HAL QCD method for hadron interactions on the lattice

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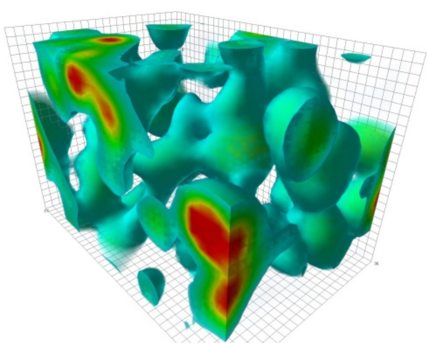
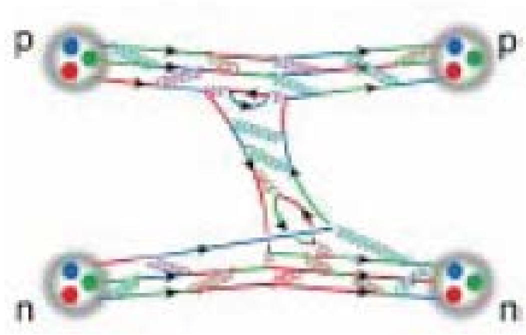
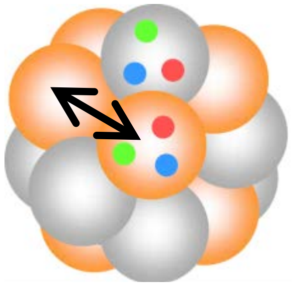
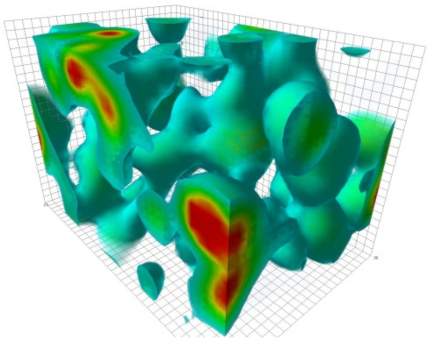
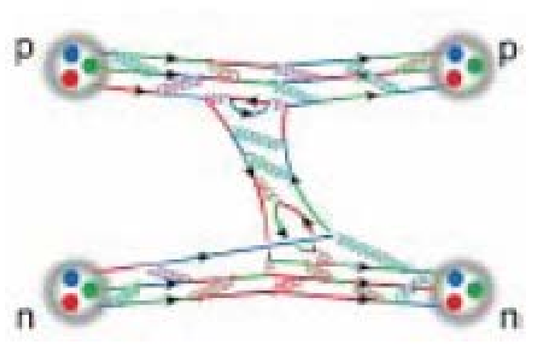
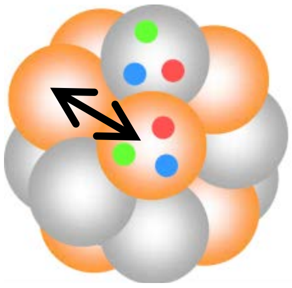






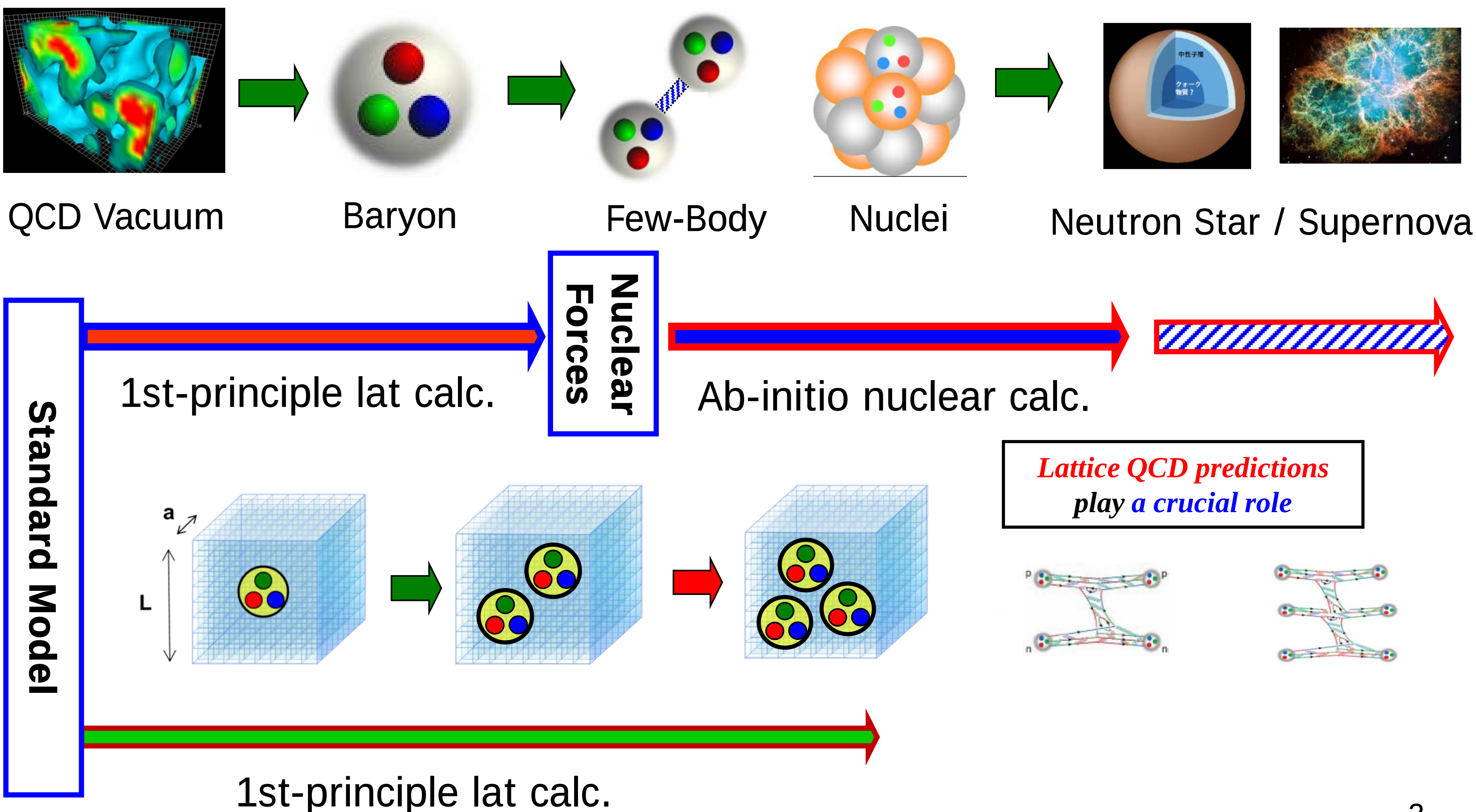


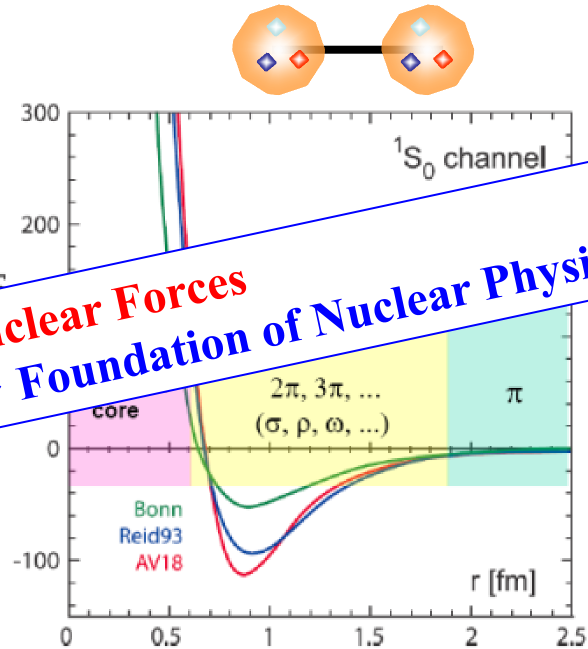




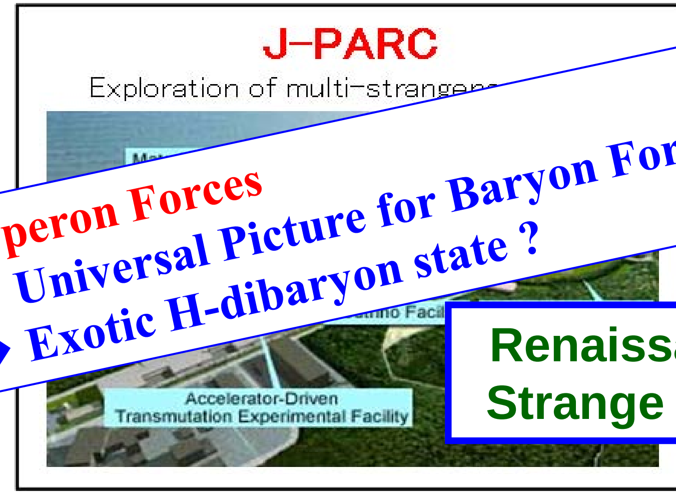
- **<u>Outline</u>**
  - Introduction
  - Theoretical framework of HAL QCD method
    - Challenges in multi-baryons
    - NBS wave functions & E-indep “potential”
  - Numerical results
  - Prospects toward physical point

Motivation:
Nuclear Physics and Astrophysics from Lat QCD





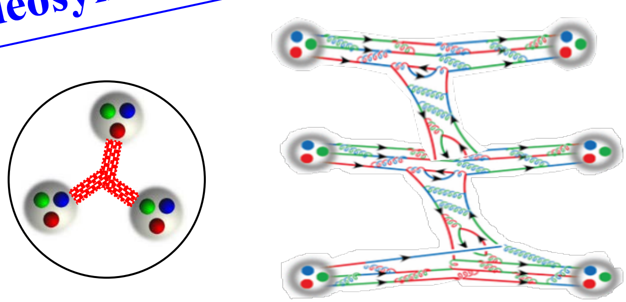
Nuclear Forces
➔ **Foundation of Nuclear Physics**



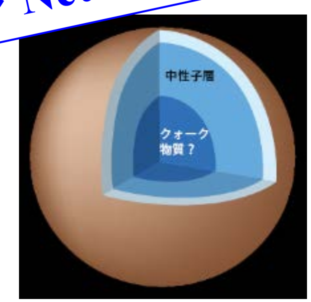
Hyperon Forces
➔ **Universal Picture for Baryon Forces**
➔ **Exotic H-dibaryon state ?**

Renaissance in Strange World !

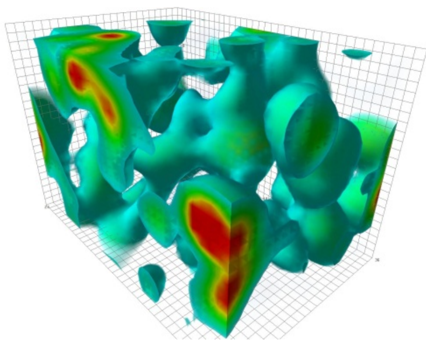
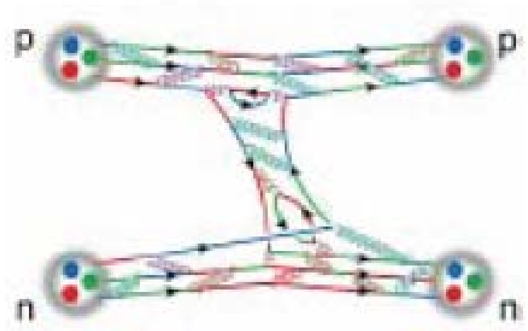
Three-Baryon Forces
➔ **Paradigm Shift in Unstable Nuclei ?**
➔ **Nucleosynthesis**

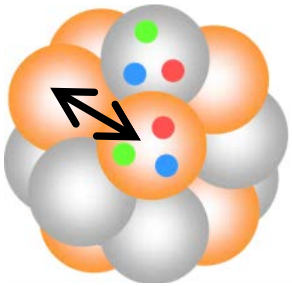


EoS of Dense Matter
➔ **Neutron Star**



Exotics w/ charm ➔ Y. Ikeda's talk



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Interactions on the Lattice

- Luscher's method (& Light nuclei)
  - Phase shift & B.E. from temporal correlation in finite V
  - (Probe interactions in indirect way)

M.Luscher, CMP104(1986)177
CMP105(1986)153
NPB354(1991)531

- HAL QCD method
  - "Potential" from spacial (& temporal) correlation
  - (Probe interactions in direct way)
  - Phase shift & B.E by solving Schrodinger eq in infinite V

