Lattice Calculation of the Hadronic Light by Light Contributions to the Muon Anomalous Magnetic Moment

by Luchang Jin

Columbia University / RBC Collaboration

Tom Blum (UConn)
Norman Christ (Columbia)
Masashi Hayakawa (Nagoya)
Taku Izubuchi (BNL/RBRC)
Christoph Lehner (BNL)
Norikazu Yamada (KEK)

Multi-Hadron and Nonlocal Matrix Elements in Lattice QCD

Brookhaven National Laboratory

2015.02.06

$$\boldsymbol{\mu}_\mu = -g_\mu \frac{e}{2m_\mu}\mathbf{s}_\mu \tag{1}$$

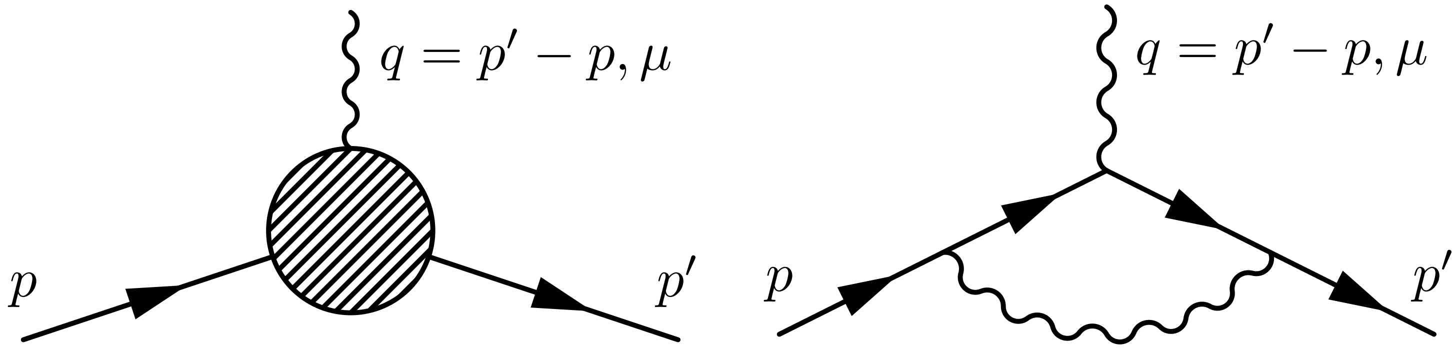


Figure 1. (L) Muon Vertex Function Diagram (R) Schwinger Term Diagram.

$$\bar{u}(p')\Gamma^\mu(p',p)u(p) = \bar{u}(p')\left[F_1(q^2)\gamma_\mu + i\frac{F_2(q^2)}{4m}[\gamma_\mu,\gamma_\nu]q_\nu\right]u(p) \tag{2}$$

$$F_2(0) = \frac{g_\mu - 2}{2} \equiv a_\mu \tag{3}$$

Figure 2. The headstone of Julian Schwinger at Mt Auburn Cemetery in Cambridge, MA.

	Value $\pm$ Error	Reference
Experiment (0.54 ppm)	116592089 ± 63	E821, The $g-2$ Collab. 2006
Standard Model	116591828 ± 50	arXiv:1311.2198
Difference ($\mathrm{Exp} - \mathrm{SM}$)	261 ± 78	
HVP LO	6949 ± 43	Hagiwara et al. 2011
Hadronic Light by Light	105 ± 26	Glasgow Consensus, 2007

Table 1. Standard model theory and experiment comparison [in units 10^{-11}]

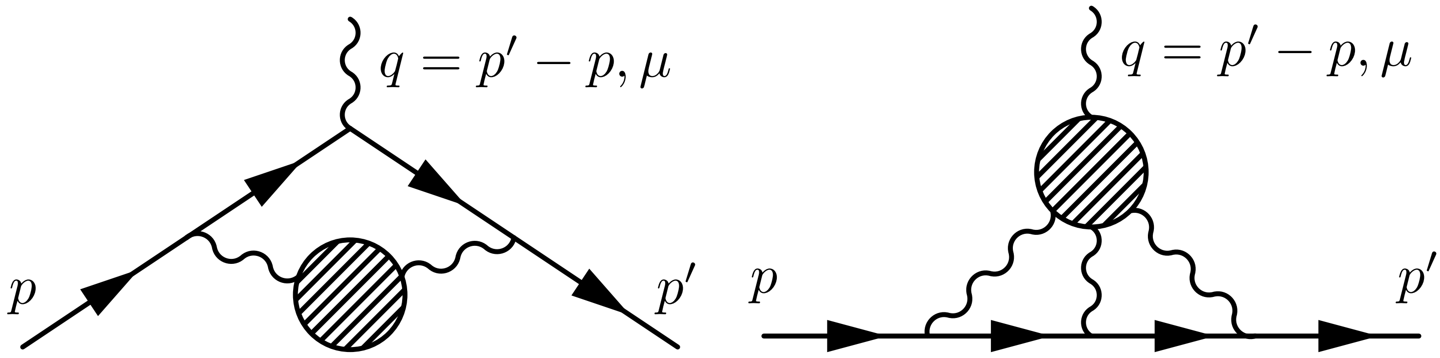


Figure 3. (L) Vaccum polarization diagram. (R) Light by light diagram.

There is 3.3σ deviation!

Figure 4. The 50-foot-wide Muon g-2 electromagnet being driven north on I-355 between Lemont and Downers Grove, Illinois, shortly after midnight on Thursday, July 25, 2013. *Credit: Fermilab.*

Almost 4 times more accurate then the previous experiment.

- In this talk, we focus on the calculation of connected light by light amplitude on lattice.
- This subject is started by T. Blum, S. Chowdhury, M. Hayakawa, T. Izubuchi more than 5 years ago. Phys. Rev. Lett. 114, 012001 (2015).

