq 0$$

Similar to the anomalous fluid ***D.T. Son, P. Surowka, PRL. 103, 191601 (2009).***

Also see ***S.Y. Li, M.A. Stephanov, H.U. Yee, appear in PRL, 2011.12318***

Entropy principle (3)

$$\partial_\mu(\mathcal{S}^\mu + \delta\mathcal{S}^\mu) = \ldots + \Delta' \geq 0 \quad \Delta' = -\partial_\lambda(u^\lambda S^{\mu\nu})\left(\frac{1}{2}\partial_\mu \frac{u_\nu}{T} + \frac{\omega_{\mu\nu}}{T}\right)$$

It reproduces the results in canonical form!

- **In global equilibrium, $\omega^{\mu\nu} = -T\omega_{\mu\nu}^{th}/2$.**
- **In local equilibrium, by the tensor decomposition,**

$$\partial_\lambda(u^\lambda S^{\rho\sigma}) = q^\rho u^\sigma - q^\sigma u^\rho + \phi^{\rho\sigma}$$

we can get the same results as in canonical form.

$$\begin{aligned} q^\mu &= \lambda[(u\cdot\partial)u^\mu + \frac{1}{T}\Delta^{\mu\nu}\partial_\nu T - 4\omega^{\mu\nu}u_\nu], \\ \phi^{\mu\nu} &= 2\gamma[T\omega_{th}^{\mu\nu} + 2(g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta)\omega_{\alpha\beta}]/T. \end{aligned}$$

We have re-discovered the equation of motion for spin by entropy principle!

Main equations for Belinfante form

- **Energy momentum conservation**

$$\boxed{\partial_\mu T^{\mu\nu}_{\text{Bel}} = 0.}$$

equivalent

$$\boxed{\partial_\mu J^{\mu\nu\alpha}_{\text{Bel}} = 0,}$$

- **Current conservation**

$$\boxed{\partial_\mu j^\mu = 0}$$

- **Equations from entropy principle**

$$\partial_\lambda(u^\lambda S^{\rho\sigma}) = q^\rho u^\sigma - q^\sigma u^\rho + \phi^{\rho\sigma}$$

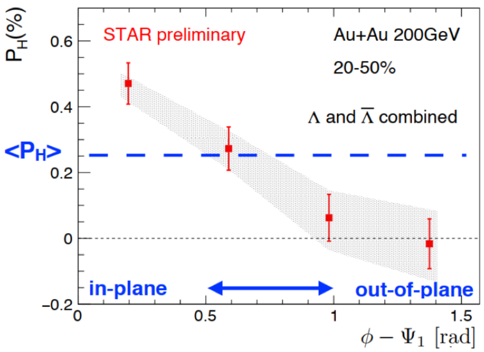
$$\begin{aligned} q^\mu &= \lambda[(u\cdot\partial)u^\mu + \frac{1}{T}\Delta^{\mu\nu}\partial_\nu T - 4\omega^{\mu\nu}u_\nu], \\ \phi^{\mu\nu} &= 2\gamma[T\omega^{\mu\nu}_{th} + 2(g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta)\omega_{\alpha\beta}]/T. \end{aligned}$$

- **Number of equations: 4+1+6= 11**
- **Variables: $T, \mu, S^{\mu\nu}, u^i$, 1+1+6+3=11**
- **Equation of state: $e = e(T, \mu, S^{\mu\nu})$ + Gibbs relation** $\omega^{\mu\nu} = \left.\frac{de}{dS^{\mu\nu}}\right|_{n,s}$

Revisit local spin polarization

Ref.
C. Yi, SP, D.L Yang
"Revisit local spin polarization beyond global equilibrium in relativistic heavy ion collisions", arXiv:2106.00238

Polarization induced by thermal vorticity



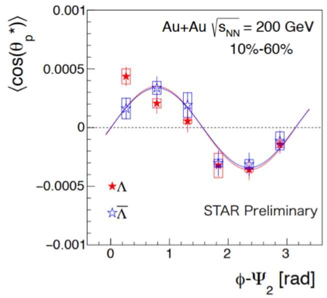
Thermal vorticity

$$\omega^{\rm th}_{\rho\sigma} = \frac{1}{2}\left[\partial_\rho\left(\frac{u_\sigma}{T}\right) - \partial_\sigma\left(\frac{u_\rho}{T}\right)\right]$$

Distribution function: f_0

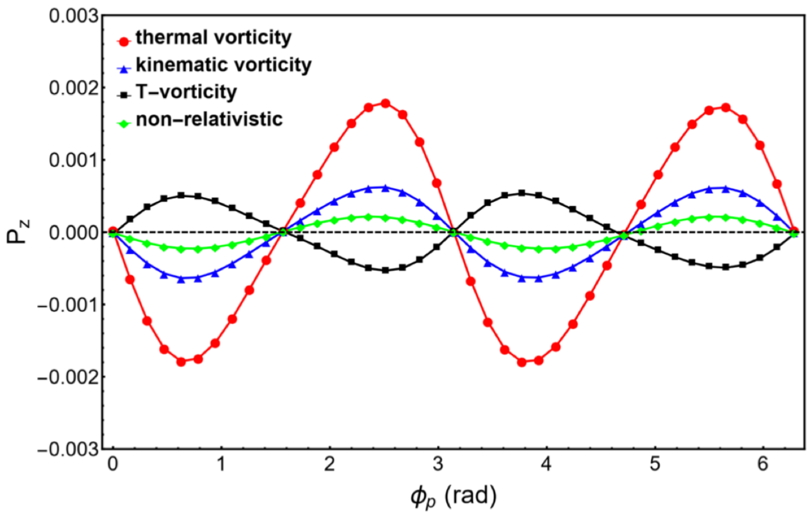
$$\mathcal{S}^\mu(p) = \frac{1}{8m_\Lambda}\epsilon^{\mu\nu\rho\sigma}p_\nu\frac{\int d\Sigma_\lambda p^\lambda f_0(1-f_0)\omega^{\rm th}_{\rho\sigma}}{\int d\Sigma_\lambda p^\lambda f_0}$$