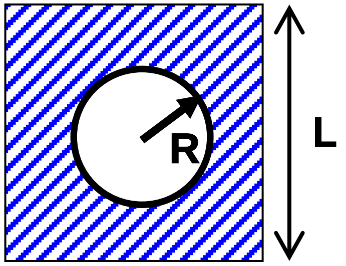
Ishii-Aoki-Hatsuda, PRL99(2007)022001, PTP123(2010)89
HAL QCD Coll., PTEP2012(2012)01A105

Luscher's formula: Scatterings on the lattice

- Consider Schrodinger eq at asymptotic region

$$(\nabla^2 + k^2)\psi_k(\boldsymbol{r}) = mV_k(\boldsymbol{r})\psi_k(\boldsymbol{r})$$

$$V_k(\boldsymbol{r}) = 0 \text{ for } r > R$$



  - (periodic) Boundary Condition in finite V ➔ constraint on energies of the system
  - Energy E ⟷ phase shift (at E)

$$k \cot \delta_{\mathbf{E}} = \frac{2}{\sqrt{\pi}L} Z_{00}(1; q^2), \quad q = \frac{kL}{2\pi}, \quad E = 2\sqrt{m^2 + k^2}$$

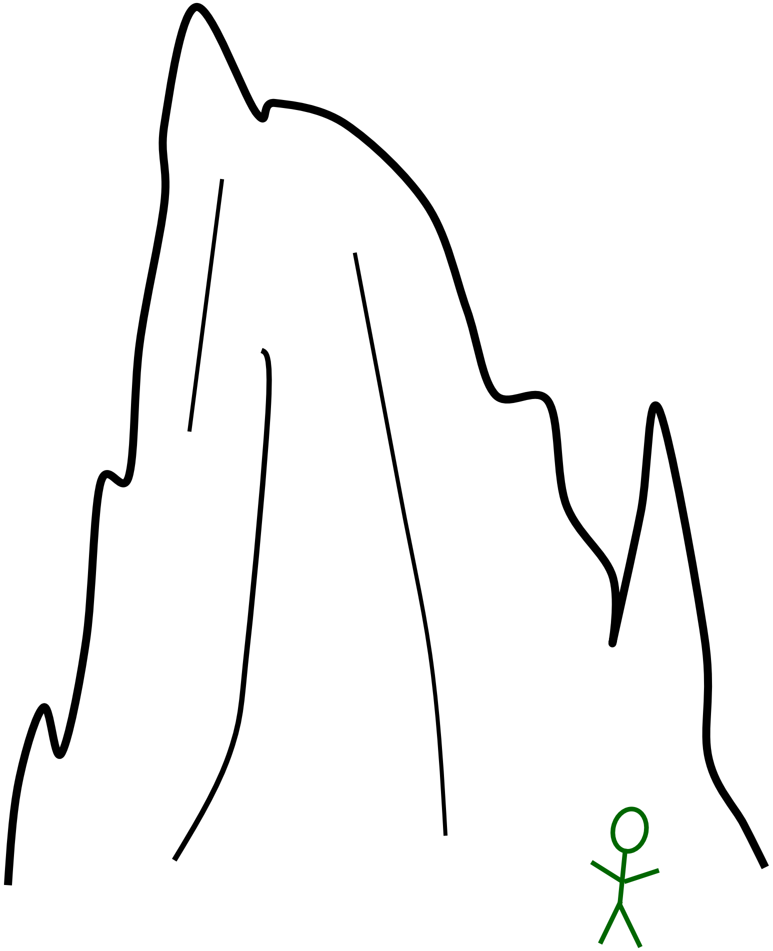
Large V: $$\Delta E = E - 2m = -\frac{4\pi \mathbf{a}}{mL^3}\left[1 + c_1 \frac{a}{L} + c_2 \left(\frac{a}{L}\right)^2 + \mathcal{O}(\frac{1}{L^3})\right]$$

- Calculate the energy spectrum of NN on (finite V) lattice
  - Temporal correlation in Euclidean time ➔ energy

$$G(t) = \langle 0|\mathcal{O}(t)\overline{\mathcal{O}}(0)|0|\rangle = \sum_n A_n e^{-E_n t} \to A_0 e^{-E_0 t} \quad (t \to \infty)$$

The Challenges

- Signal / Noise issue



Challenges in multi-baryons on the lattice

- Signal / Noise estimate

Parisi, Lepage(1989)

In usual LQCD calc.,
G.S. saturation is necessary by t ➔ ∞

– pion

$$\frac{\text{Signal}}{\text{Noise}} \sim \frac{\exp(-m_\pi t)}{\sqrt{\exp(-2m_\pi t)}} \sim \boxed{\text{const.}}$$

– nucleon

$$\frac{\text{Signal}}{\text{Noise}} \sim \frac{\exp(-m_N t)}{\sqrt{\exp(-3m_\pi t)}} \sim \boxed{\exp[-(m_N - 3/2 m_\pi)t]}$$

$$\frac{\text{Signal}}{\text{Noise}} \sim \boxed{\exp[-\mathbf{A}(m_N - 3/2 m_\pi)t]}$$ (for mass number = A)

Challenges in multi-baryons on the lattice

- Signal / Noise estimate

Parisi, Lepage(1989)

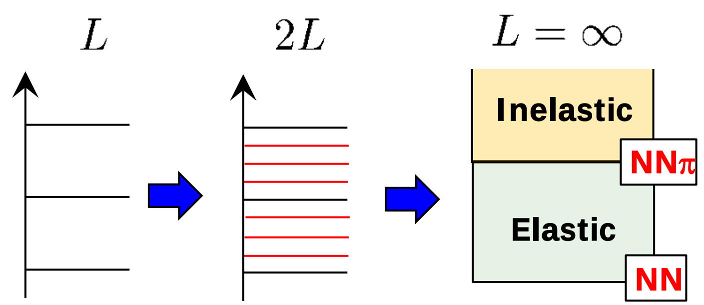
  - S/N gets worse
    for larger mass number A & light quark mass & $t \to \infty$

$$S/N \sim \exp[-\mathbf{A} \times (\mathbf{m_N} - \mathbf{3/2m_\pi}) \times \mathbf{t}]$$

Variational method ?

  - Larger spectral density
    ➔ larger t required

$$\Delta E \simeq \frac{\vec{p}^2}{m_N} \simeq 15\mathrm{MeV} \quad \text{for } L = 10\mathrm{fm}$$
$$\longrightarrow t \gg 100 \times (0.1\mathrm{fm}/a)$$



G.S. saturation becomes more and more difficult for larger V & lighter mass

$$S/N \sim 10^{-42} \ !?$$

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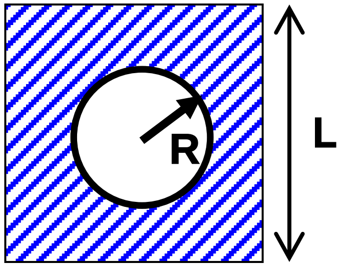


  – (periodic) Boundary Condition in finite V ➔ constraint on energies of the system

  – Energy E ←➔ phase shift (at E)

$$k \cot \delta_{\mathbf{E}} = \frac{2}{\sqrt{\pi} L} Z_{00}(1; q^2), \quad q = \frac{kL}{2\pi}, \quad E = 2\sqrt{m^2 + k^2}$$

Large V: $\Delta E = E - 2m = -\frac{4\pi \mathbf{a}}{mL^3}\left[1 + c_1 \frac{a}{L} + c_2 \left(\frac{a}{L}\right)^2 + \mathcal{O}(\frac{1}{L^3})\right]$

G.S. Saturation required

- Calculate the energy spectrum of (finite V) lattice
  – Temporal correlation in Euclidean time ➔ energy

$$G(t) = \langle 0|\mathcal{O}(t)\overline{\mathcal{O}}(0)|0|\rangle = \sum_n A_n e^{-E_n t} \to A_0 e^{-E_0 t} \quad (t \to \infty)$$

Further Challenges in multi-baryons on Lat

Is energy the only quantity from which we can extract interactions ?

- Phase shift:
  - Obtained at only one (or several) energies
    (➔ we have to repeat the calc w/ different V, etc.)

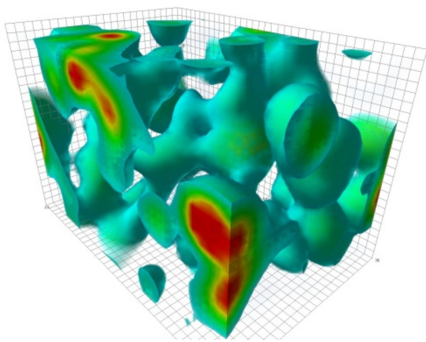
- Coupled channel
  - Unknown: 2 phases shifts + 1 mixing parameter (for 2x2)
    Obtained on Lat: 2 energies
  - ➔ "Parametrize" E-dependence

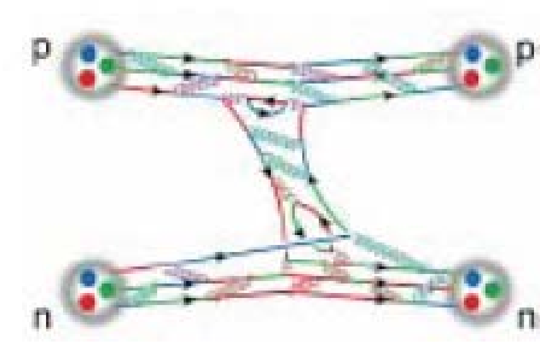
S. He et al., JHEP07(2005)011
Hansen-Sharpe, PRD86(2012)016007
J. Dudek et al., PRL113(2014)182001

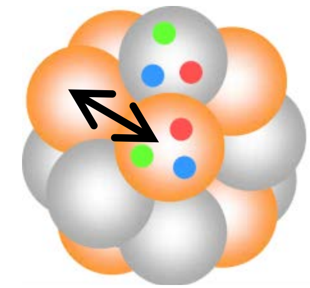
- Many-body forces
  - Embedded in E in inefficient way
  - 2-body: ~ $1/L^3$, 3-body: ~$1/L^6$, 4-body: ~$1/L^9$

$$\Delta E = E - 2m = -\frac{4\pi \mathbf{a}}{mL^3}\left[1 + c_1\frac{a}{L} + c_2\left(\frac{a}{L}\right)^2 + \mathcal{O}(\frac{1}{L^3})\right]$$ (for 2-body)

(the situation may be better for bound states)







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Hadrons to Atomic nuclei from Lattice QCD
(HAL QCD Collaboration)

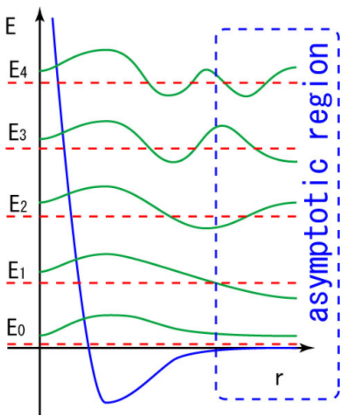
S. Aoki, T. Iritani (YITP)
B. Charron (Univ. of Tokyo)
T. Doi, T.Hatsuda, Y. Ikeda, V. Krejcirik (RIKEN)
F. Etminan (Univ. of Birjand)
T. Inoue (Nihon Univ.)
N. Ishii, K. Murano (RCNP)
T. Miyamato, H. Nemura, K. Sasaki, M. Yamada (Univ. of Tsukuba)

HAL QCD method for 2-body elastic scatt.