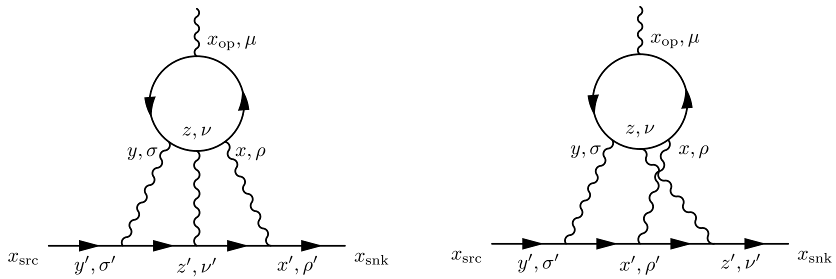


Figure 5. Light by Light diagrams. There are 4 other possible permutations.

$$\begin{aligned}
\mathcal{M}_\mu^{\mathrm{LbL}} &= -(-ie)^6 \sum_{x,y,z} \left\langle \sum_q \mathrm{tr}[\gamma_\mu S_q(x_{\mathrm{op}}, x)\gamma_\rho S_q(x,z)\gamma_\nu S_q(z,y)\gamma_\sigma S_q(y,x_{\mathrm{op}})] \right\rangle_{\mathrm{QCD}} \\
&\cdot \sum_{x',y',z'} G_{\rho\rho'}(x;x')G_{\sigma\sigma'}(y;y')G_{\nu\nu'}(z;z') \\
&\cdot \Big[S(x_{\mathrm{snk}}, x')\gamma_{\rho'} S(x',z')\gamma_{\nu'} S(z',y')\gamma_{\sigma'} S(y',x_{\mathrm{src}}) \\
&\quad + S(x_{\mathrm{snk}}, z')\gamma_{\nu'} S(z',x')\gamma_{\rho'} S(x',y')\gamma_{\sigma'} S(y',x_{\mathrm{src}}) \\
&\quad + \text{other 4 permutations} \Big] \qquad (4)
\end{aligned}$$

1. Muon Anomalous Magnetic Moment
2. BNL E821 (0.54 ppm) and Standard Model Prediction
3. **Lattice QED Schwinger Term as an Example**
   i. Stochastic Photon
   ii. Exact Photon
   iii. Finite Volume
   iv. Discretization Errors
4. Stochastic Photon Light by Light
5. Point Source Photon Light by Light
6. Current Status and Outlook

We would like to do a standard Euclidean-space lattice calculation with a muon source and sink, well separated in Euclidean time.

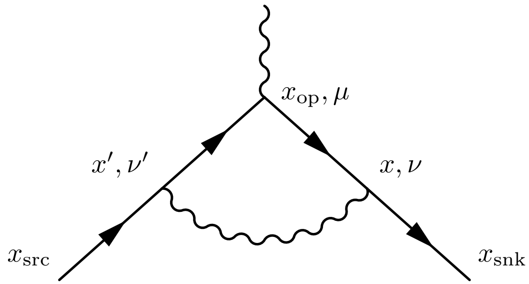


Figure 6. Schwinger term diagram.

$$\begin{aligned}\mathcal{M}_{\mu}^{\text{1-loop}} &= (-ie)^2 \sum_{x,x'} S(x_{\mathrm{snk}}, x)\gamma_{\nu} S(x, x_{\mathrm{op}})\gamma_{\mu} S(x_{\mathrm{op}}, x')\gamma_{\nu'} S(x', x_{\mathrm{src}}) \\ &\cdot \; G_{\nu\nu'}(x, x')\end{aligned} \tag{5}$$

Naively, the sum would require $\mathcal{O}(\mathrm{Volume}^2)$ computation, which is not affordable. We discuss two strategies:

- Calculate the sum stochastically.
- Fast Fourier Transformation.

Both approaches make the problem $\mathcal{O}(\mathrm{Volume})$.

Evaluate the photon propagator with N stochastic sample.

$$G_{\mu\nu}(x,y) \approx \frac{1}{M}\sum_{m=1}^{M} A_{\nu}^{m}(x)A_{\nu'}^{m}(y) \tag{6}$$

$$A_{\mu}^{m}(x) = \frac{1}{\sqrt{V}}\sqrt{2}\mathrm{Re}\sum_{k}\frac{\epsilon_{\mu}^{m}(k)}{\sqrt{|k^2|}}e^{ik\cdot x} \qquad \frac{1}{M}\sum_{m=1}^{M}\epsilon_{\mu}^{m}(k)\epsilon_{\nu}^{m*}(k') \approx \delta_{\mu\nu}\delta_{kk'} \tag{7}$$

$$\begin{aligned}\mathcal{M}_{\mu}^{\text{1-loop}} &= (-ie)^2\frac{1}{M}\sum_{m=1}^{M} \\ &\cdot \left[\sum_{x} S(x_{\mathrm{snk}},x)\gamma_{\nu}A_{\nu}^{m}(x)S(x,x_{\mathrm{op}})\right]\gamma_{\mu}\left[\sum_{x'} S(x_{\mathrm{op}},x)\gamma_{\nu'}A_{\nu'}^{m}(x')S(x',x_{\mathrm{src}})\right]\end{aligned} \tag{8}$$

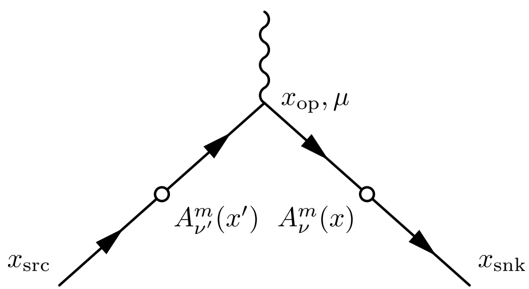


Figure 7. Schwinger term diagram calculated with stochastic photon.

$$G_{\mu\nu}(x;y) = \frac{1}{V}\sum_k \frac{\delta_{\mu\nu}}{k^2} e^{ik\cdot(x-y)} \tag{9}$$

$$\begin{aligned}\mathcal{M}_\mu^{\text{1-loop}} &= (-ie)^2 \frac{1}{V}\sum_k \frac{\delta_{\nu\nu'}}{k^2} \\ &\cdot \left[\sum_x S(x_{\text{snk}}, x)\gamma_\nu e^{ik\cdot x} S(x, x_{\text{op}})\right]\gamma_\mu\left[\sum_{x'} S(x_{\text{op}}, x')\gamma_{\nu'} e^{-ik\cdot x'} S(x', x_{\text{src}})\right]\end{aligned} \tag{10}$$

Evaluate the expression in brackets with Fast Fourier Transformation.