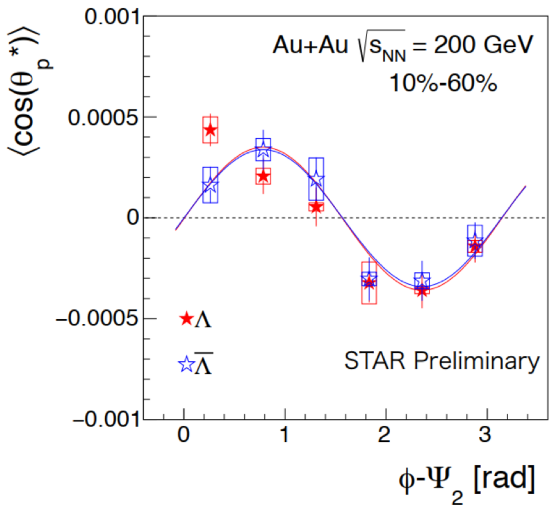
Freezeout surface



Karpenko, F. Becattini, Eur. Phys. J. C 77 (2017) 213
R.-H. Fang, L.-G. Pang, Q. Wang, X.-N. Wang, Phys. Rev. C94, 024904 (2016)

Local polarization from different vorticities





Kinematic vorticity:

$$\omega_{\mu\nu}^{(K)} = -\frac{1}{2}(\partial_\mu u_\nu - \partial_\nu u_\mu)$$

T-vorticity:

$$\omega_{\mu\nu}^{(T)} = -\frac{1}{2}[\partial_\mu(Tu_\nu) - \partial_\nu(Tu_\mu)]$$

Non-Relativistic vorticity

$$\omega_{\mu\nu}^{(\mathrm{NR})} = \epsilon_{\nu\mu\rho\eta}u^\rho\omega^\eta$$

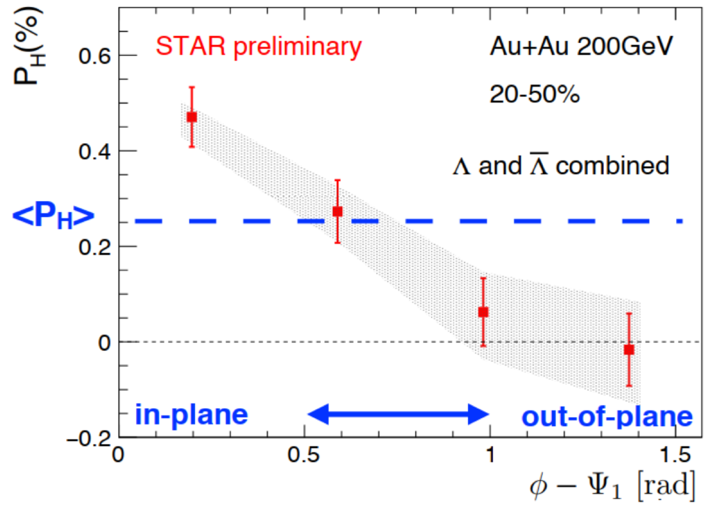
Thermal vorticity:

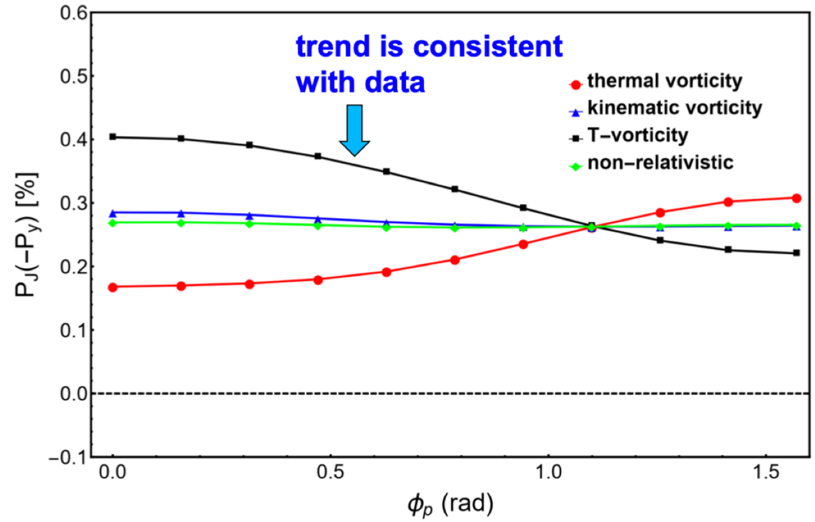
$$\omega_{\rho\sigma}^{\mathrm{th}} = \frac{1}{2}\left[\partial_\rho\left(\frac{u_\sigma}{T}\right) - \partial_\sigma\left(\frac{u_\rho}{T}\right)\right]$$

Wu, Pang, Huang, Wang, PRR 1, 033058(2019)

- **Only T-vorticity gives the right trend for both Pz and Py**
- **Why T-vorticity? Out-of-equilibrium effects?**

Local polarization from different vorticities





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Wu, Pang, Huang, Wang, PRR 1, 033058(2019)

