The Curious Case of Life on the Lattice

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CERN
Standard Model

Classification of Elementary Particles

**Force Carriers**
- Photon
- $W^\pm$ and $Z^0$
- Gluon

**Quarks**
- Up
- Down
- Charm
- Strange
- Top
- Bottom

**Leptons**
- Electron
- Electron Neutrino
- Muon
- Muon Neutrino
- Tau
- Tua Neutrino

**Higgs**
Quantum Field Theory

Mathematical Formulation of Particle Physics

Quantum field theory = special relativity + quantum mechanics

*Path Integral Formulation of Quantum Field Theory*

Sum over all possible trajectories (field configurations)

\[
Z = \int D\Phi \exp \left[ i \int d^4x \mathcal{L}(\Phi[x]) \right]
\]
Quantum field theory = special relativity + quantum mechanics

Path Integral Formulation of Quantum Field Theory

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- **Partition Function**
- **Sum over all paths**
- **Phase accumulated for specific path**
Quantum field theory = special relativity + quantum mechanics

Path Integral Formulation of Quantum Field Theory
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- Partition Function
- Sum over all paths
- Phase accumulated for specific path

But how do we actually calculate anything?
The Lattice

Definition of “The Lattice”

Numerical Monte-Carlo Finite Volume Discretized Field Theory
**The Lattice**

*Extremely Powerful Toolkit*

**Definition of “The Lattice”**

Numerical Monte-Carlo

- Importance sampling of path integral, using large scale computing resources
Definition of “The Lattice”

Numerical Monte-Carlo  Finite Volume

Boundary effects cannot be ignored and are used to extract infinite volume scattering

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**The Lattice**

*Extremely Powerful Toolkit*

**Definition of “The Lattice”**

- **Numerical Monte-Carlo**
- **Finite Volume**
- **Discretized Field Theory**

Boundary effects cannot be ignored and are used to extract infinite volume scattering.

Importance sampling of path integral, using large scale computing resources.

QFT defined on discretized Euclidean spacetime instead of continuous Minkowski.
Discretized Field Theory

Euclidean Spacetime

*Idea:* Map continuous functions into discrete ones.

$f(x)$  \[ \rightarrow \]  $f(x_n)$

$x$  \[ \rightarrow \]  $x_n$
**Discretized Field Theory**

*Euclidean Spacetime*

**Idea:** Map continuous functions into discrete ones

\[ f(x) \rightarrow f(x_n) \]

**Path Integral Formulation of (Euclidean) QFT**

Sum over all possible discretized trajectories

\[
Z = \int \prod_x d\Phi(x) \exp \left[ -\alpha^4 \sum_x \mathcal{L}(\Phi[x]) \right]
\]

Partition Function  
Sum over all paths  
Accumulated weight
Numerical Monte Carlo

Importance Sampling

**Monte Carlo**: Computational method using random sampling to numerically approximate an observable
**Numerical Monte Carlo**

*Importance Sampling*

**Monte Carlo:** Computational method using random sampling to numerically approximate an observable

*Ex:* Find approximate value of $\pi$ by randomly firing at a square
**Numerical Monte Carlo**

*Evaluating Correlation Functions*

**Ex:** Running of the strong couplings constant, $\alpha_s$, with energy

**Notice:** QCD becomes strongly coupled at low energies

*Lattice is only method for calculating low energy QCD observables from first principles*

**ALPHA collaboration; arXiv:1706.03821**
Numerical Monte Carlo

*Computationally Intense*

**Ex:** Cost of simulating QCD with Wilson fermions in 2001

\[
\text{cost} \approx 2.8 \left[ \frac{\text{#of conf.}}{1000} \right] \left[ \frac{M_\pi/M_\rho}{0.6} \right]^{-6} \left[ \frac{L}{3 \text{fm}} \right] \left[ \frac{a^{-1}}{2 \text{GeV}} \right]^7 \text{Tflops} \cdot \text{year}
\]

Numerical Monte Carlo

Computationally Intense

**Example Simulation**

Physical point, 100 configs, 196x96³ lattice points with a=0.064fm

Cost in 2001: 640 billion core hours
Cost in 2020: 20 million core hours

This dramatic decrease is mainly due to algorithmic advances, not Moore’s Law!