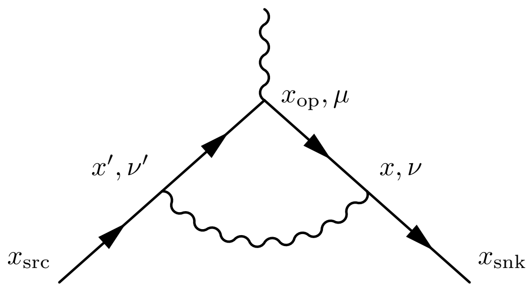


Figure 8. Schwinger term diagram calculated with exact photon.

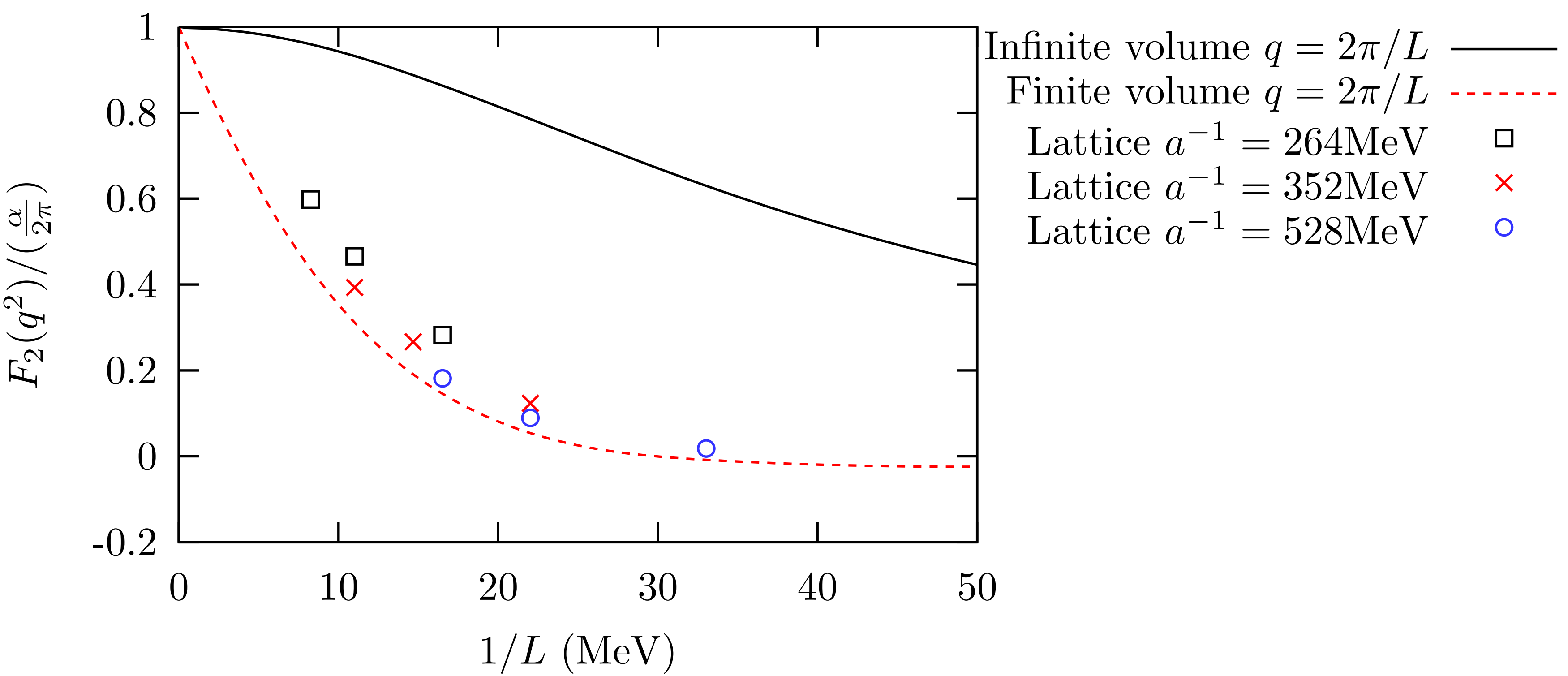


Figure 9. Finite volume effects on F_2. The data points are obtained using exact photon method.

- The solid line represents the continuum result in infinite volume and momentum transfer $q = 2\pi/L$. The dashed line represents the continuum result in L^3 volume and momentum transfer $q = 2\pi/L$.
- Lattice sizes are $32^3 \times 128$, $24^3 \times 96$, $16^3 \times 64$ with $L_s = 8$ and $t_{\mathrm{snk}} - t_{\mathrm{op}} = t_{\mathrm{op}} - t_{\mathrm{src}} = T/4$.
- Muon mass is $m_\mu = 106\text{MeV}$. a is the lattice spacing.