- Potential is constructed so as to reproduce the NN phase shifts (or, S-matrix)

- Nambu-Bethe-Salpeter (NBS) wave function

$$\psi(\vec{r}) = \langle 0 | N(\vec{x}+\vec{r}) N(\vec{x}) | 2N, W \rangle$$

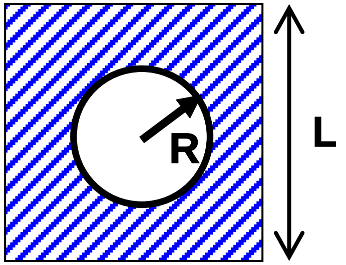


$$W = 2\sqrt{m^2 + k^2}$$

$$(\nabla^2 + k^2)\psi(\vec{r}) = 0, \quad r > R$$

  – Wave function ←→ phase shifts

$$\psi(r) \simeq A \frac{\sin(kr - l\pi/2 + \delta(k))}{kr}$$



M.Luscher, NPB354(1991)531
C.-J.Lin et al., NPB619(2001)467
CP-PACS Coll., PRD71(2005)094504
Ishizuka, Pos LAT2009 (2009) 119
Aoki-Hatsuda-Ishii PTP123(2010)89
S.Aoki et al., PRD88(2013)014036

Asymptotic form of BS wave function

[C.-J.D.Lin et al., NPB619,467(2001)]

For simplicity, we consider BS wave function of two pions

$$\psi_{\vec{q}}(\vec{x}) \equiv \left\langle 0 \middle| N(\vec{x})\, N(\vec{0}) \middle| N(\vec{q}) N(-\vec{q}), in \right\rangle$$

complete set

$$1 = \int \frac{d^3 p}{(2\pi)^3 2E_N(\vec{p})} |N(\vec{p})\rangle\langle N(\vec{p})| + \cdots$$

$$= \int \frac{d^3 p}{(2\pi)^3 2E_N(\vec{p})} \langle 0 | N(\vec{x}) | N(\vec{p}) \rangle \left\langle N(\vec{p}) \middle| N(\vec{0}) \middle| N(\vec{q}) N(-\vec{q}), in \right\rangle + I(\vec{x})$$

$$Z^{1/2} e^{i\vec{p}\cdot\vec{x}}$$

$$disc. + Z^{1/2} \frac{T(\vec{p};\vec{q})}{m_N^2 - (2E_N(\vec{q}) - E_N(\vec{p}))^2 + \vec{p}^2 - i\varepsilon}$$

$$= Z\left(e^{i\vec{q}\cdot\vec{x}} + \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2E_N(\vec{p})} \frac{T(\vec{p};\vec{q})}{4E_N(\vec{q}) \cdot (E_N(\vec{p}) - E_N(\vec{q}) - i\varepsilon)} e^{i\vec{p}\cdot\vec{x}} \right)$$

Integral is dominated by the on-shell contribution $E_N(\vec{p}) \approx E_N(\vec{q})$

➔ T-matrix becomes the on-shell T-matrix

$$T^{(s-wave)}(s) = \frac{E(\vec{q})}{2|\vec{q}|}(-i)\left(e^{2i\delta_0(s)} - 1\right)$$

$$= Z\left(e^{i\vec{q}\cdot\vec{x}} + \frac{1}{2i}\left(e^{2i\delta_0(s)} - 1\right)\frac{e^{iqr}}{qr} \right) + \cdots$$

The asymptotic form

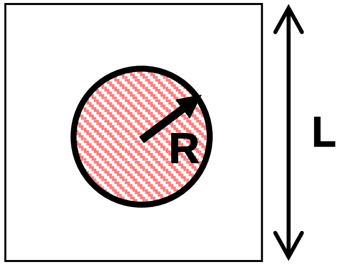
$$\boxed{\psi_{\vec{q}}(\vec{x}) = Z e^{i\delta_0(s)} \frac{\sin(qr + \delta_0(s))}{qr} + \cdots \quad \text{(s-wave)}}$$

This is analogous to a non-rela. wave function

"Potential" as a representation of S-matrix

- Consider the wave function at "interacting region"

$$(\nabla^2 + k^2)\psi(\boldsymbol{r}) = m \int d\boldsymbol{r'} U(\boldsymbol{r}, \boldsymbol{r'})\psi(\boldsymbol{r'}), \quad r < R$$



Probe interactions in "direct" way

  – U(r,r'): faithful to the phase shift by construction
    - U(r,r'): NOT an observable, but well defined
    - U(r,r'): E-independent, while non-local in general

Proof of Existence of E-independent potential

$$V_W(\boldsymbol{r})\psi_W(\boldsymbol{r}) = (E_W - H_0)\psi_W(\boldsymbol{r})$$ **[START] local but E-dep pot. (L³xL³ dof)**

- We consider the linear-indep wave functions and define

$$\mathcal{N}_{W_1 W_2} = \int d\boldsymbol{r} \overline{\psi_{W_1}(\boldsymbol{r})} \psi_{W_2}(\boldsymbol{r})$$

- We define the non-local potential

$$U(\boldsymbol{r}, \boldsymbol{r}') = \sum_{W_1, W_2}^{W_{\rm th}} (E_{W_1} - H_0)\psi_{W_1}(\boldsymbol{r}) \mathcal{N}^{-1}_{W_1 W_2} \overline{\psi_{W_2}(\boldsymbol{r}')}$$

- The above potential trivially satisfy Schrodinger eq.

$$\begin{aligned}
\int d\boldsymbol{r}' U(\boldsymbol{r}, \boldsymbol{r}')\psi_W(\boldsymbol{r}') &= \int d\boldsymbol{r}' \sum_{W_1, W_2}^{W_{\rm th}} (E_{W_1} - H_0)\psi_{W_1}(\boldsymbol{r}) \mathcal{N}^{-1}_{W_1 W_2} \overline{\psi_{W_2}(\boldsymbol{r}')} \psi_W(\boldsymbol{r}') \\
&= \sum_{W_1, W_2}^{W_{\rm th}} (E_{W_1} - H_0)\psi_{W_1}(\boldsymbol{r}) \mathcal{N}^{-1}_{W_1 W_2} \mathcal{N}_{W_2 W} \\
&= (E_W - H_0)\psi_W(\boldsymbol{r})
\end{aligned}$$

Intuitive understanding

[GOAL] non-local but E-indep pot. (L³xL³ dof)

c.f. Krolikowski-Rzewuski, Nuovo Cimento, 4, 1212 (1956)

"Potential" as a representation of S-matrix

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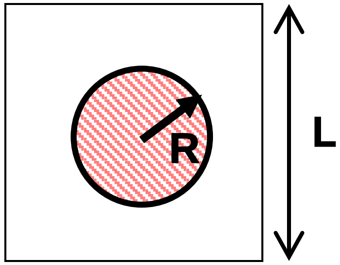


Probe interactions in "direct" way

- U(r,r'): faithful to the phase shift by construction
  - U(r,r'): NOT an observable, but well defined
  - U(r,r'): E-independent, while non-local in general
- Phase shifts at all E (below inelastic threshold) obtained by solving Scrodinger eq in infinite V

  - Non-locality ➔ derivative expansion   Okubo-Marshak(1958)

$$U(\vec{r},\vec{r}') = \underbrace{V_c(r)}_{\text{LO}} + \underbrace{S_{12}V_T(r)}_{\text{LO}} + \underbrace{\vec{L}\cdot\vec{S}V_{LS}(r)}_{\text{NLO}} + \underbrace{\mathcal{O}(\nabla^2)}_{\text{NNLO}}$$

Check on convergence: K.Murano et al., PTP125(2011)1225

Control the E-dependence of phase shifts

Taming of S/N issue w/ E-indep potential

- Original (t-indep) HAL QCD method

$$R(\boldsymbol{r},t) \equiv C_{NN}(\boldsymbol{r},t)/C_N(t)^2 = \sum_i A_{W_i}\psi_{W_i}(\boldsymbol{r})e^{-(W_i-2m)t}$$

$$\begin{aligned}
\int d\boldsymbol{r}' U(\boldsymbol{r},\boldsymbol{r}')\psi_{W_0}(\boldsymbol{r}') &= (E_{W_0} - H_0)\psi_{W_0}(\boldsymbol{r}) \\
\int d\boldsymbol{r}' U(\boldsymbol{r},\boldsymbol{r}')\psi_{W_1}(\boldsymbol{r}') &= (E_{W_1} - H_0)\psi_{W_1}(\boldsymbol{r}) \\
\int d\boldsymbol{r}' U(\boldsymbol{r},\boldsymbol{r}')\psi_{W_2}(\boldsymbol{r}') &= (E_{W_2} - H_0)\psi_{W_2}(\boldsymbol{r})
\end{aligned}$$

G.S. saturation necessary

- New (t-dep) HAL QCD method
  - All equations can be combined in

$$\int d\boldsymbol{r}' U(\boldsymbol{r},\boldsymbol{r}')R(\boldsymbol{r}',t) = \left(-\frac{\partial}{\partial t} + \frac{1}{4m}\frac{\partial^2}{\partial t^2} - H_0\right)R(\boldsymbol{r},t)$$

Relativistic correction

G.S. saturation in R(r,t) NOT necessary

Extract the signal from excited states

N.Ishii et al. (HAL QCD Coll.) PLB712(2012)437

E-indep of potential U(r,r') ➔ *(excited) scatt states share the same U(r,r')*

*They are **not contaminations**, **but signals***

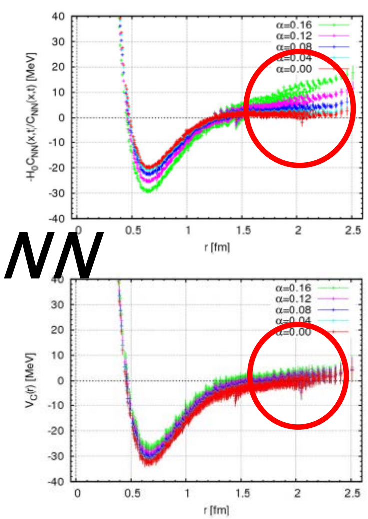
➔ **Time-dependent Schrodinger Eq.**

$$2\sqrt{m^2+k_n^2} = E_n = -\frac{\partial}{\partial_t}$$

$$\left(-\frac{\partial}{\partial t} + \frac{1}{4m}\frac{\partial^2}{\partial t^2} - H_0\right) R(\boldsymbol{r},t) = \int d\boldsymbol{r}' U(\boldsymbol{r},\boldsymbol{r}') R(\boldsymbol{r}',t) \qquad R(\boldsymbol{r},t) \equiv C_{NN}(\boldsymbol{r},t)/C_N(t)^2$$

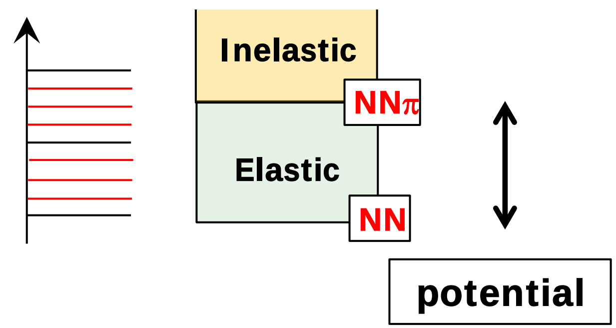
Ground State (G.S.) saturation is NOT necessary !

NN



[OLD] "contaminations" from excited states

[NEW] "signals" from excited states



N.B. Inelastic scattering suppression still necessary

Coupled Channel

(beyond inelastic threshold)

- Asymptotic behavior of NBS wave func

Ex.) A + B ⇔ C + D

$$\psi_{AB}(\boldsymbol{r},\boldsymbol{k}) = 1/\sqrt{Z_A Z_B}\cdot\langle 0|\phi_A(\boldsymbol{x}+\boldsymbol{r})\phi_B(\boldsymbol{x})|W\rangle$$
$$\psi_{CD}(\boldsymbol{r},\boldsymbol{q}) = 1/\sqrt{Z_C Z_D}\cdot\langle 0|\phi_C(\boldsymbol{x}+\boldsymbol{r})\phi_D(\boldsymbol{x})|W\rangle$$

$$|W\rangle = c_{AB}|AB,W\rangle_{\text{in}} + c_{CD}|CD,W\rangle_{\text{in}}$$

$$W = \sqrt{m_A^2+k^2}+\sqrt{m_B^2+k^2} = \sqrt{m_C^2+q^2}+\sqrt{m_D^2+q^2}$$

$$\psi^l_{AB}(r,k) = c_{AB}\left[j_l(kr)+\frac{k}{4\pi}H_l^{AB,AB}(k,k)(n_l(kr)+ij_l(kr))\right]+c_{CD}\left[\frac{k}{4\pi}H_l^{AB,CD}(k,q)(n_l(kr)+ij_l(kr))\right]$$
$$\psi^l_{CD}(r,q) = c_{CD}\left[j_l(qr)+\frac{q}{4\pi}H_l^{CD,CD}(q,q)(n_l(qr)+ij_l(qr))\right]+c_{AB}\left[\frac{q}{4\pi}H_l^{CD,AB}(q,k)(n_l(qr)+ij_l(qr))\right]$$

where

$$H^{AB,AB(CD)}(\boldsymbol{k};\boldsymbol{k}(\boldsymbol{q})) = \frac{1}{2W}T^{AB,AB(CD)}(k_A,k_B;k_A,k_B(q_C,q_D))$$
$$H^{CD,AB(CD)}(\boldsymbol{q};\boldsymbol{k}(\boldsymbol{q})) = \frac{1}{2W}T^{CD,AB(CD)}(q_C,q_D;k_A,k_B(q_C,q_D))$$

S.Aoki et al. (HAL Coll.), Proc. Jpn. Acad. Ser. B87(2011)

Coupled Channel

- T-matrix parametrization by unitarity

$$T_l^{I,J}(W) = \frac{8\pi W}{p_I}\left[O(W)\begin{pmatrix} \underline{e^{i\delta_l^1(W)}\sin\delta_l^1(W)} & 0 \\ 0 & \underline{e^{i\delta_l^2(W)}\sin\delta_l^2(W)} \end{pmatrix} O^{-1}(W)\right]^{I,J}$$

$$O(W) = \underline{\begin{pmatrix} \cos\theta(W) & -\sin\theta(W) \\ \sin\theta(W) & \cos\theta(W) \end{pmatrix}} \qquad (p_1 = k, p_2 = q)$$