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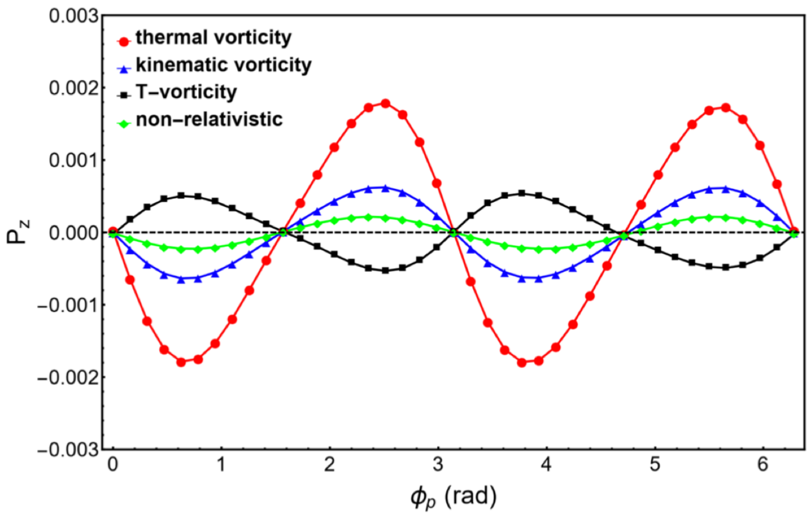

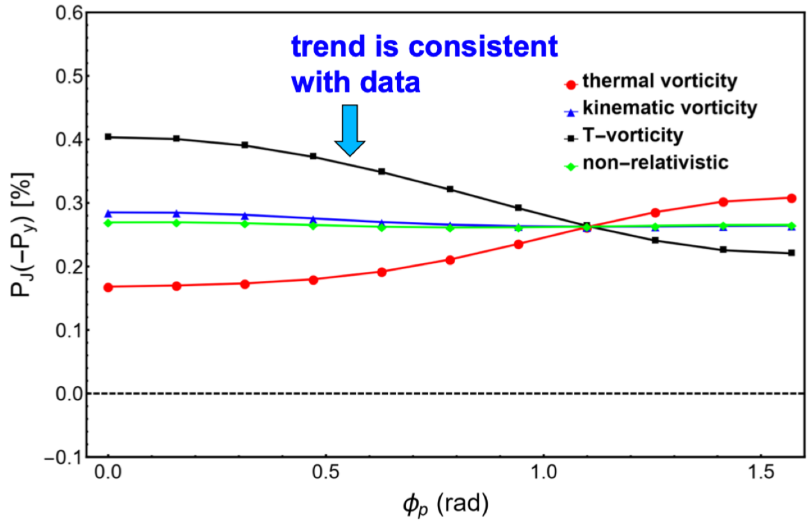

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Non-Relativistic vorticity

$$\omega_{\mu\nu}^{(\mathrm{NR})} = \epsilon_{\nu\mu\rho\eta} u^\rho \omega^\eta$$

Thermal vorticity:

$$\omega_{\rho\sigma}^{\mathrm{th}} = \frac{1}{2}\left[\partial_\rho\left(\frac{u_\sigma}{T}\right) - \partial_\sigma\left(\frac{u_\rho}{T}\right)\right]$$

Wu, Pang, Huang, Wang, PRR 1, 033058(2019)

- **Only T-vorticity gives the right trend for both Pz and Py**
- **Why T-vorticity? Out-of-equilibrium effects?**

Polarization and axial current

- **Recalling the original equations**

$$\mathcal{S}^{\mu}(\mathbf{p}) = \frac{\int d\Sigma \cdot p\mathcal{J}_5^{\mu}(p, X)}{2m_{\Lambda}\int d\Sigma \cdot \mathcal{N}(p, X)},$$

- **For massless fermions, the left and right handed currents read**

$$\begin{aligned}\mathcal{J}^{\mu}_{\lambda}(p, X) \;=\; & 2\pi\mathrm{sign}(u\cdot p)\left\{p^{\mu} + \lambda\frac{\hbar}{2}\delta(p^2)\left[u^{\mu}(p\cdot\omega) - \omega^{\mu}(u\cdot p)\right.\right. \\ & \left.\left. -2S^{\mu\nu}_{(u)}\tilde{E}_{\nu}\right]\partial_{u\cdot p} + \lambda\frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\partial^{p}_{\nu}\delta(p^2)\right\} f^{(0)}_{\lambda},\end{aligned}$$

$$S^{\mu\nu}_{(u)} \;=\; \epsilon^{\mu\nu\alpha\beta}p_{\alpha}u_{\beta}/(2u\cdot p),$$

$\lambda = \pm$

+: right

-: left

$$\tilde{E}_{\nu} \;=\; E_{\nu} + T\partial_{\nu}\frac{\mu_{\lambda}}{T} + \frac{(u\cdot p)}{T}\partial_{\nu}T - p^{\sigma}\left[\partial_{<\sigma}u_{\nu>} + \frac{1}{3}\Delta_{\sigma\nu}(\partial\cdot u) + u_{\nu}Du_{\sigma}\right].$$

$$f^{(0)}_{\lambda} = 1/(e^{(u\cdot p-\mu_{\lambda})/T} + 1),$$

Y. Hidaka, SP, and D.L. Yang, Phys. Rev. D97, 016004 (2018)

Polarization induced by different sources

- **Axial currents can be decomposed as**

$$\mathcal{J}^{\mu}_{5} = \mathcal{J}^{\mu}_{\text{thermal}} + \mathcal{J}^{\mu}_{\text{shear}} + \mathcal{J}^{\mu}_{\text{accT}} + \mathcal{J}^{\mu}_{\text{chemical}} + \mathcal{J}^{\mu}_{\text{EB}},$$

where they are related to:

Thermal vorticity

$$\mathcal{J}^{\mu}_{\text{thermal}} = a\frac{1}{2}\boxed{\epsilon^{\mu\nu\alpha\beta}p_{\nu}\partial_{\alpha}\frac{u_{\beta}}{T}},$$

Shear viscous tensor

$$\mathcal{J}^{\mu}_{\text{shear}} = -a\frac{1}{(u\cdot p)T}\epsilon^{\mu\nu\alpha\beta}p_{\alpha}u_{\beta}p^{\sigma}\boxed{\partial_{<\sigma}u_{\nu>}}$$