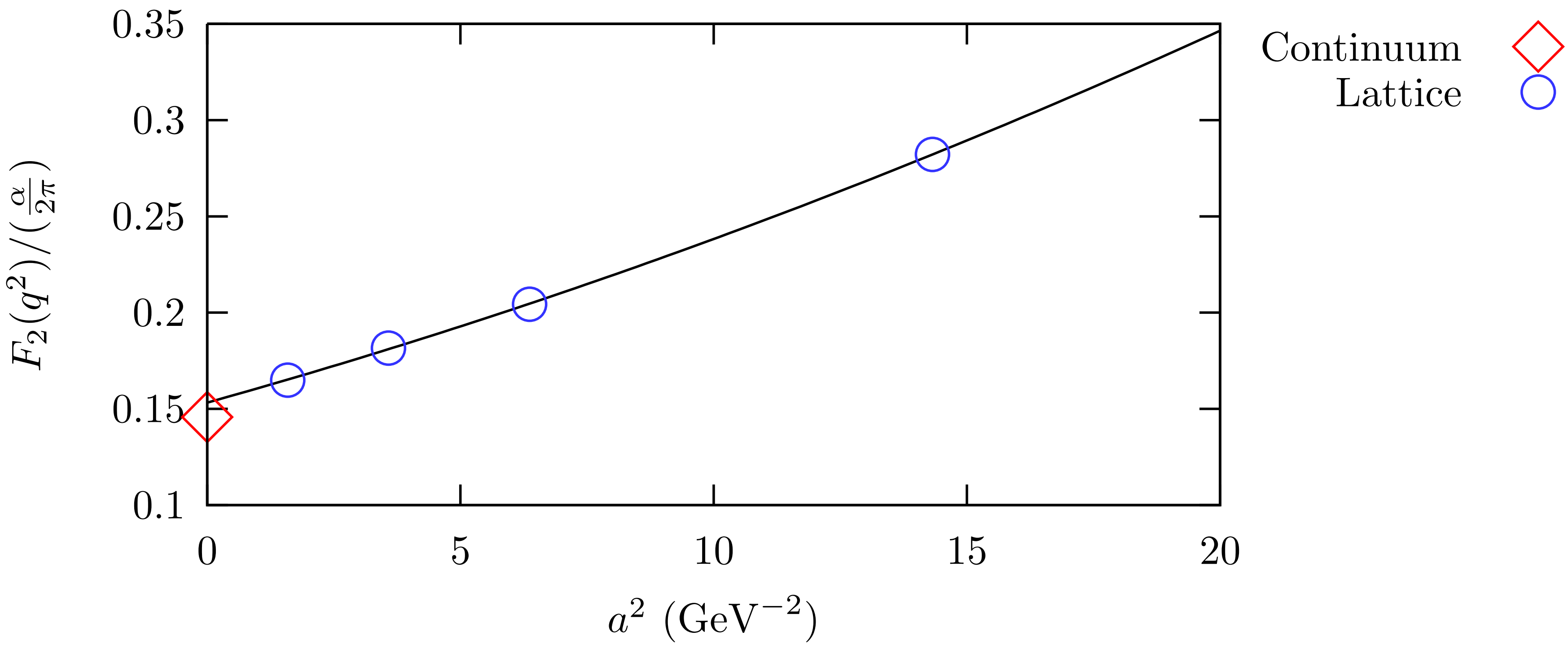


Figure 10. Discretization errors on F_2. The data points are obtained using exact photon method.

- $m_\mu L = 6.4$ and lattice sizes are $32^3 \times 128$, $24^3 \times 96$, $16^3 \times 64$, $12^3 \times 48$ with $L_s = 8$ and $t_{\mathrm{snk}} - t_{\mathrm{op}} = t_{\mathrm{op}} - t_{\mathrm{src}} = T/4$.
- $q = 2\pi/L$ is the momentum of the external photon.
- The line is 2nd order polynomial obtained by fitting the results from lattice calculations.
- Muon mass is $m_\mu = 106\mathrm{MeV}$. a is the lattice spacing. An a^4 term is visible.

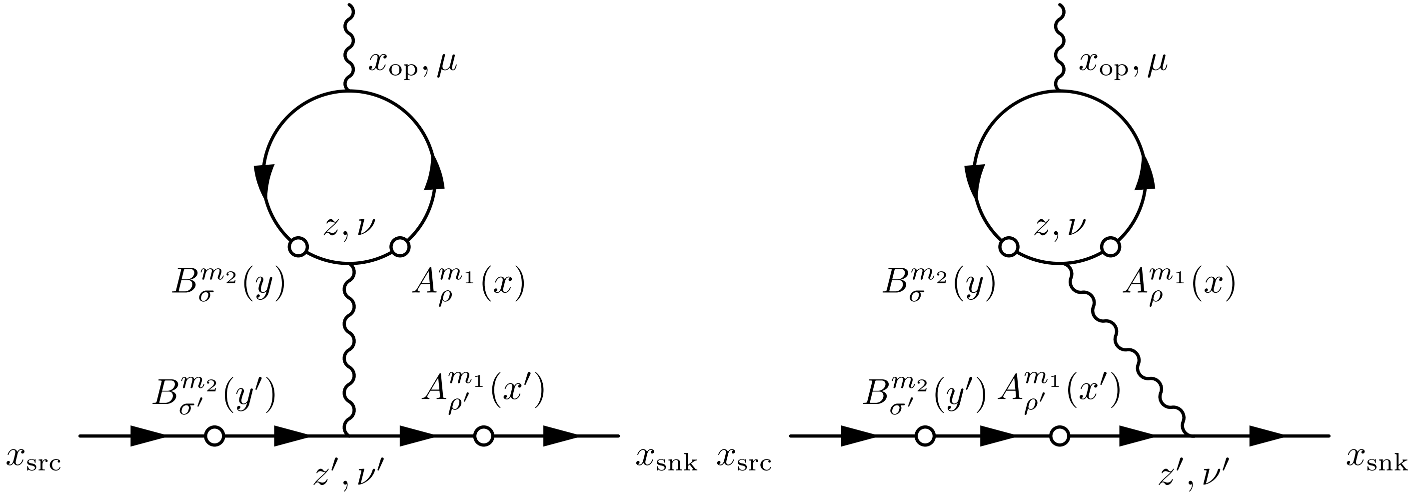
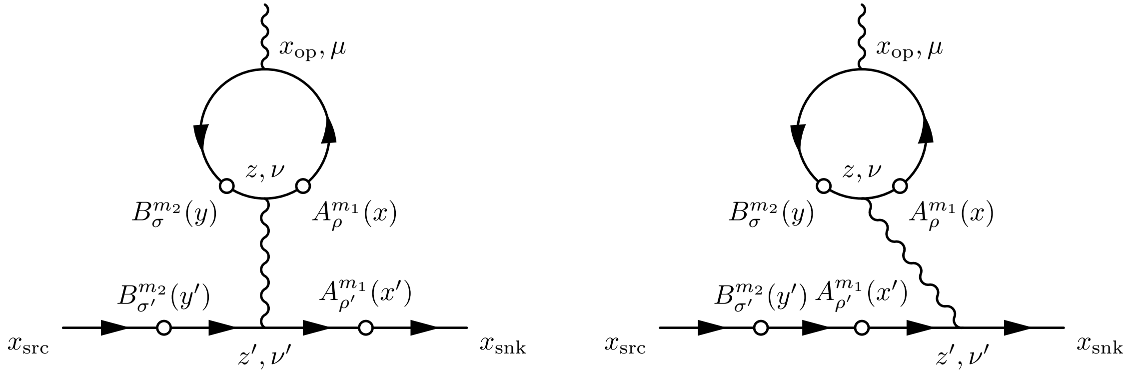


Figure 11. Light by Light diagrams calculated with one exact photon and two stochastic photon. There are 4 other possible permutations.