- Asymptotic behavior ←→ $\delta_l^1(W), \delta_l^2(W), \theta(W)$

$$\begin{pmatrix} \psi_{AB}(r,k) \\ \psi_{CD}(r,q) \end{pmatrix} \simeq \begin{pmatrix} j_l(kr) & 0 \\ 0 & j_l(qr) \end{pmatrix} \begin{pmatrix} c_{AB} \\ c_{CD} \end{pmatrix}$$

$$+ \begin{pmatrix} n_l(kr) + i j_l(kr) & 0 \\ 0 & n_l(qr) + i j_l(qr) \end{pmatrix} O(W) \begin{pmatrix} e^{i\delta_l^1(W)}\sin\delta_l^1(W) & 0 \\ 0 & e^{i\delta_l^2(W)}\sin\delta_l^2(W) \end{pmatrix} O^{-1}(W) \begin{pmatrix} c_{AB} \\ c_{CD} \end{pmatrix}$$

- → Coupled channel potentials can be defined

$$(E_{k_i}^{AB} - H_0^{AB})\psi_{AB}(\boldsymbol{r}, k_i) = \int d\boldsymbol{r}' U_{AB,AB}(\boldsymbol{r}, \boldsymbol{r}')\psi_{AB}(\boldsymbol{r}', k_i) + \int d\boldsymbol{r}' U_{AB,CD}(\boldsymbol{r}, \boldsymbol{r}')\psi_{CD}(\boldsymbol{r}', q_i)$$

$$(E_{q_i}^{CD} - H_0^{CD})\psi_{CD}(\boldsymbol{r}, q_i) = \int d\boldsymbol{r}' U_{CD,AB}(\boldsymbol{r}, \boldsymbol{r}')\psi_{AB}(\boldsymbol{r}', k_i) + \int d\boldsymbol{r}' U_{CD,CD}(\boldsymbol{r}, \boldsymbol{r}')\psi_{CD}(\boldsymbol{r}', q_i)$$

Coupled Channel

S.Aoki et al. (HAL Coll.), PRD87(2013)034512

- Proof of Existence of E-indep potential

W_{th}^2

$\Delta_1 = [W_{th}^1, W_{th}^2)$

$W_{th}^1 = m_C + m_D$

$\Delta_0 = [W_{th}^0, W_{th}^1)$

$W_{th}^0 = m_A + m_B$

NBS wave func.

$$\begin{aligned}
\psi_{AB,AB}(\boldsymbol{r}) &= 1/\sqrt{Z_A Z_B}\cdot\langle 0|\phi_A(\boldsymbol{x}+\boldsymbol{r})\phi_B(\boldsymbol{x})|AB,W\rangle_{\rm in}\\
\psi_{AB,CD}(\boldsymbol{r}) &= 1/\sqrt{Z_A Z_B}\cdot\langle 0|\phi_A(\boldsymbol{x}+\boldsymbol{r})\phi_B(\boldsymbol{x})|CD,W\rangle_{\rm in}\\
\psi_{CD,AB}(\boldsymbol{r}) &= 1/\sqrt{Z_C Z_D}\cdot\langle 0|\phi_C(\boldsymbol{x}+\boldsymbol{r})\phi_D(\boldsymbol{x})|AB,W\rangle_{\rm in}\\
\psi_{CD,CD}(\boldsymbol{r}) &= 1/\sqrt{Z_C Z_D}\cdot\langle 0|\phi_C(\boldsymbol{x}+\boldsymbol{r})\phi_D(\boldsymbol{x})|CD,W\rangle_{\rm in}
\end{aligned}$$

Vector of NBS

$$\Psi_{XY} = (\psi_{AB,XY}, \psi_{CD,XY})^T, \quad (XY = AB \text{ or } CD) \quad \text{for } W\in\Delta_1$$

$$\Psi_{XY} = (\psi_{AB,XY}, \psi_{CD,XY})^T, \quad (XY = AB \text{ only}) \quad \text{for } W\in\Delta_0$$

Norm

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}_{AB,AB}(\Delta_0,\Delta_0) & \mathcal{N}_{AB,AB}(\Delta_0,\Delta_1) & \mathcal{N}_{AB,CD}(\Delta_0,\Delta_1)\\ \mathcal{N}_{AB,AB}(\Delta_1,\Delta_0) & \mathcal{N}_{AB,AB}(\Delta_1,\Delta_1) & \mathcal{N}_{AB,CD}(\Delta_1,\Delta_1)\\ \mathcal{N}_{CD,AB}(\Delta_1,\Delta_0) & \mathcal{N}_{CD,AB}(\Delta_1,\Delta_1) & \mathcal{N}_{CD,CD}(\Delta_1,\Delta_1)\end{pmatrix} \qquad \mathcal{N}_{XY,X'Y'} = (\Psi_{XY}, \Psi_{X'Y'})$$

E-indep pot.

$$U = \sum (E - H_0)\Psi\mathcal{N}^{-1}\overline{\Psi}$$

- Generalization to A+B ←→ C+D+E, etc. possible
  - 2-body relativistic, otherwise non-rela approx. necessary

Extension to multi-particle systems (n>=3)

- Unitarity of S-matrix

S.Aoki et al. (HAL Coll.), PRD88(2013)014036

$$T^\dagger - T = iT^\dagger T$$

Hyper-spherical func in D=3(n-1) dim

$$T([\boldsymbol{q}^A]_n, [\boldsymbol{q}^B]_n) = \sum_{[L],[K]} T_{[L][K]}(Q_A, Q_B) Y_{[L]}(\Omega_{Q_A}) \overline{Y_{[K]}(\Omega_{Q_B})}$$

$$[L] = L, M_1, M_2, \ldots$$

diagonalization

$$T_{[L][K]}(Q,Q) = \sum_{[N]} U_{[L][N]}(Q) T_{[N]}(Q) U^\dagger_{[N][K]}(Q)$$

$$(Q = Q_A = Q_B)$$

$$T_{[L]}(Q) = -\frac{2n^{3/2}}{mQ^{3n-5}} \boxed{e^{i\delta_{[L]}(Q)} \ \sin \delta_{[L]}(Q)}$$

c.f. R.B. Newton (1974) for n = 3

Similar formula to 2-body system

(w/ diagonalization matrix U which includes dynamics)

(non-rela approx.)

Extension to multi-particle systems (n>=3)

S.Aoki et al. (HAL Coll.), PRD88(2013)014036

- NBS wave function

$$\psi_\alpha([\boldsymbol{x}]) = _{\text{in}} \langle 0|\phi([\boldsymbol{x}])|\alpha\rangle_{\text{in}} = _{\text{in}} \langle 0|N(\vec{x}_1)N(\vec{x}_2)\cdots N(\vec{x}_n)|\alpha\rangle_{\text{in}}$$

Lippmann-Schwinger eq.

$$\psi_\alpha([\boldsymbol{x}]) = _{\text{in}} \langle 0|\phi([\boldsymbol{x}])|\alpha\rangle_0 + \int d\beta \frac{_{\text{in}}\langle 0|\phi([\boldsymbol{x}])|\beta\rangle_0 T_{\beta\alpha}}{E_\alpha - E_\beta + i\epsilon}$$

Expansion w/ hyper-coordinate

$$\psi(\boldsymbol{R}, \boldsymbol{Q}_A) = \sum_{[L],[K]} \psi_{[L][K]}(R, Q_A) Y_{[L]}(\Omega_R) \overline{Y_{[K]}(\Omega_{Q_A})}$$

$$\psi_{[L],[K]}(R, Q_A) \propto \sum_{[N]} U_{[L][N]}(Q_A) \boxed{e^{i\delta_{[N]}(Q_A)} \frac{\sin(Q_A R - \Delta_L + \delta_{[N]}(Q_A))}{(Q_A R)^{(D-1)/2}}} U^\dagger_{[N][K]}(Q_A)$$

Similar asymptotic behavior to 2-body system

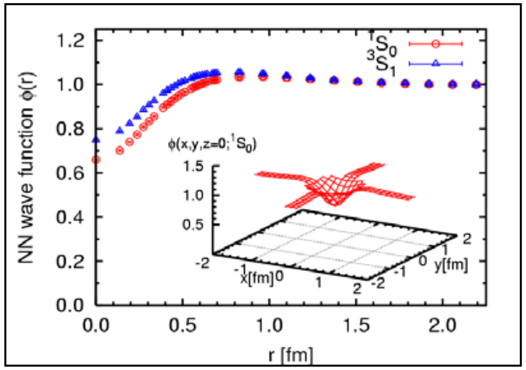
(non-rela approx.)
(no bound state in subsystem assumed)

c.f. Finite V spectrum, n=3 only, relativistic: Hansen-Sharpe, arXiv:1408.5933

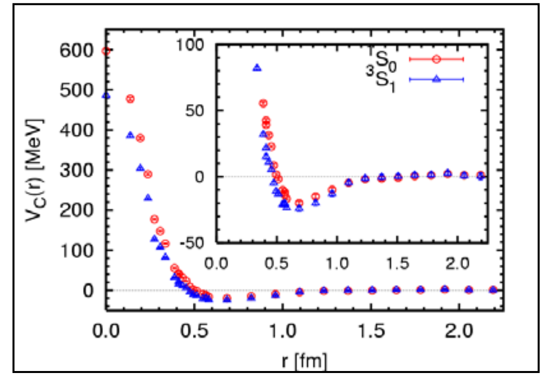
Prescription in HAL QCD method

Lattice QCD

NBS wave func.



Lat Nuclear Force



$$\psi_{NBS}(\vec{r}) = \langle 0|N(\vec{r})N(\vec{0})|N(\vec{k})N(-\vec{k}), in\rangle$$
$$\simeq e^{i\delta_l(k)} \sin(kr - l\pi/2 + \delta_l(k))/(kr)$$

(at asymptotic region)

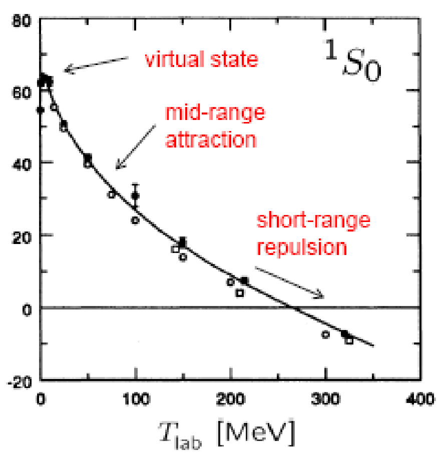
$$(k^2/m_N - H_0)\,\psi(\vec{r}) = \int d\vec{r}\,' U(\vec{r},\vec{r}\,')\psi(\vec{r}\,')$$

Lat potential is faithful to phase shift by construction

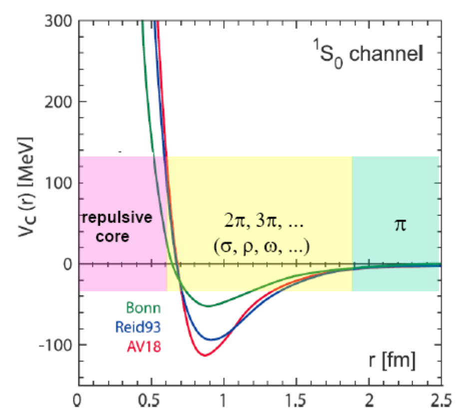
Analog to …

Scattering Exp.

Phase shifts



Phen. Potential



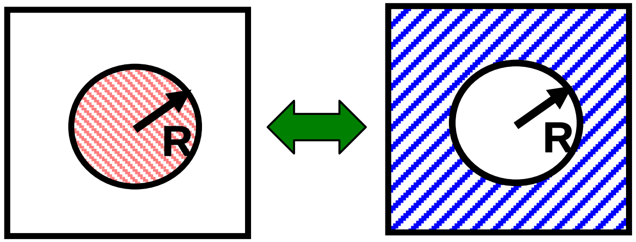
A few remarks on the Lattice Potential

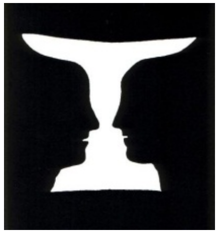
- Potential is NOT an observable and is not unique:
  They are, however, phase-shift equivalent potentials.
  - Choosing the pot. ⇔ choosing the “scheme” (sink op.)