Fluid acceleration

$$\mathcal{J}^{\mu}_{\text{accT}} = -a\frac{1}{2T}\epsilon^{\mu\nu\alpha\beta}p_{\nu}u_{\alpha}\boxed{\left(Du_{\beta} - \frac{1}{T}\partial_{\beta}T\right)}.$$

Gradient of chemical potential

$$\mathcal{J}^{\mu}_{\text{chemical}} = a\frac{1}{(u\cdot p)}\epsilon^{\mu\nu\alpha\beta}p_{\alpha}u_{\beta}\boxed{\partial_{\nu}\frac{\mu}{T}},$$

Electromagnetic fields

$$\mathcal{J}^{\mu}_{\text{EB}} = a\frac{1}{(u\cdot p)T}\epsilon^{\mu\nu\alpha\beta}p_{\alpha}u_{\beta}\boxed{E_{\nu} + a\frac{B^{\mu}}{T}},$$

Y. Hidaka, SP, and D.L. Yang, PRD97, 016004 (2018); C. Yi, SP, D.L. Yang, arXiv:2106.00238

Out-of-equilibrium corrections

- **Polarization vector**

$$\mathcal{P}^z(p) = \int_{-1}^{+1} dY \mathcal{S}^z(p),$$
$$\mathcal{P}^y(p) = \int_{-1}^{+1} dY \mathcal{S}^y(p),$$

- **Polarization induced by thermal vorticity, shear viscous tensor and residual part of fluid acceleration**

$$\mathcal{S}^\mu_{\rm thermal}(\mathbf{p}) = \frac{\hbar}{8m_\Lambda N}\int d\Sigma^\sigma p_\sigma f_V^{(0)}(1-f_V^{(0)})\epsilon^{\mu\nu\alpha\beta}p_\nu\partial_\alpha\frac{u_\beta}{T},$$
$$\mathcal{S}^\mu_{\rm shear}(\mathbf{p}) = -\frac{\hbar}{4m_\Lambda N}\int d\Sigma\cdot p f_V^{(0)}(1-f_V^{(0)})\frac{\epsilon^{\mu\nu\alpha\beta}p_\alpha u_\beta}{(u\cdot p)T}\frac{1}{2}\{p^\sigma(\partial_\sigma u_\nu+\partial_\nu u_\sigma)-Du_\nu\}$$
$$\mathcal{S}^\mu_{\rm accT}(\mathbf{p}) = -\frac{\hbar}{8m_\Lambda N}\int d\Sigma\cdot p f_V^{(0)}(1-f_V^{(0)})\frac{1}{T}\epsilon^{\mu\nu\alpha\beta}p_\nu u_\alpha(Du_\beta-\frac{1}{T}\partial_\beta T),$$

Recent related works

- **Shear induced polarization draws some attentions.**
- **Shear induced Polarization from massless fermions (Theory):**

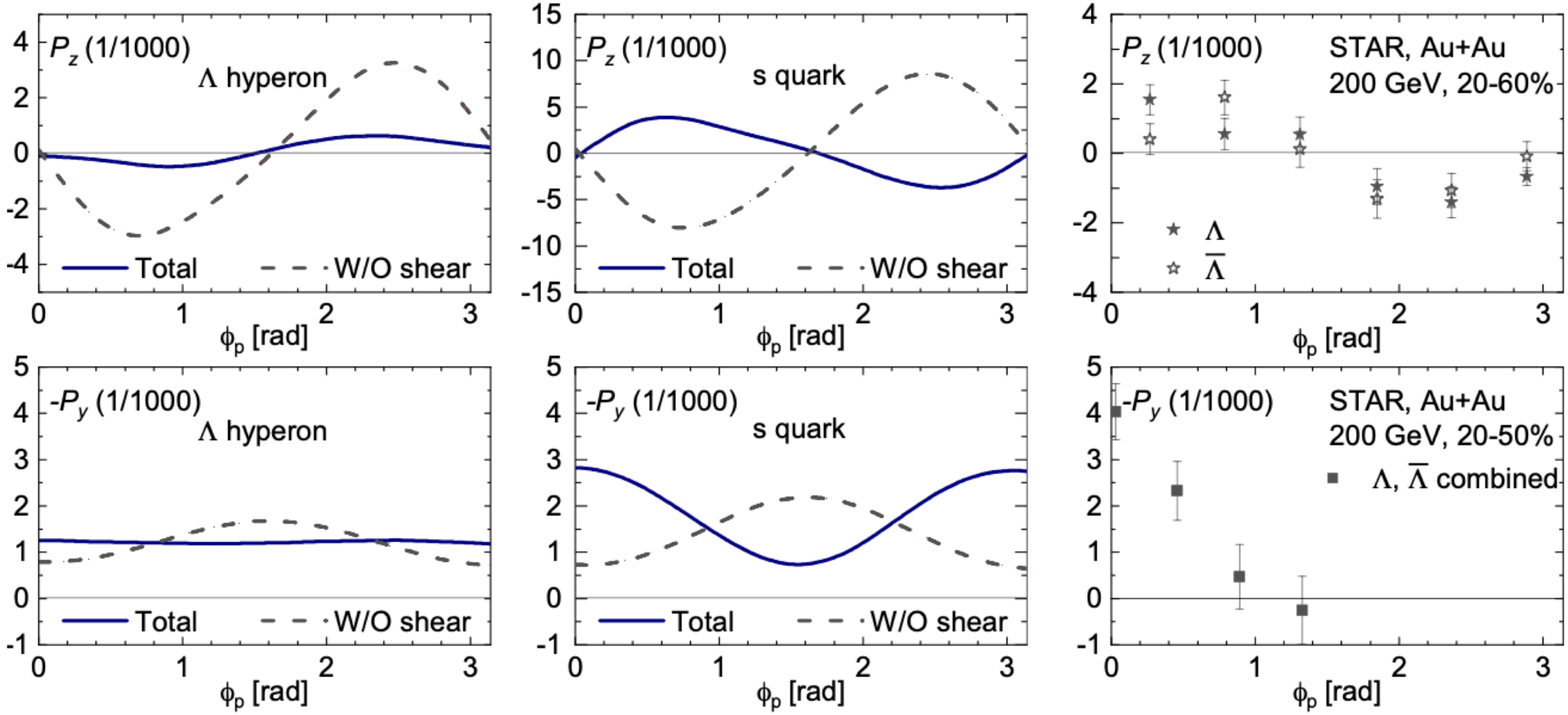
  Y. Hidaka, SP, and D.L. Yang, PRD97, 016004 (2018);
- **Shear induced Polarization from massive fermions:**
  - **Theory:**