- $M = 12$ stochastic photon fields for both A and B.
- $S = 18$ random wall sources for the external local current.

Computation Cost

- $2 \times S \times M$ times inversion for the quark loop.
- $8 \times M^2$ times inversion for muon line.
- Statistics is roughly proportional to $S \times M^2$.
- Cost grows as $\mathcal{O}(\text{Volume})$ not $\mathcal{O}(\text{Volume}^2)$.



$$
\begin{aligned}
= \; & -(-ie)^6 \frac{1}{M^2} \sum_{m_1,m_2=1}^{M} \frac{1}{V} \sum_k \frac{\delta_{\nu\nu'}}{k^2} \\
\cdot \; & \sum_z \left\langle \sum_q \mathrm{tr} \left\{ \gamma_\mu \left[\sum_x S_q(x_{\rm op}, x) \gamma_\rho A^{m_1}_\rho(x) S_q(x,z) \right] \gamma_\nu e^{ik\cdot z} \left[\sum_y S_q(z,y) \gamma_\sigma B^{m_2}_\sigma(y) S_q(y, x_{\rm op}) \right] \right\} \right\rangle_{\rm QCD} \\
\cdot \; & \sum_{z'} \left\{ \left[\sum_{x'} S(x_{\rm snk}, x') \gamma_{\rho'} A^{m_1}_{\rho'}(x') S(x', z') \right] \gamma_{\nu'} e^{-ik\cdot z'} \left[\sum_{y'} S(z', y') \gamma_{\sigma'} B^{m_2}_{\sigma'}(y') S(y', x_{\rm src}) \right] \right. \\
& + S(x_{\rm snk}, z') \gamma_{\nu'} e^{-ik\cdot z'} \left[\sum_{x'} S(z', x') \gamma_{\rho'} A^{m_1}_{\rho'}(x') \left(\sum_{y'} S(x', y') \gamma_{\sigma'} B^{m_2}_{\sigma'}(y') S(y', x_{\rm src}) \right) \right] \\
& \left. + \, \text{other 4 permutations} \right\}
\end{aligned} \tag{11}
$$

- QED test: Replace quark loop by muon loop.

Lattice Size	t_{sep}	$m_\mu a$	$\frac{F_2 \pm \text{Err}}{(\alpha/\pi)^3}$	$N \times S \times M^2$ confs	$\frac{\sqrt{\text{Var}}}{(\alpha/\pi)^3}$
$16^3 \times 64$	32	0.2	0.2228 ± 0.0046	$548 \times 18 \times 12^2$	5.5
$16^3 \times 64$	32	0.2	0.1962 ± 0.0368	$1024 \times 18 \times 1^2$	5.0
$16^3 \times 64$ (point src)	32	0.2	0.232 ± 0.033	$1508 \times 12 \times 6^2$	28.4
$16^3 \times 64$	32	0.1	0.1666 ± 0.0069	$88 \times 18 \times 12^2$	3.3
$16^3 \times 64$ ($\mathbf{p}_1 = 0$)	32	0.1	0.2278 ± 0.0265	$285 \times 36 \times 24^2$	64.4

Figure 12. M stands for the number of stochastic A or B fields, S stands for the number of random wall sources that we use to calculate the external current. The calculation is repeated N times. $\sqrt{\text{Var}} = \text{Err} \times \sqrt{N \times S \times M^2}$ stands for the projected variance deduced from to the statistical uncertainty of the averaged result and the total number of samples.

- Average over different combinations of A, B electromagnetic field helps reducing the statistical errors.
- Random wall source at the location of the external current works very well.
- Using symmetric kinematics significantly reduces the statistical error as both the initial and the final state are the lowest energy state possible. For muon, we use anti-periodic boundary condition in z direction and set the momenta of initial and final state to be $\pm\pi/L$.