- Potential approach has some benefits:
  - Convenient to understand physics
  - Essential to study many-body

$$Lat \rightarrow \boxed{\delta_E \rightarrow U(\boldsymbol{r})} \rightarrow \text{many-body}$$

$$\boxed{Lat \rightarrow \underset{\delta_E}{\rightarrow} \underset{\leftarrow}{\rightarrow} U(\boldsymbol{r})} \rightarrow \text{many-body}$$





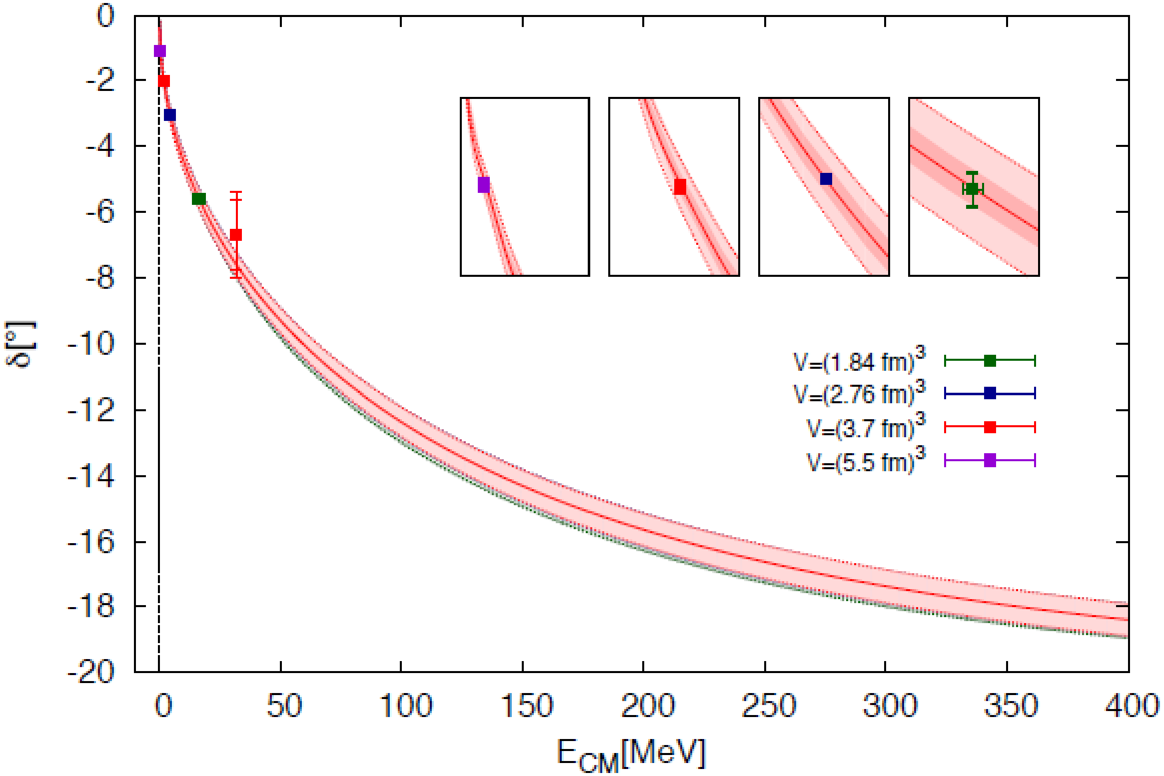
  - Phase shifts at various E obtained
    ➔ Coupled channel straightforward
  - Finite V artifact better under control
  - Excited states better under control

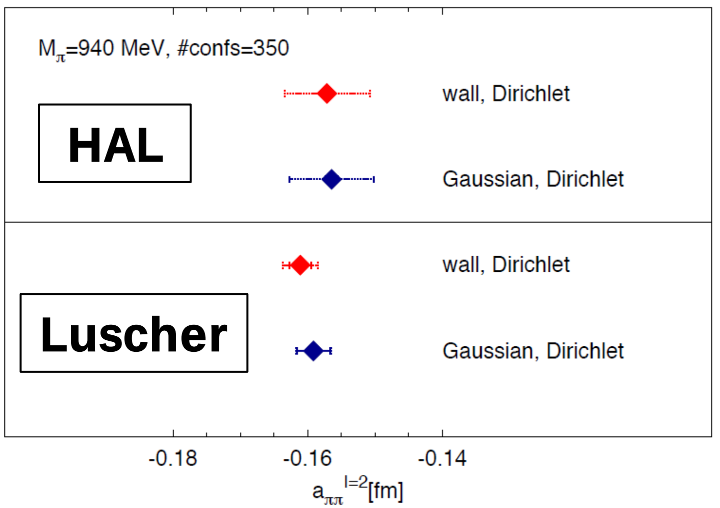
Luscher's method vs. HAL method

I=2 $\pi\pi$ system

Best S/N on the lattice
G.S. saturation can be achieved in this case

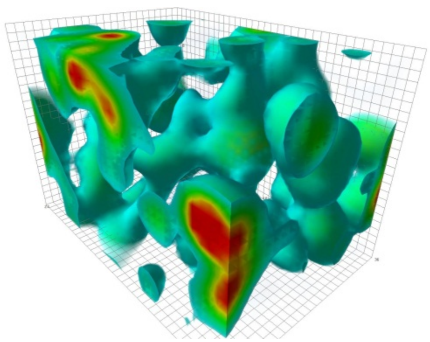
Beautiful Agreement !

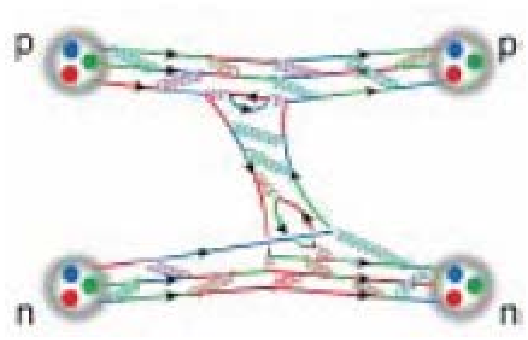


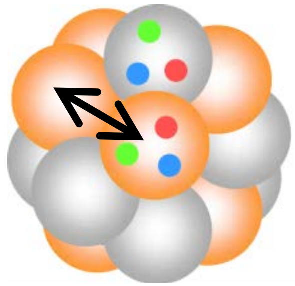


(HAL = "time-dependent" HAL method)

Kurth et al., JHEP1312(2013)015







- **<u>Outline</u>**
  - Introduction
  - Theoretical framework of HAL QCD method
    - Challenges in multi-baryons
    - NBS wave functions & E-indep "potential"
  - Numerical results
    - Unified Contraction Algorithm (UCA)
    - NN, YN/YY (H-dibaryon), NNN
  - Prospects toward physical point

Challenges in multi-baryons on the lattice (2)

- Enormous computational cost for correlators
  - # of Wick contraction (permutation)

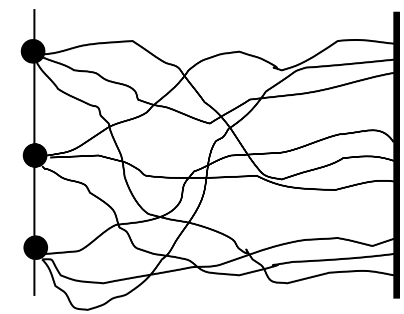
    $$N_{\rm perm} = N_u! \times N_d! \sim [\,(\tfrac{3}{2}A)!\,]^2$$ for mass number A

    (🡐 can be reduced by 2^A by inner-baryon exchange)

  - # of color / spinor contractions

    $$N_{\rm loop} = \underset{\text{(color)}}{6^A} \cdot \underset{\text{(spinor)}}{4^A} \quad \text{or} \quad 6^A \cdot 2^A$$

  - Total cost: $N_{\rm perm} \times N_{\rm loop}$
    - ^{2}H : 9 x 144 = 1 x 10^3
    - ^{3}H : 360 x 1728 = 6 x 10^5
    - ^{4}He : 32400 x 20736 = 7 x 10^8



c.f. T.Yamazaki et al., PRD81(2010)111504

$N_{\rm perm} = 1107$ for ^{4}He in the isospin limit

Solution: Unified contraction algorithm

TD, M.Endres, Comput. Phys. Comm.184(2013)117

- Traditional algorithm

color/spinor contractions (ξ')

$$\Pi^{2N} \simeq \langle qqqqqq(t)\bar{q}(\xi'_1)\bar{q}(\xi'_2)\bar{q}(\xi'_3)\bar{q}(\xi'_4)\bar{q}(\xi'_5)\bar{q}(\xi'_6)(t_0)\rangle \times \mathrm{Coeff}^{2N}(\xi'_1, \cdots, \xi'_6)$$

Permutations

color ϵ_{abc}, spinor $(C\gamma_5)$, etc.

- New algorithm

[impose the same spacial label at source]

  – Permutation applies to color/spinor indices at “Coeff”

Permuted Sum

$$\Pi^{2N} \simeq \langle qqqqqq(t)\bar{q}(\xi'_1)\bar{q}(\xi'_2)\bar{q}(\xi'_3)\bar{q}(\xi'_4)\bar{q}(\xi'_5)\bar{q}(\xi'_6)(t_0)\rangle \times \mathrm{Coeff}^{2N}(\xi'_1, \cdots, \xi'_6)$$

Sum over color/spinor unified list

  – **Permutation DONE beforehand**
    - (Wick contraction and color/spinor contractions are unified)
  – Significant improvement

$$\boxed{\times 192 \text{ for } {}^3\mathrm{H}/{}^3\mathrm{He}, \ \times 20736 \text{ for } {}^4\mathrm{He}, \ \times 10^{11} \text{ for } {}^8\mathrm{Be}}$$

(x add'l. speedup)

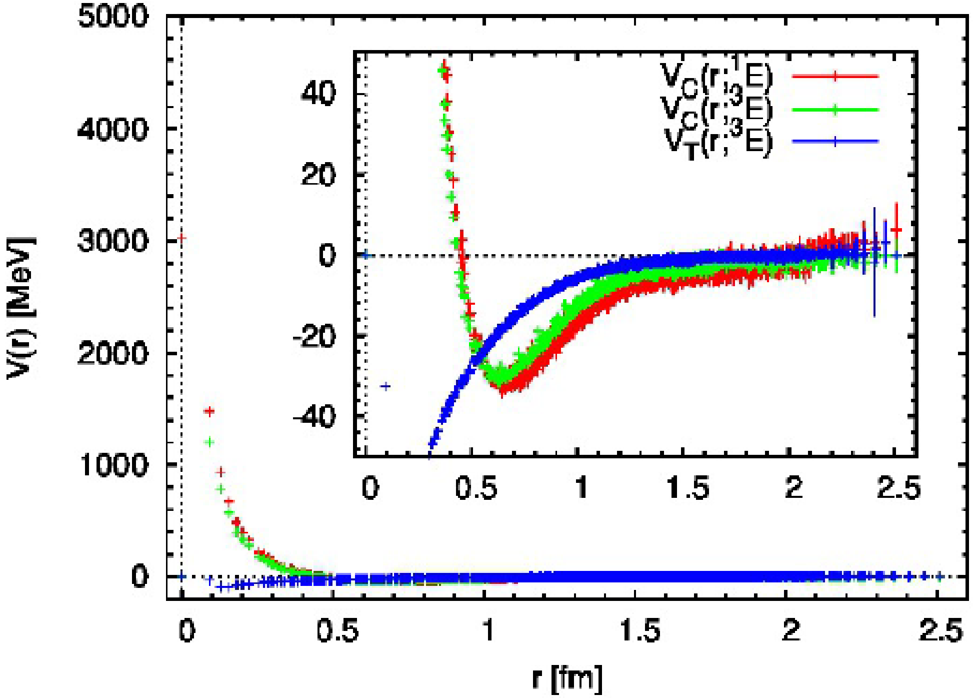
See also subsequent works:

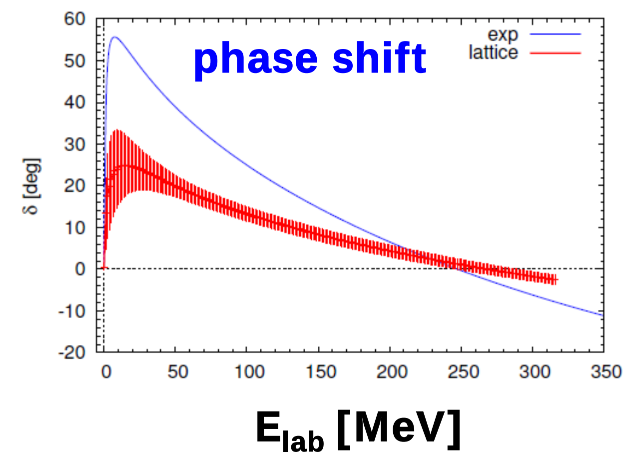
Detmold et al., PRD87(2013)114512
Gunther et al., PRD87(2013)094513

(1) NN potential on the lattice
(positive parity)

$^{2S+1}L_J$

- "di-neutron" channel $^1S_0$ ➔ central force
- "deuteron" channel $^3S_1 - {}^3D_1$ ➔ central & tensor force