    S. Y. F. Liu, Y. Yin, 2103.09200
    F. Becattini, M. Buzzegoli, A. Palermo, 2103.10917
  - **Hydrodynamic simulations:**

    B. Fu, S. Y. F. Liu, L. Pang, H. Song, Y. Yin, 2103.10403
    F. Becattini, M. Buzzegoli, A. Palermo, G. Inghirami, I. Karpenko, 2103.14621
    C. Yi, SP, D.L. Yang, arXiv:2106.00238
- **Global polarization induced by shear and gradient of chemical potential**

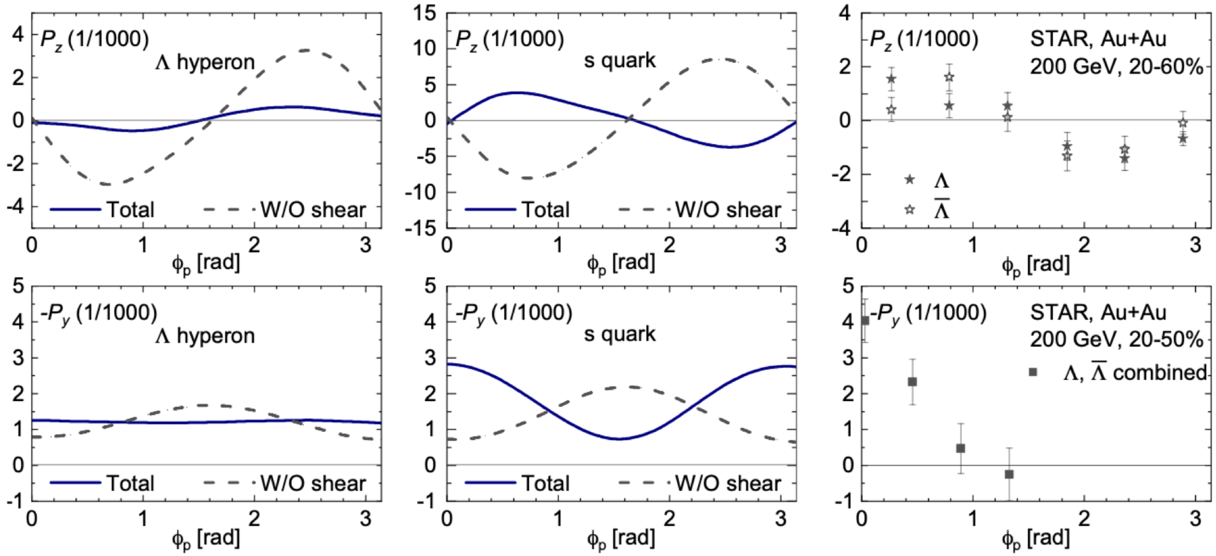
  S. Ryu, V. Jupic, C. Shen, arXiv:2106.08125

s quark scenario



B. Fu, S. Y. F. Liu, L. Pang, H. Song, Y. Yin, 2103.10403

s quark scenario: why it may works?



$$\mathcal{S}^{\mu}_{\rm thermal}(\mathbf{p}) = \frac{\hbar}{8\boxed{m_\Lambda}N}\int d\Sigma^{\sigma}p_{\sigma}f_V^{(0)}(1-f_V^{(0)})\epsilon^{\mu\nu\alpha\beta}p_{\nu}\partial_{\alpha}\frac{u_\beta}{T},$$

$$\mathcal{S}^{\mu}_{\rm shear}(\mathbf{p}) = -\frac{\hbar}{4\boxed{m_\Lambda}N}\int d\Sigma\cdot p f_V^{(0)}(1-f_V^{(0)})\frac{\epsilon^{\mu\nu\alpha\beta}p_{\alpha}u_{\beta}}{\boxed{(u\cdot p)}T}\frac{1}{2}\{p^{\sigma}(\partial_{\sigma}u_{\nu}+\partial_{\nu}u_{\sigma})-Du_{\nu}\}$$