Ex: Cost of simulating QCD with Wilson fermions in 2001 *

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Finite Volume

**Necessary for Simulations**

**Idea:** Dependence of energies at finite volume can be used to extract infinite volume scattering

**Ex:** Extraction of infinite volume $\pi\pi$ scattering phase shift from finite volume energy shifts

*Dudek, Edwards, Thomas; arXiv:1212.0830*
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**Phase Shift**

Dudek, Edwards, Thomas; arXiv:1212.0830
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Outline

1. Introduction: What is “the Lattice”

2. Chiral fermions on the lattice

3. Finite volume effects
Main Messages

*Dual Sides of the Lattice*

**First Main Message**

The lattice provides both a deeper theoretical understanding of quantum field theories and the only way to extract many numbers of experimental relevance.

**Second Main Message**

Life on the lattice is full of complications that must be overcome, but it is incredibly worthwhile to do so.
Fermions on the Lattice
Chiral Symmetry in Standard Model

Property of Fermions

**Chiral Symmetry:** is a mirror image identical to the original?

- **Left Handed:** Spin and momentum are anti-parallel
- **Right Handed:** Spin and momentum are parallel

Handiness is frame independent only for massless fermions

*Chiral symmetry plays in incredibly important role in the Standard Model*
Anomaly

Mystery of the Heavy $\eta'$

**Anomaly:** Classical symmetry broken at the quantum level

**Ex:** Symmetries of the QCD Lagrangian

Two symmetries of the Lagrangian

- $U(1)_V$ counts sum of left handed and right handed fermions
- $U(1)_A$ counts difference between number of left handed and right handed fermions

These two symmetries, plus flavor symmetries, predict 9 light mesons. But we only have 8, as $m_{\eta'} \sim 960$ MeV

**Solution:** $U(1)_A$ is not a symmetry of the quantum theory
Chiral Symmetry in Standard Model

Gauge Theories

Electroweak interactions distinguish left from right

Ex: decay of $^{60}\text{Co}$

Set-Up

1) Align spins of cobalt atoms using strong magnetic field
2) Unstable cobalt decays
3) Measure distribution of emitted electrons and gamma rays
4) Compare two distributions

$^{60}\text{Co} \rightarrow ^{60}\text{Ni} + e^- + \bar{\nu}_e + 2\gamma$

Goal: construct a discretized theory that has the necessary symmetries to faithfully reflect the continuum theory

Wu et al.; Phys. Rev. 105, 1413
No-Go Theorem

Nielson-Ninomiya Theorem

No-Go Theorem: No lattice fermion operator can satisfy all four conditions simultaneously

1. Fourier transform of operator is local

2. At least one Dirac fermion in the continuum limit

3. No more than one Dirac fermion in the continuum limit

4. Full chiral symmetry from the continuum is preserved
No-Go Theorem

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Fermions on the lattice must violate at least one condition
Fermion Doubling Problem

*Nielson-NinomiyTheorem*

**Recall 3rd condition:** no more than one fermion in the continuum

**Continuum Theory**

\[
\mathcal{L} = \int d^4 x \bar{\psi} \phi \psi
\]

Spectrum has single massless fermion

**Discrete Theory**

\[
\mathcal{L} = \sum_n \bar{\psi}_n \gamma_\mu \left( \frac{\psi_{n+\mu} - \psi_{n-\mu}}{2} \right)
\]

Spectrum has 16 massless fermion!

Number of Doublers grows like \(2^d\)
**Ex:** Massless electrons in two dimensions

![Diagram showing initial ground state for right and left moving electrons](image-url)
**Ex:** Massless electrons in two dimensions

- **Initial Ground State**
- **Final Ground State (Infinite Dirac Sea)**

- **Right Moving**
- **Left Moving**

Electric field on, off
**Anomalies and the Dirac Sea**

**Ex:** Massless electrons in two dimensions

- **Initial Ground State**
  - Right Moving
  - Left Moving

- **Final Ground State** (Infinite Dirac Sea)
  - \(U(1)_V\) preserved
  - \(U(1)_A\) violated