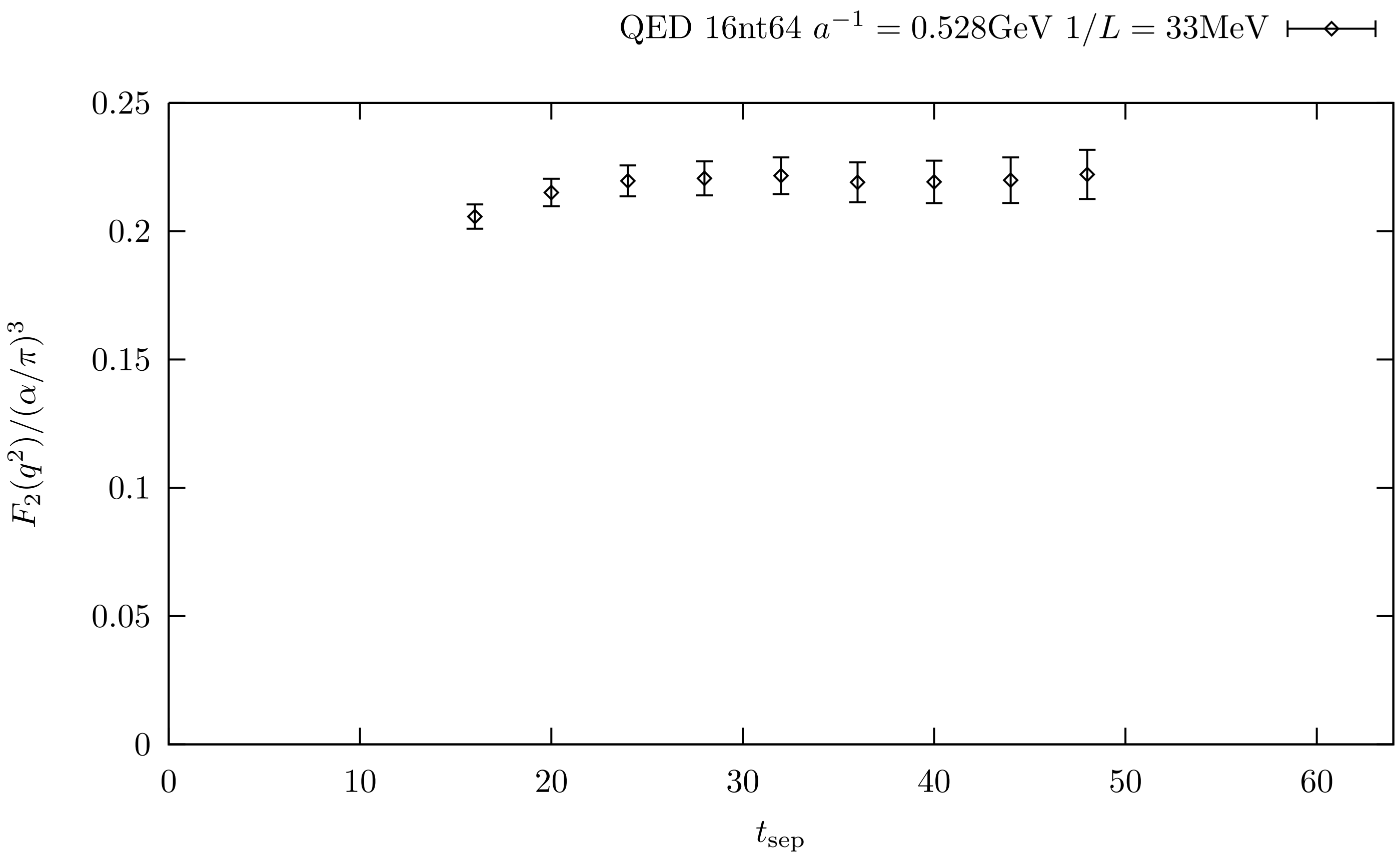


Figure 13. Excited effects on F_2.

- We have good control of the excited state effects.
- The simulations were done in L^3 volume and momentum transfer $q = 2\pi / L$.
- Muon mass is $m_\mu = 106\text{MeV}$. a is the lattice spacing.

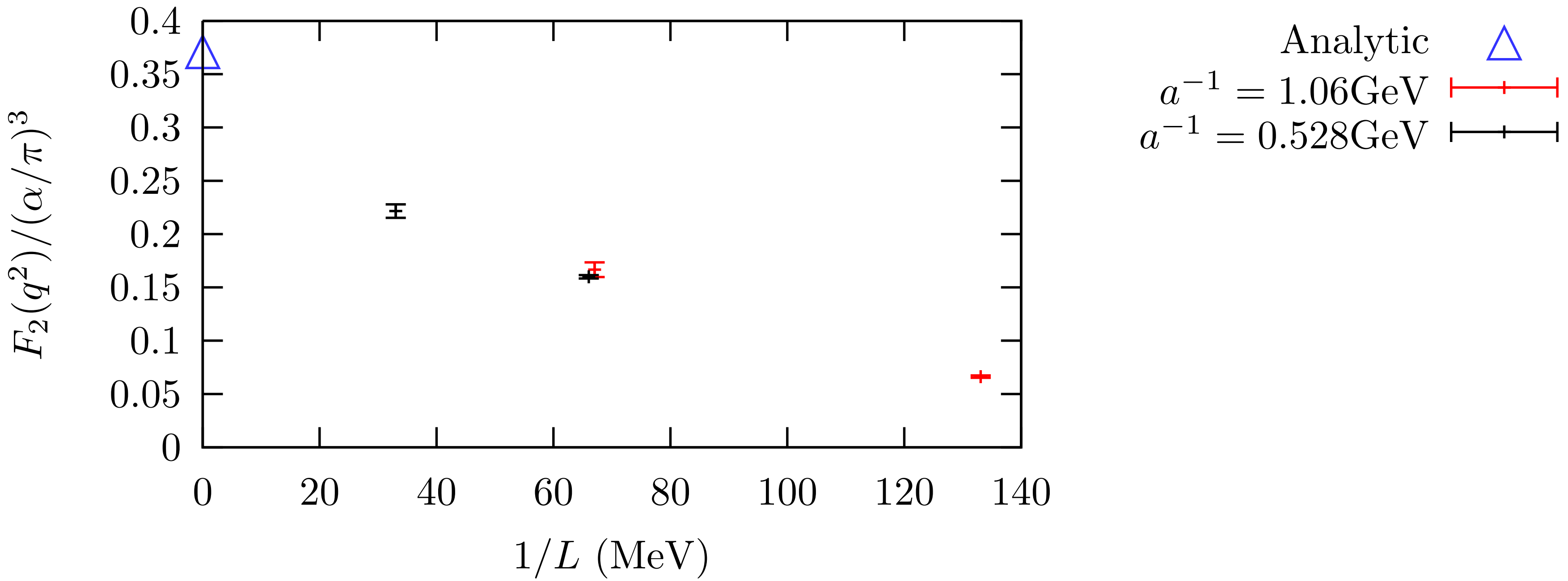


Figure 14. Finite volume effect on F_2.

- Lattice sizes are $16^3 \times 64$, $8^3 \times 32$ with $L_s = 8$ and $t_{\text{snk}} - t_{\text{op}} = t_{\text{op}} - t_{\text{src}} = T/4$.
- The simulations were done in L^3 volume and momentum transfer $q = 2\pi/L$.
- Muon mass is $m_\mu = 106\text{MeV}$. a is the lattice spacing.

	Model	Subtraction
$\frac{F_2 \pm \mathrm{Err}}{(\alpha/\pi)^3}$	0.08 ± 0.02	0.29 ± 0.06

Table 2. Hadronic Light by Light Estimates, values were also shown in section 2.