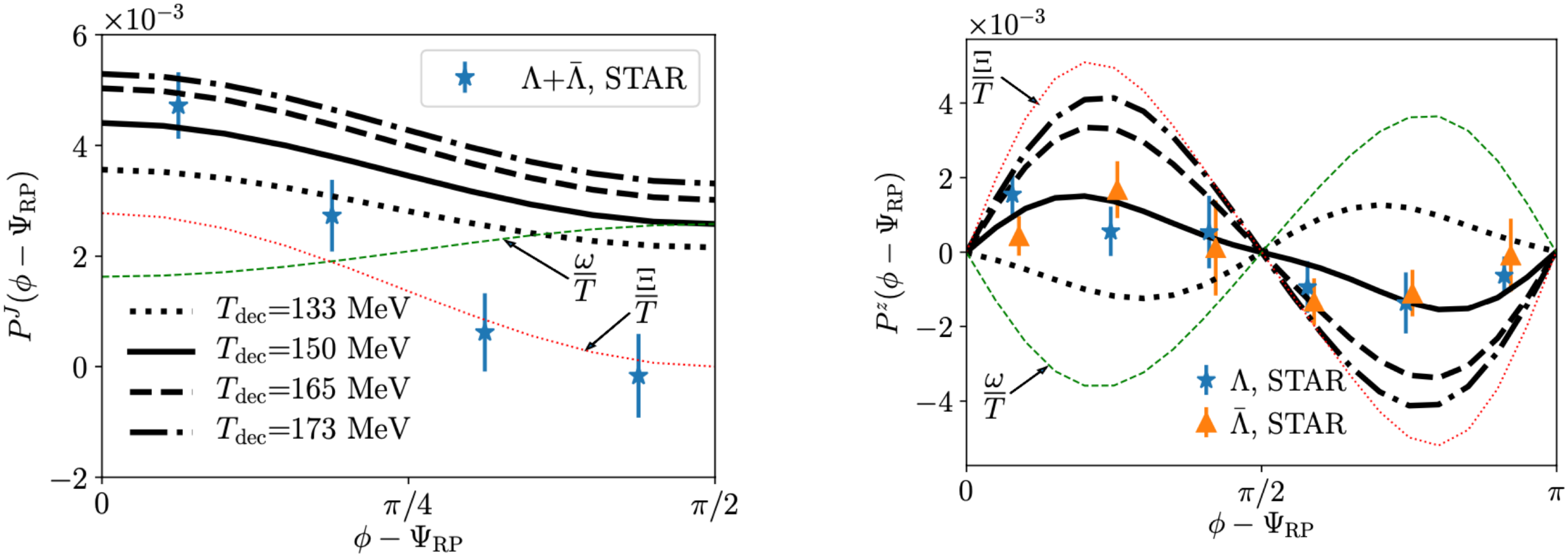
$$(u\cdot p)\sim m$$

$$m_\Lambda \to m_s$$

$$m_s \simeq 0.3\mathrm{GeV}$$

$$m_\Lambda \simeq 1.116\mathrm{GeV}$$

Isothermal local equilibrium



$$S^{\mu}_{\rm ILE}(p) = -\,\epsilon^{\mu\rho\sigma\tau} p_\tau \frac{\int_\Sigma {\rm d}\Sigma\cdot p\, n_F(1-n_F)\left[\omega_{\rho\sigma} + 2\,\hat{t}_\rho \frac{p^\lambda}{\varepsilon}\Xi_{\lambda\sigma}\right]}{8mT_{\rm dec}\int_\Sigma {\rm d}\Sigma\cdot p\, n_F}$$

$$\omega_{\rho\sigma} = \frac{1}{2}\left(\partial_\sigma u_\rho - \partial_\rho u_\sigma\right)$$

All gradient of temperature are neglected!

$$\Xi_{\rho\sigma} = \frac{1}{2}\left(\partial_\sigma u_\rho + \partial_\rho u_\sigma\right)$$

F. Becattini, M. Buzzegoli, A. Palermo, G. Inghirami, I. Karpenko, 2103.14621

Hydrodynamic setup

- **(3+1) dimensional viscous hydrodynamic CLVisc**

  L.G. Pang, H. Petersen, and X.N. Wang, Phys. Rev. C 97, 064918 (2018)

- **AMPT initial conditions**

  Z.W. Lin, C. M. Ko, B.A. Li, B. Zhang, and S. Pal, Phys. Rev. C 72, 064901

- **EoS “sp95-pce”**

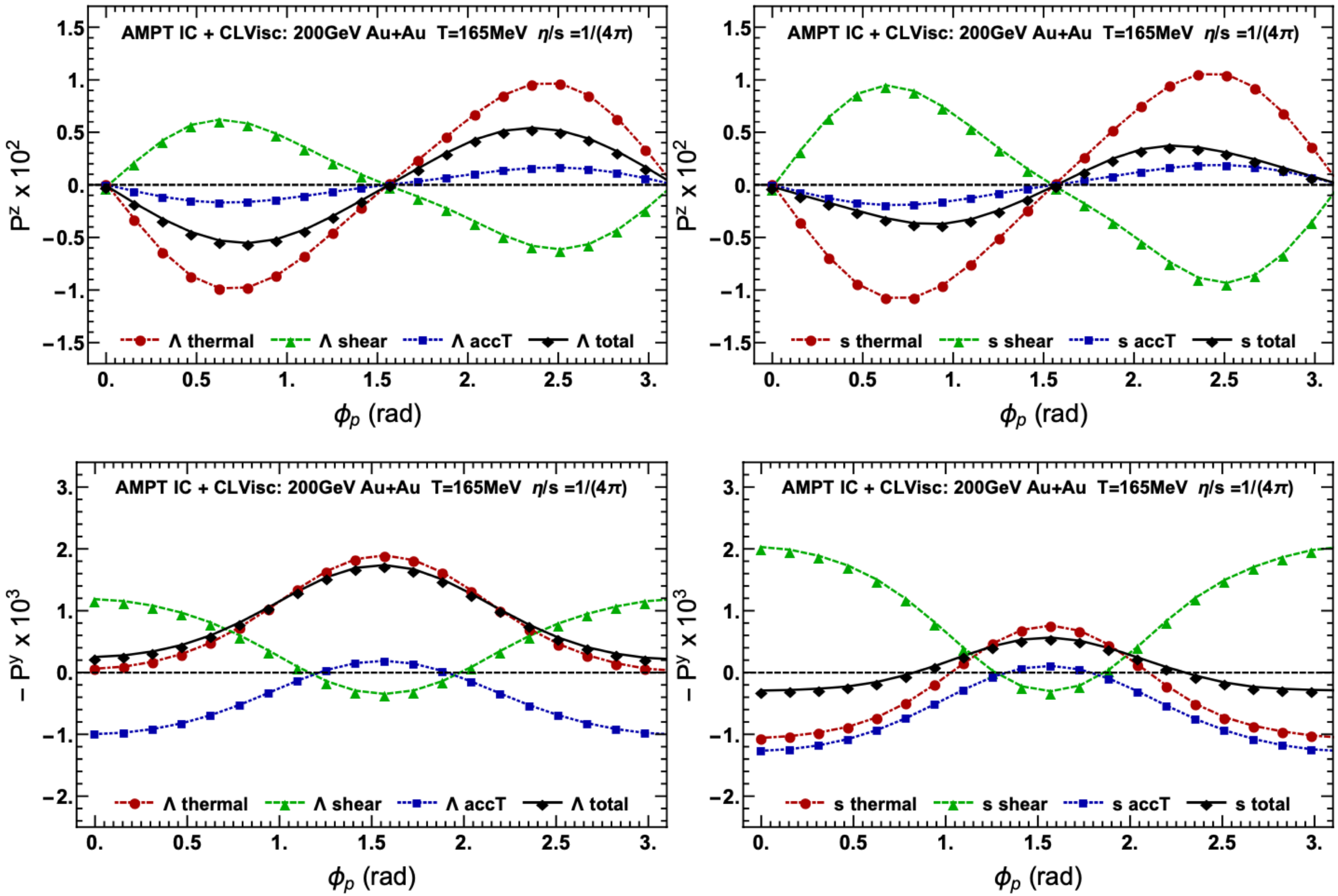
  P. Huovinen and P. Petreczky, Nucl. Phys. A 837, 26 (2010)