**Electric field on, off**
Ex: Massless electrons in two dimensions

Initial Ground State

Electric field on, off

Final Ground State
(Finite Dirac Sea)
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Initial Ground State

Electric field on, off

Final Ground State (Finite Dirac Sea)

- $U(1)_V$ preserved
- $U(1)_A$ preserved

Anomalies require infinite number of degrees of freedom

The lattice always has a finite number of degrees of freedom
**Domain Wall Fermions**

*Extra Dimension*

**Idea:** Utilize an extra dimension to localize light fermion modes on the boundaries

\[
\mathcal{L}_F = \bar{\Psi}(x, s) \left[ \gamma^\mu D_\mu(x) + \Gamma \partial_s + m(s) \right] \Psi(x, s)
\]

Kaplan, arXiv: 0912.2560
Domain Wall Fermions

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U(1)$_A$ is broken *explicitly* by particles flowing between walls

*Kaplan, arXiv: 0912.2560*
Flowed Domain Wall Fermions

*Chiral Gauge Theories on the Lattice*

**Idea:** Decouple unwanted chiral modes by turning off gauge field in gauge invariant way

\[ \partial_s A_\mu(x, s) = \tau \text{sgn}(s) D_\nu F_{\mu\nu} \]

*DMG + Kaplan arXiv: 1610.02151, 1511.03649*
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Important open questions remain

*DMG + Kaplan arXiv: 1610.02151, 1511.03649
Numerical Limitations

Finite Volume Effects
QFT Picture for Scattering

Infinite Volume

Asymptotically Free Initial Single Particle State

Asymptotically Free Final Single Particle State

Time Dependent Phenomena
Scattering Experiments

\( J/\psi \)-pair mass spectrum

LHCb, arXiv: 2006.16957

Extraction of scattering information from experimental data requires an assumption about the spectral function form

Broad structure next to threshold

Narrow Resonance

Hint for another structure

\[ p_T^{\text{di-}\psi} > 5.2 \text{ GeV}/c \]

\[ 3.065 < M_{\mu\mu} < 3.135 \text{ GeV}/c^2 \]

\[ 3.00 < M_{\mu\mu} < 3.05 \text{ GeV}/c^2 \text{ or } 3.15 < M_{\mu\mu} < 3.20 \text{ GeV}/c^2, \text{ normalised} \]
Scattering With Boundary Conditions

**Toy Model:** elastic scattering of two identical spinless bosons in 1D in non-relativistic quantum mechanics

finite range potential

Infinite Volume
Scattering With Boundary Conditions

**Toy Model:** elastic scattering of two identical spinless bosons in 1D in non-relativistic quantum mechanics

finite range potential

\[ \psi(x) \sim \cos(p|x| + \delta(p)) \]

**Infinite Volume:** No constraint on \( p \)

\( p \in \mathbb{R} \)
Scattering With Boundary Conditions

**Toy Model:** elastic scattering of two identical spinless bosons in 1D in non-relativistic quantum mechanics

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**Toy Model:** elastic scattering of two identical spinless bosons in 1D in non-relativistic quantum mechanics

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**Finite Volume:** Allowed momenta depend on both infinite volume scattering phase shift and finite volume size

\[ \psi(x) = \psi(x + L) \quad \Rightarrow \quad p = \frac{2\pi}{L} n + \frac{2\delta(p)}{L} \]
Quantization Condition

Two to Two

Idea: Energy levels of particles in a box map to infinite volume scattering amplitude

\[ E_2(L) \quad E_1(L) \quad E_0(L) \]

Solution for Two to Two: Discrete energy levels satisfy the Luscher quantization condition

\[
\det \left[ 1 + \mathcal{M}_2 F_2 \right] = 0
\]

Lattice Energy Levels

Must solve quantization condition to extract scattering information

(Usually numerically)

Note that $M_n$ and $F_n$ are infinite dimensional matrices

*DMG + Hansen, to appear
Lattice Energy Levels

Must solve quantization condition to extract scattering information

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Note that $\mathcal{M}_n$ and $F_n$ are infinite dimensional matrices

Current work

Solve quantization condition to have analytic control over energy splitting

• Two to Two and Three to Three
• Center of Mass and Moving Frame

Energy levels

Solution


DMG + Hansen, to appear
Conclusions

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