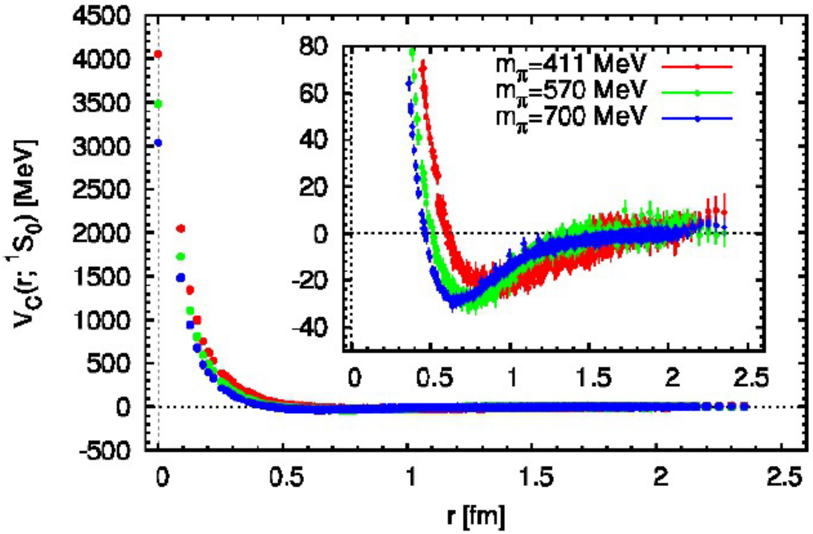
Not Bound $a({}^1S_0) = 1.6(1.1)$ fm

Nf=2+1 clover (PACS-CS), 1/a=2.2GeV,
L=2.9fm, m_π=0.7GeV, m_N=1.6GeV

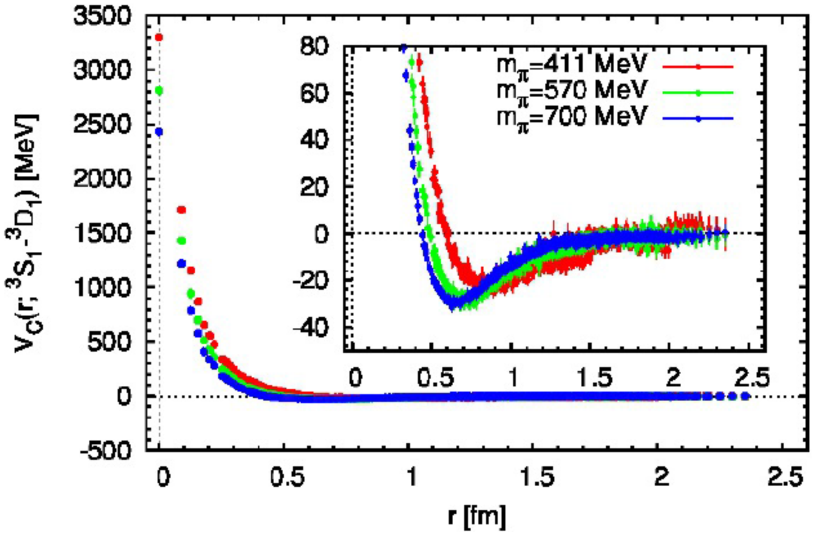
N.Ishii et al. (HAL QCD Coll.)
PLB712(2012)437

Quark mass dependence

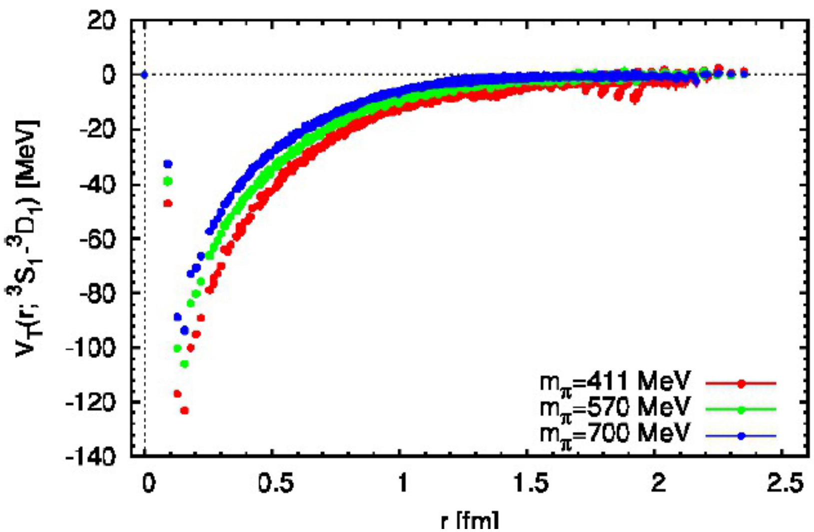
Central in $^1S_0$



$^3S_1$-$^3D_1$ channel



Central



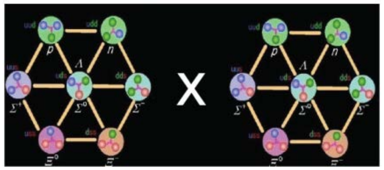
Tensor

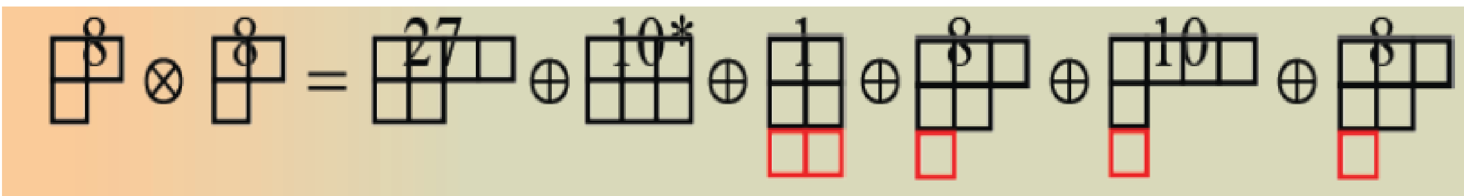
Lighter mass corresponds to…

- **Longer interaction range**
- **Larger Repulsive Core**
- **Stronger Tensor Force**

N.Ishii @ Lat2012

Hyperon Forces





8 X 8 = 27 + 8s + 1 + 10* + 10 + 8a

symmetric

anti-symmetric

NN channel

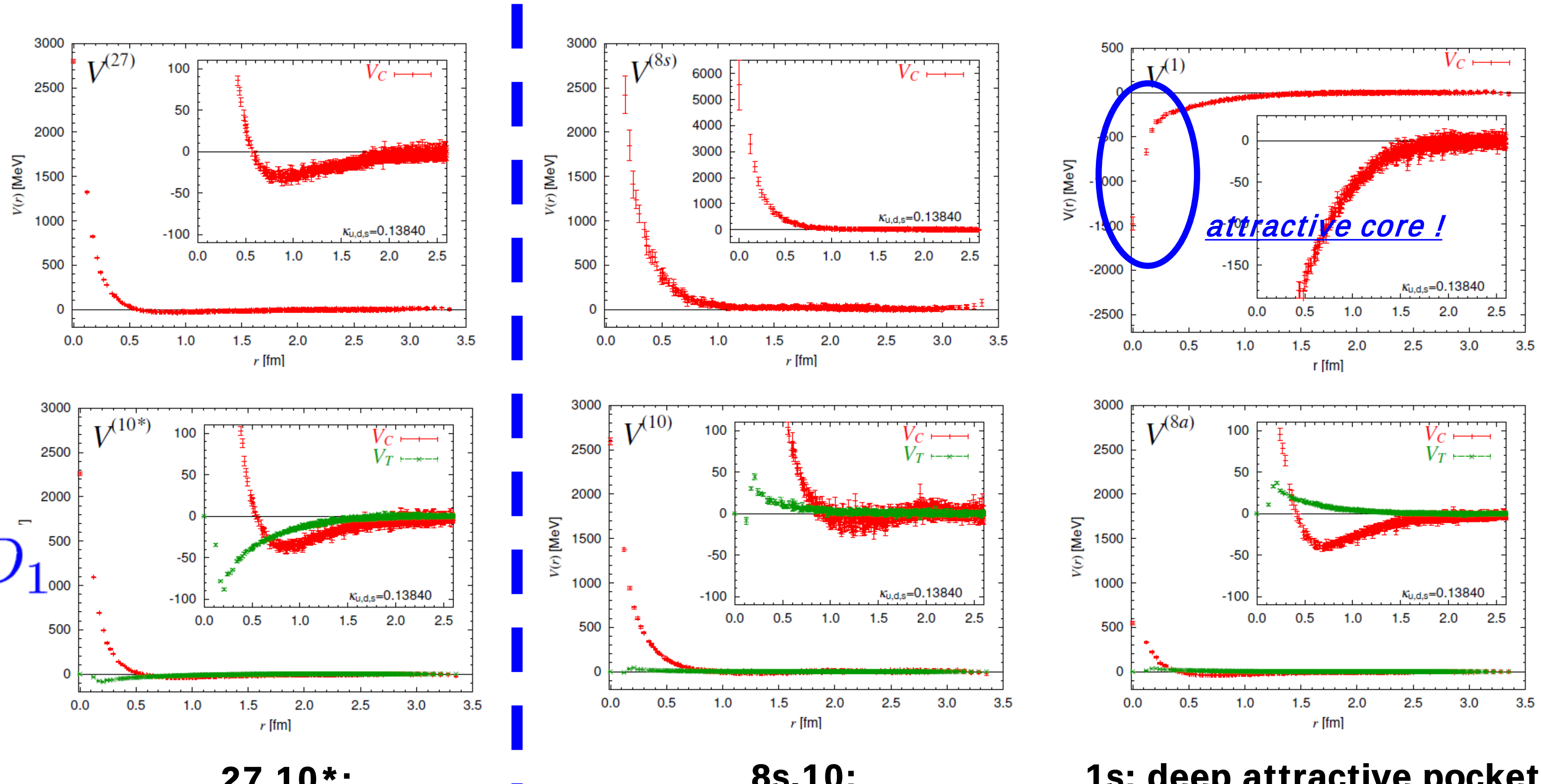
SU(3) study

(2) BB potentials

a=0.12fm, L=3.9fm,
m(PS)= 0.47-1.2GeV

$^1S_0$

$^3S_1-^3D_1$



27,10*:
Same as NN

8s,10:
strong repulsive core

1s: deep attractive pocket
8a: weak repulsive core

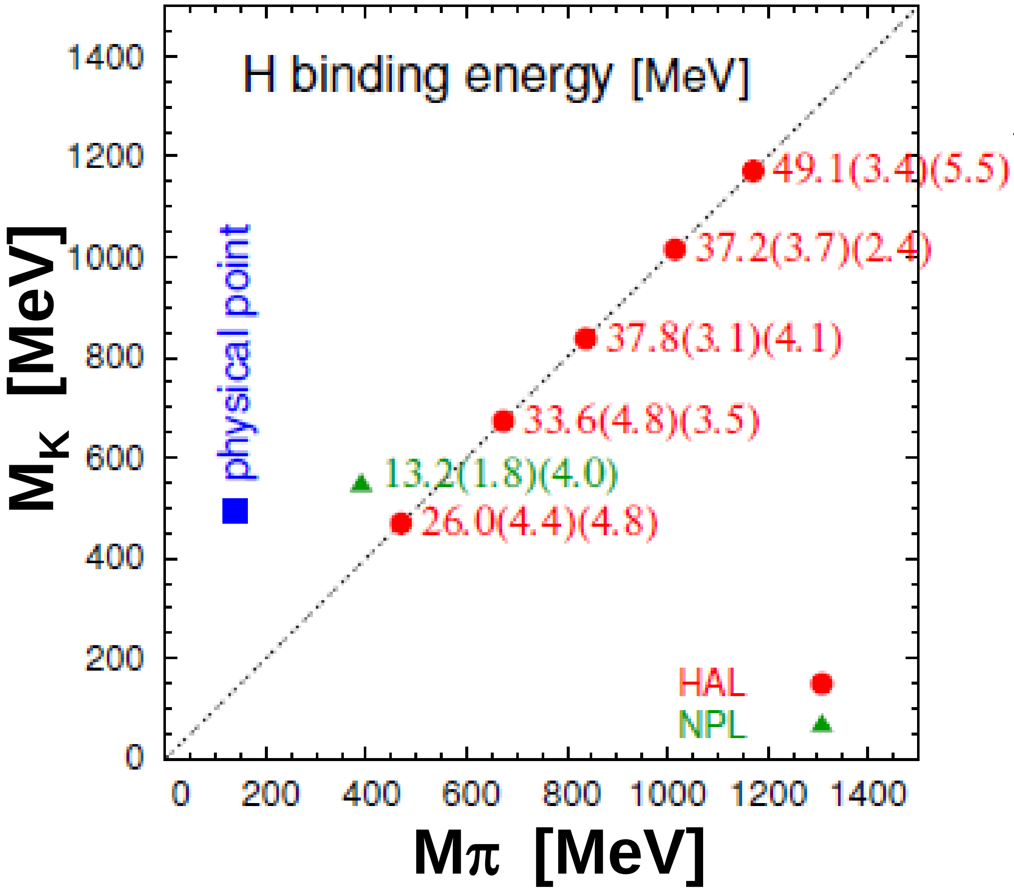
Repulsive core
← Pauli principle !