- **Two scenarios**

  B. Fu, S. Y. F. Liu, L. Pang, H. Song, and Y. Yin, (2021), 2103.10403

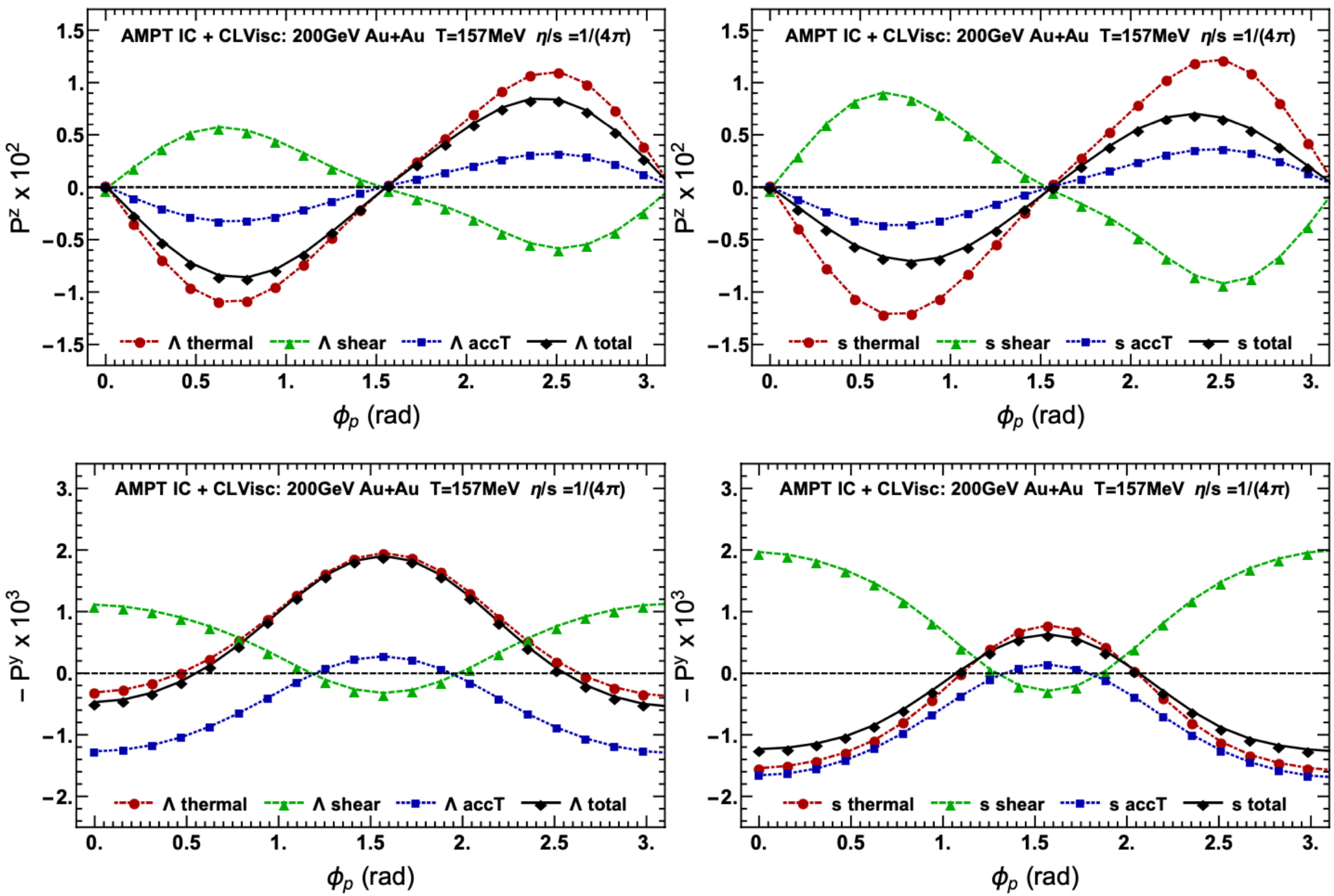
  - **Lambda equilibrium scenario**
  - **s quark equilibrium scenario**

Main result (I)