- Model: Glasgow Consensus, 2007
- Subtraction: Using the experimental $g-2$ value, E821, The $g-2$ Collab. 2006 and subtract the theoretical values of other contributions.
- First lattice attempt: T. Blum, S. Chowdhury, M. Hayakawa, T. Izubuchi, Phys. Rev. Lett. 114, 012001 (2015).

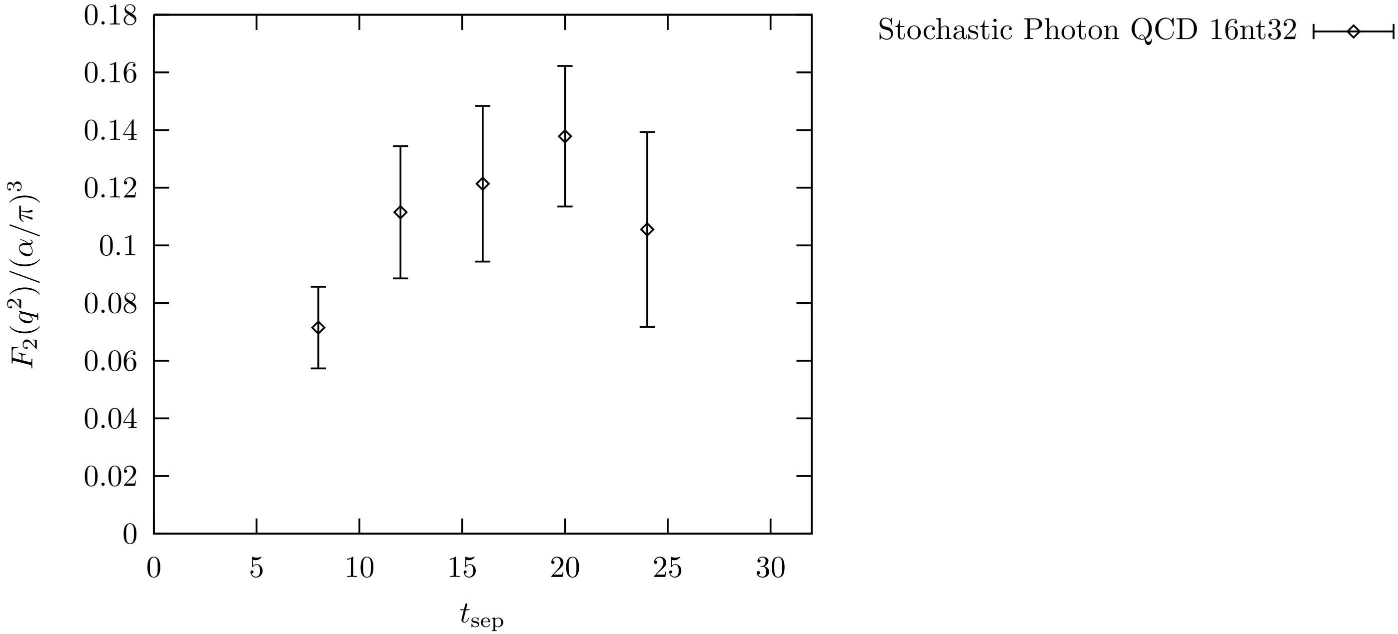


Figure 15. Excited effects on F_2. $16^3 \times 32$ lattice, with $a^{-1} = 1.747\text{GeV}$, $m_\pi = 424\text{MeV}$, $m_K = 613\text{MeV}$, $m_\mu = 332\text{MeV}$.

Outline

1. Muon Anomalous Magnetic Moment
2. BNL E821 (0.54 ppm) and Standard Model Prediction
3. Lattice QED with Schwinger Term as an Example
4. Stochastic Photon Light by Light
5. **Point Source Photon Light by Light**
   i. Comparison
   ii. Formula
   iii. Features
   iv. Distribution 16I
   v. Distribution 32ID
6. Current Status and Outlook