M.Oka et al., NPA464(1987)700

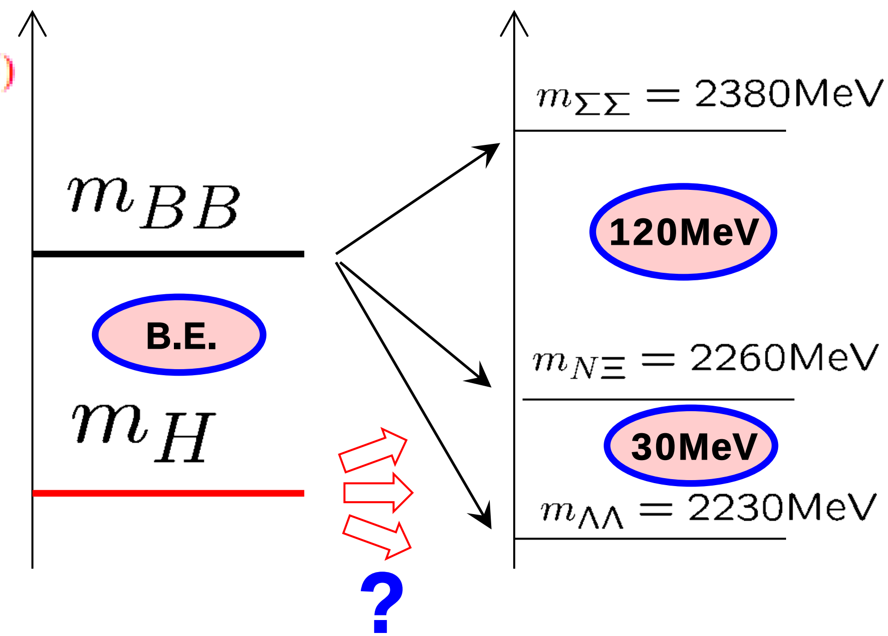
T.Inoue et al. (HAL QCD Coll.), NPA881(2012)28

Also seen in SU(2)c , Takahashi et al.,, PRD82(2010)094506
Charmonium-N, Kawanai-Sasaki, PRD82(2010)091501
Meson-baryon, Y.Ikeda et al., arXiv:1111.2663

H-dibaryon (uuddss, I=0,$^1S_0$)



SU(3) lat → **Physical point**



Coupled channel study is essential

$$\Lambda\Lambda\text{–}N\Xi\text{–}\Sigma\Sigma$$

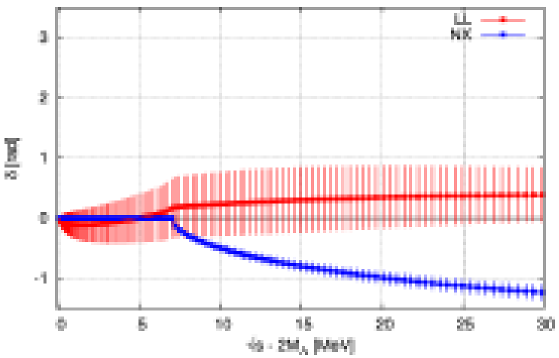
Beane et al. (NPLQCD Coll.) PRL106(2011)162001
Inoue et al. (HAL QCD Coll.) PRL106(2011)162002

Coupled channel formalism in HAL

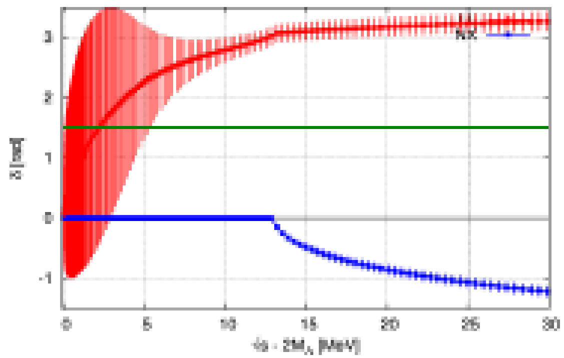
$\Lambda\Lambda - N\Xi - \Sigma\Sigma$

$\Lambda\Lambda$ and $N\Xi$ phase shifts

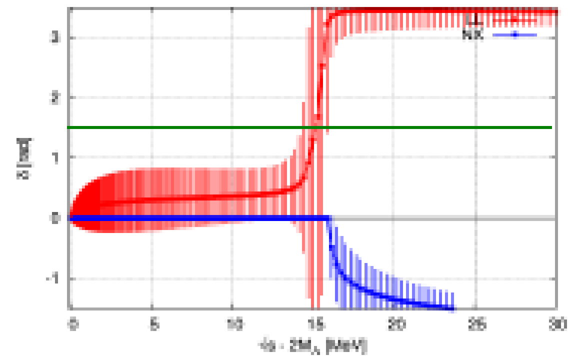
Esb1 : m_π = 701 MeV



Esb2 : m_π = 570 MeV



Esb3 : m_π = 411 MeV



Preliminary!

In unit of MeV	Esb 1	Esb 2	Esb 3
π	701±1	570±2	411±2
K	789±1	713±2	635±2
m_π/m_K	0.89	0.80	0.65
N	1585±5	1411±12	1215±12
Λ	1644±5	1504±10	1351± 8
Σ	1660±4	1531±11	1400±10
Ξ	1710±5	1610± 9	1503± 7

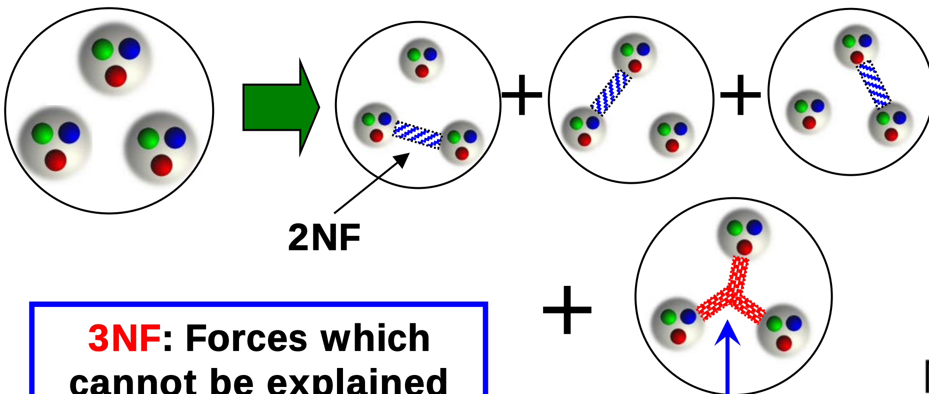
u,d quark masses lighter

- Esb1:
  - Bound H-dibaryon
- Esb2:
  - H-dibaryon is near the $\Lambda\Lambda$ threshold
- Esb3:
  - The H-dibaryon resonance energy is close to $N\Xi$ threshold..
- We can see the clear resonance shape in $\Lambda\Lambda$ phase shifts for Esb2 and 3.
- The "binding energy" of H-dibaryon from $N\Xi$ threshold becomes smaller as decreasing of quark masses.

[K. Sasaki]

Frontier in Hadron-Hadron Interactions
⇒Three-Nucleon Forces (3NF)

What is 3NF ?

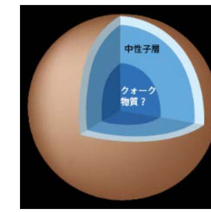


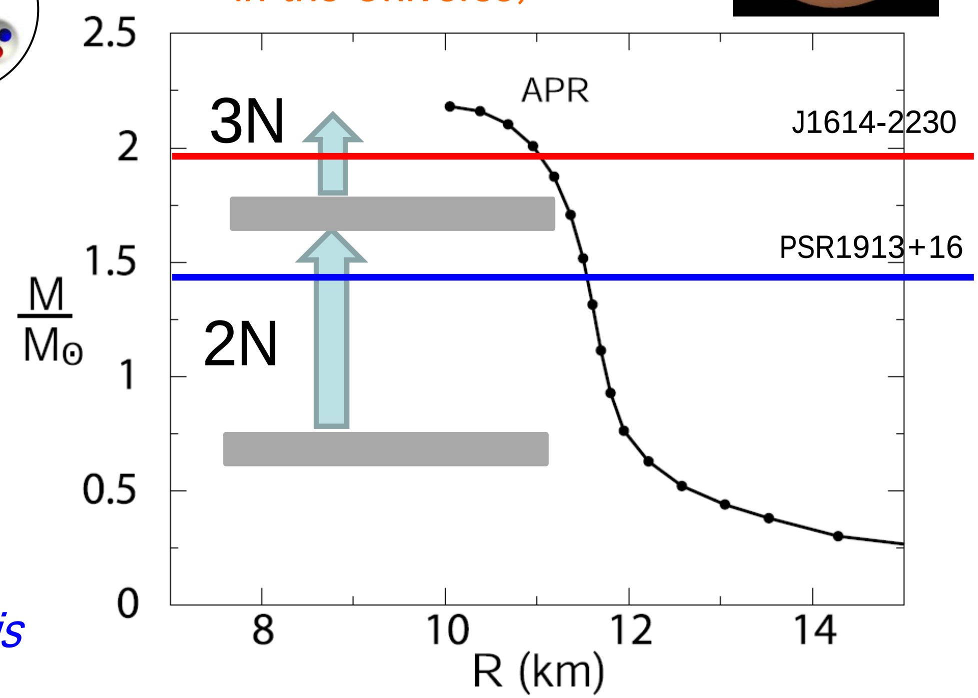
3NF: Forces which cannot be explained by pair-wise 2NF

- *EoS of high density matter*
- *Neutron rich nuclei / Nucleosynthesis*
- *B.E. of light nuclei*

Neutron Star

（Densest system in the Universe）





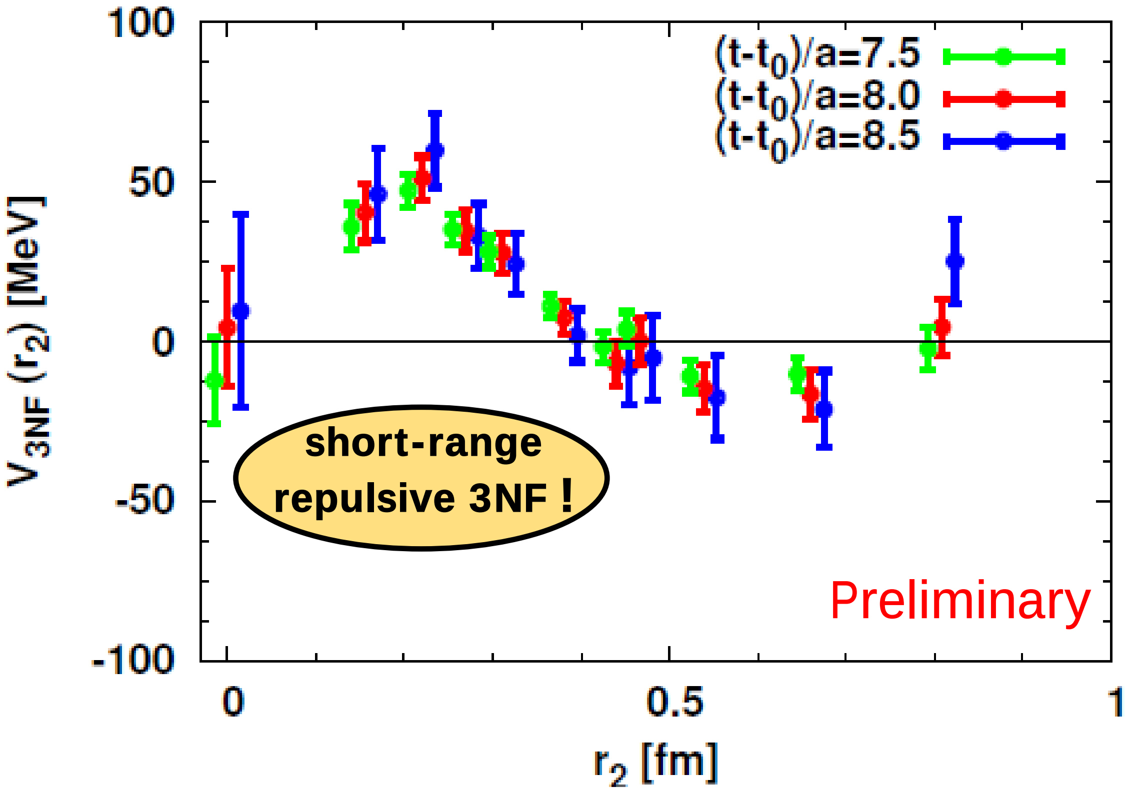
Short-range repulsive 3NF is required

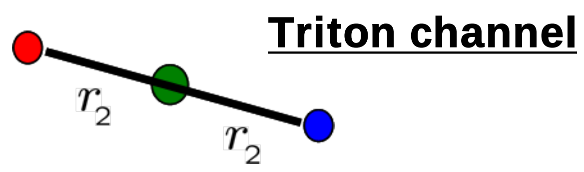
Can we understand it from QCD ?