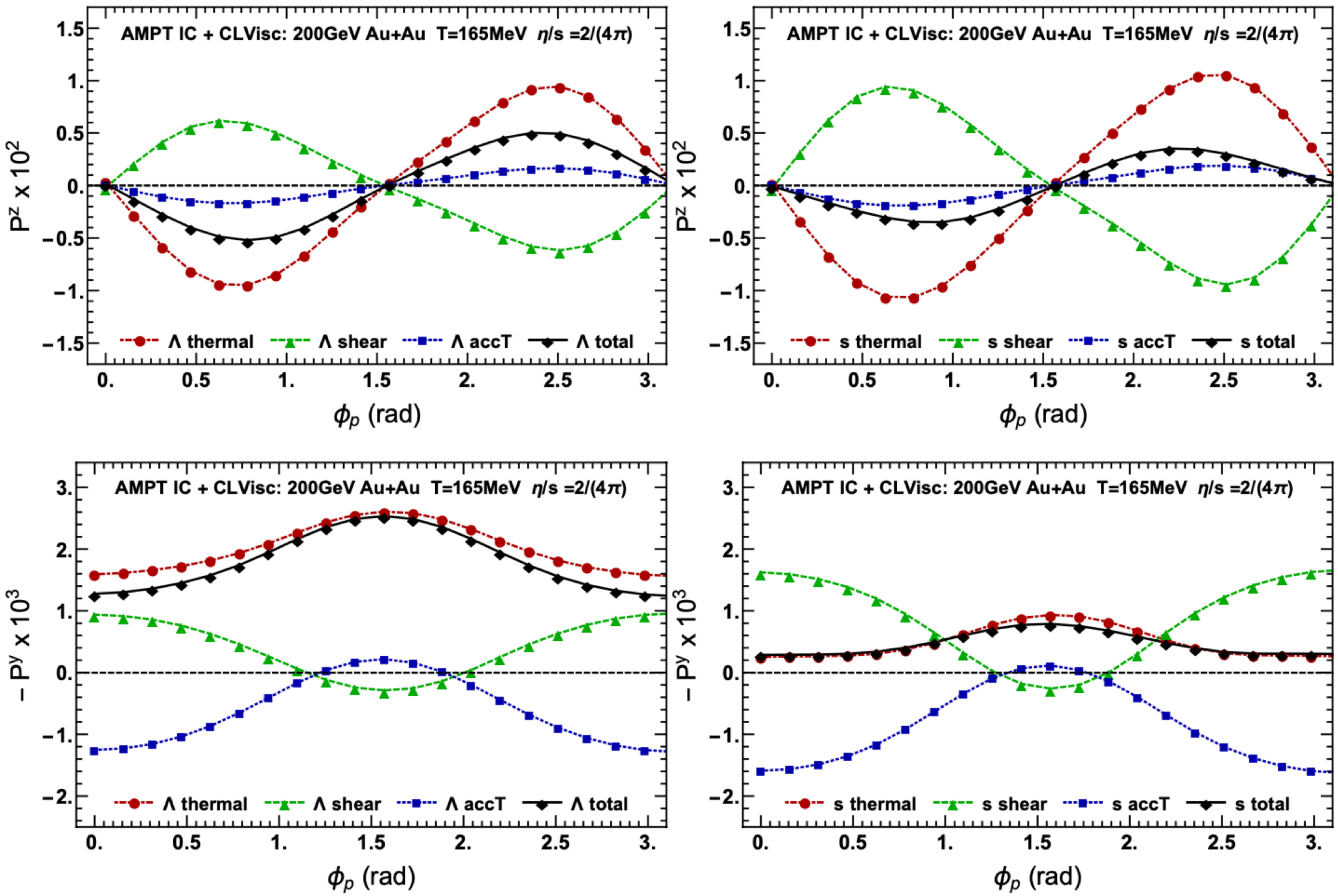
C. Yi, SP, D.L. Yang, arXiv:2106.00238

Reduce the freezeout temperature



C. Yi, SP, D.L. Yang, arXiv:2106.00238

Increase the eta/s



C. Yi, SP, D.L. Yang, arXiv:2106.00238

Conclusion

- Shear induced polarization always give a “correct” sign.
- Total local polarization is very sensitive to EoS, freeze out temperature and eta / s.
- The local spin polarization is still an open question. We still need to consider the out-of-equilibrium effects carefully through the spin hydrodynamics and the quantum kinetic theory with collisions.

Summary

Summary (I)

Strong magnetic fields

- The strongest magnetic fields are generated in HIC. 10^{17}-10^{18}G
- Chiral magnetic effect

$$\mathbf{j} = \frac{e^2}{2\pi^2}\mu_5 \mathbf{B},$$

- Chiral kinetic theory for massless fermion

$$\sqrt{G}\partial_t f + \sqrt{G}\dot{\mathbf{x}} \cdot \nabla_x f + \sqrt{G}\dot{\mathbf{p}} \cdot \nabla_p f = C[f].$$

- Non-trival Lorentz symmetry: side-jump

Summary (II)

Spin hydro in Belinfante form

- We have discussed the Belinfante energy momentum tensor, which is symmetric and gauge invariant.
- We have found the spin corrections to the dissipative terms, including quantum spin vorticity.
- By redefining the entropy flow,
  - we can reproduce the well-known results "in global equilibrium the spin chemical potential is related to thermal vorticity $\omega^{\mu\nu} = -T\omega^{th}_{\mu\nu}/2$."
  - In Local equilibrium, we can rediscover the evolution equations for the spin effects, which is consistent with the one derived in canonical form.

Revisit local spin polarization

- The local spin polarization is still an open question. We still need to consider the out-of-equilibrium effects carefully through the spin hydrodynamics and the quantum kinetic theory with collisions.

Thank you for your time!

Any comments are welcome!

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