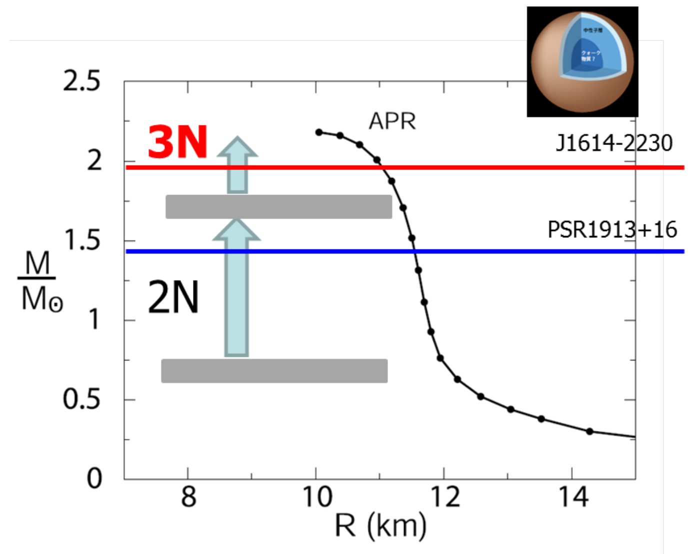
(3) 3N-forces (3NF) on the lattice

T.D. et al. (HAL QCD Coll.) PTP127(2012)723

+ t-dep method updates etc.





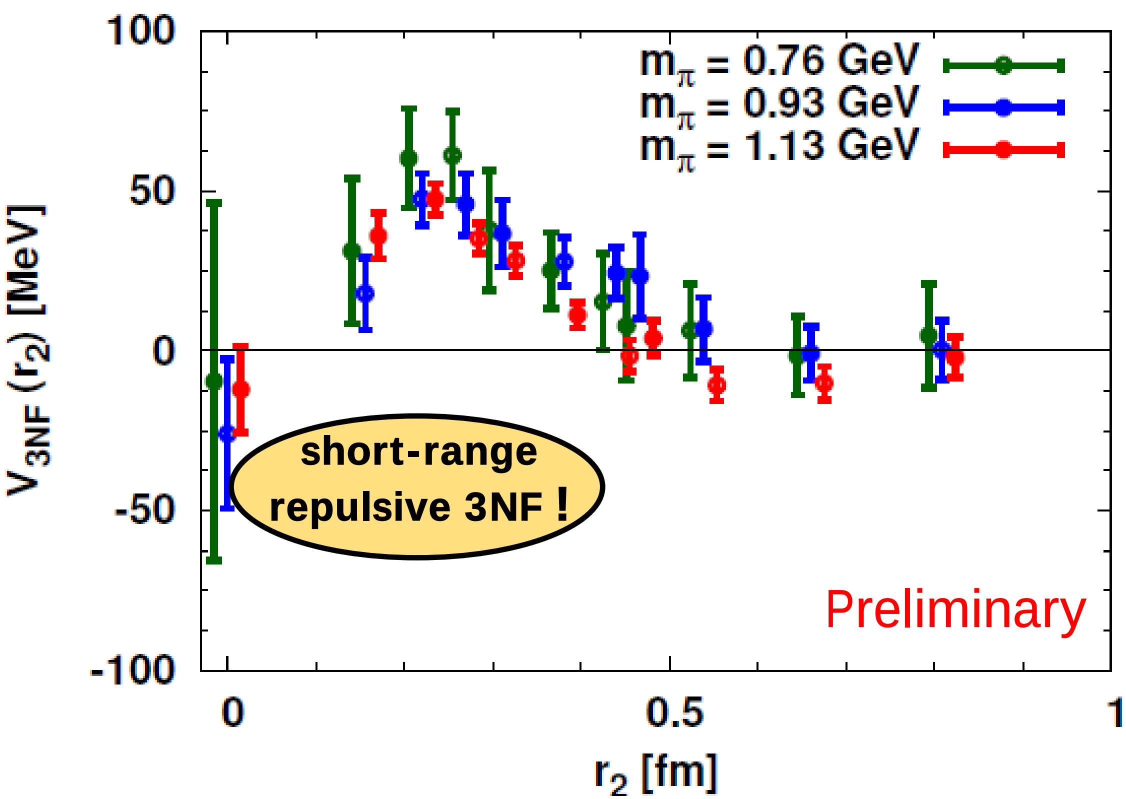


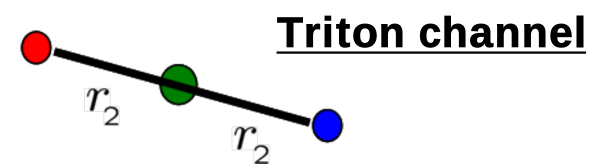
Nf=2 clover (CP-PACS), 1/a=1.27GeV,
L=2.5fm, m_π=1.1GeV, m_N=2.1GeV

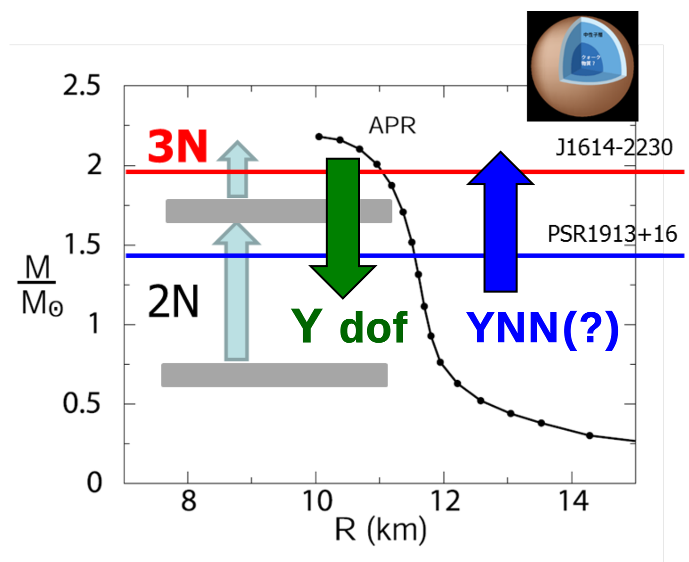
(3) 3N-forces (3NF) on the lattice

T.D. et al. (HAL QCD Coll.) PTP127(2012)723

+ t-dep method updates etc.



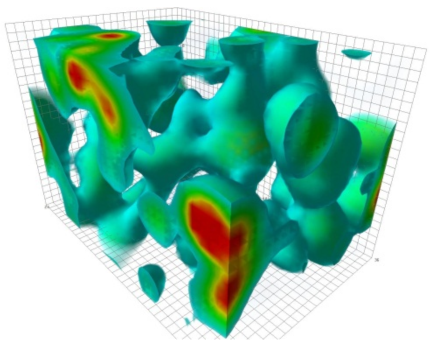


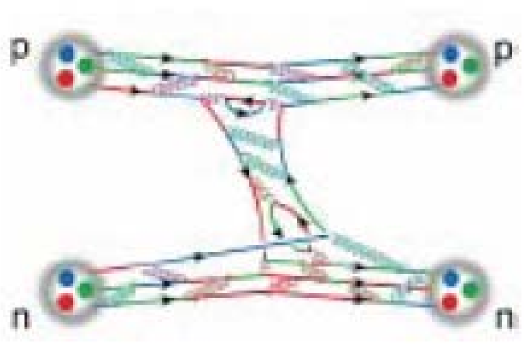


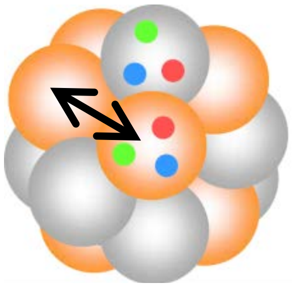
Nf=2 clover (CP-PACS), 1/a=1.27GeV,
L=2.5fm, m_π=0.76-1.1GeV, m_N=1.6-2.1GeV

How about other geometries ?

How about YNN, YYN, YYY ?



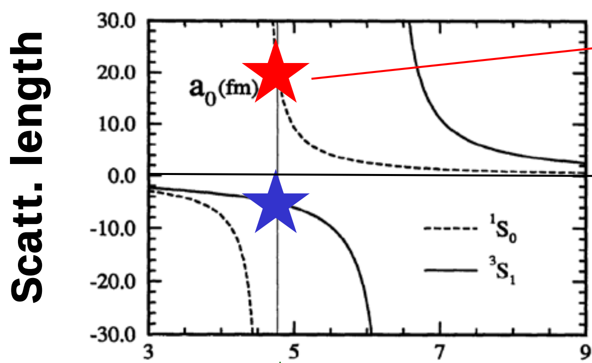




- **<u>Outline</u>**
  - – Introduction
  - – Theoretical framework of HAL QCD method
    - Challenges in multi-baryons
    - NBS wave functions & E-indep “potential”
  - – Numerical results
  - – Prospects toward physical point

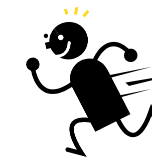
Towards realistic potential

- **Physical mass point**, **Infinite V limit**, **continuum limit**
  - Physical $m\pi$ crucial for OPEP, chiral extrapolation won't work



$a_0(^1S_0) \sim 20\text{fm}$

"Unitary Region"



➔ m_q

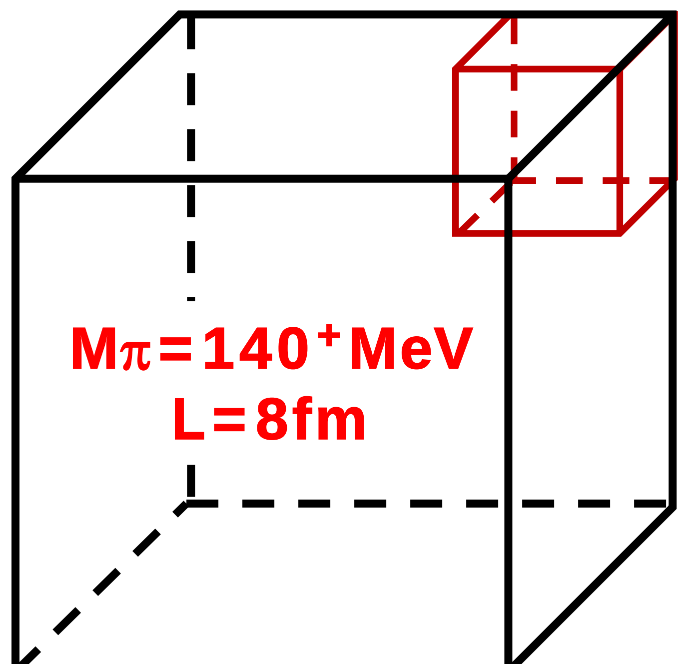
m_q

Y.Kuramashi, PTPS122(1996)153

Phys. point

We are here

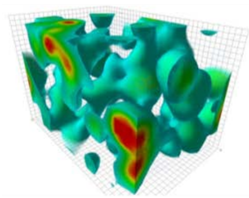
K computer

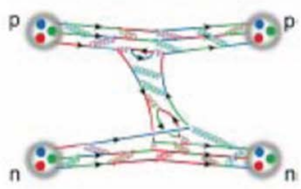


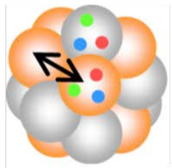
$M\pi$>=400MeV
L=3fm

$M\pi$=140$^+$MeV
L=8fm

Summary and Prospects







- HAL QCD method
  - Asymptotic behavior of NBS wave functions
  - Energy-independent potential
    - Avoid S/N issue by ground state saturation (t-dep HAL method)
    - Extended to coupled channel systems, many-body forces

- Lattice QCD results for NN, **YN/YY**, NNN, etc.
  - Intriguing physics even at heavy quark masses

- Toward physical quark mass point:
  - Unified Contraction Algorithm: breakthrough in comput. cost

➔ Realistic hadron interactions
➔ Nuclear Physics on the Lattice !