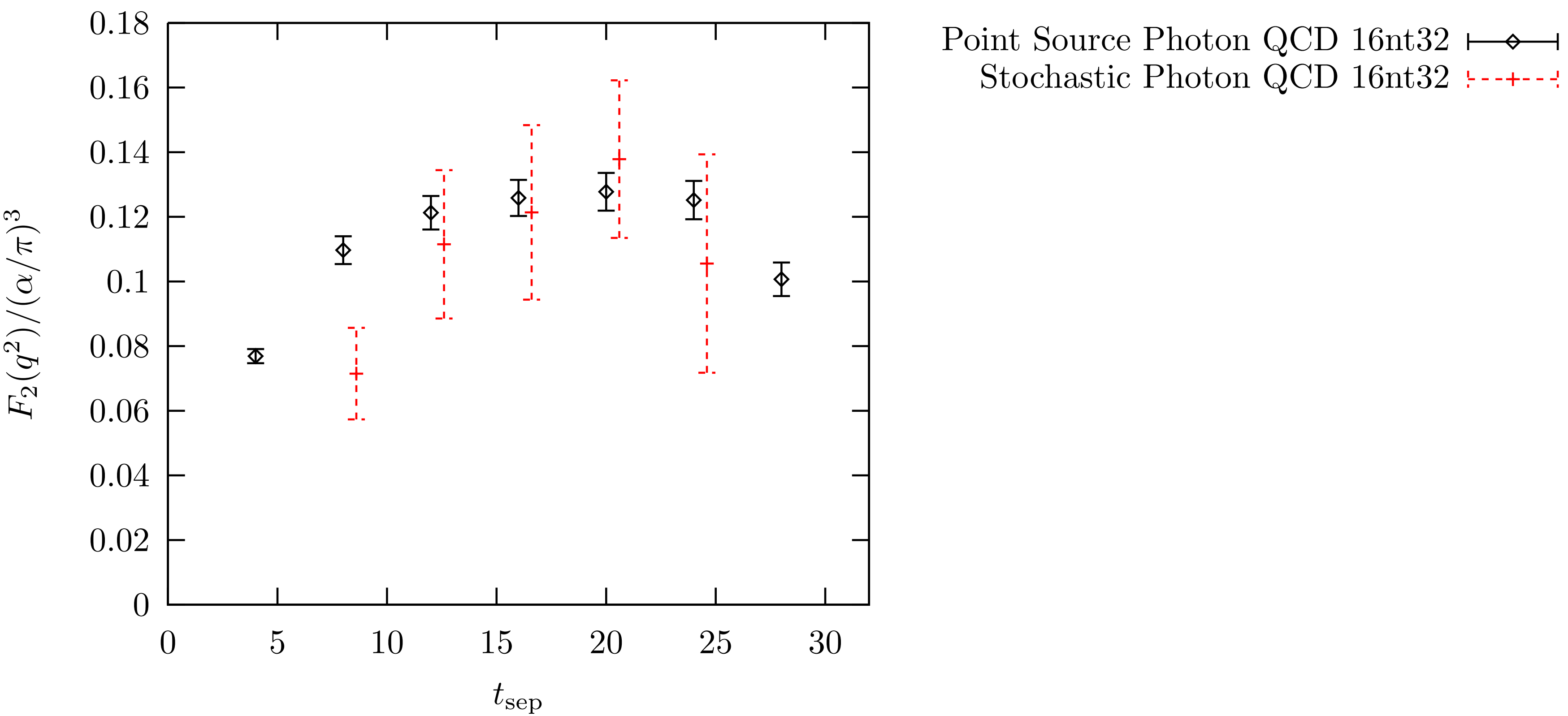


Figure 16. Excited effects on F_2. $16^3 \times 32$ lattice, with $a^{-1} = 1.747\mathrm{GeV}$, $m_\pi = 424\mathrm{MeV}$, $m_\mu = 332\mathrm{MeV}$. Here we compare the new point source method with the old stochastic photon method.

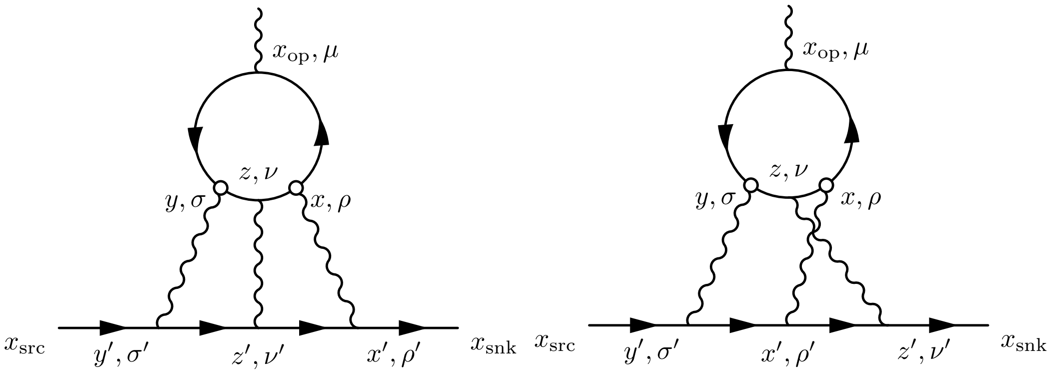


Figure 17. Light by Light diagrams. There are 4 other possible permutations.

$$
\begin{aligned}
\mathcal{M}_{\mu}^{\mathrm{LbL}}(\mathbf{q}) &= e^{i\mathbf{q}\cdot\mathbf{x}_{\mathrm{op}}}\mathcal{M}_{\mu}^{\mathrm{LbL}}(\mathbf{q}; x_{\mathrm{op}}) \\
&= \sum_{x,y}\sum_{z} F_{\mathbf{q}}(x, y; z, x_{\mathrm{op}}) \qquad (12) \\
&= \sum_{r}\left[\sum_{z,x_{\mathrm{op}}} F'_{\mathbf{q}}(x_{\mathrm{ref}}, r; z, x_{\mathrm{op}})\right] \qquad (13)
\end{aligned}
$$

$$
x_{\mathrm{ref}} = \frac{x+y}{2} \qquad r = y - x \qquad (14)
$$

$$
F'_{\mathbf{q}}(x_{\mathrm{ref}}, r; z, x_{\mathrm{op}}) = F_{\mathbf{q}}(x, y; z, x_{\mathrm{op}}) \qquad (15)
$$

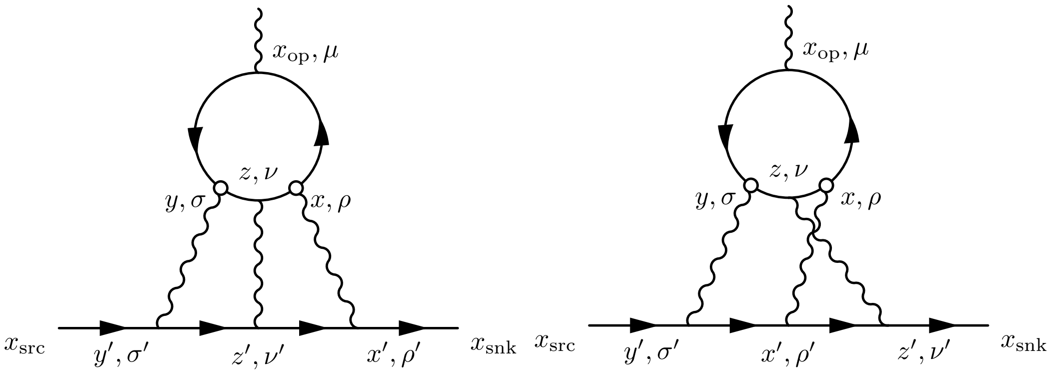


Figure 18. Light by Light diagrams. There are 4 other possible permutations.

$$
\begin{aligned}
F_{\mathbf{q}}(x, y; z, x_{\rm op}) \;=\; & e^{i\mathbf{q}\cdot\mathbf{x}_{\rm op}} \lim_{t_{\rm src}\to-\infty, t_{\rm snk}\to+\infty} e^{E_{\mathbf{q}/2} t_{\rm sep}} \\
\cdot & \sum_{\mathbf{x}_{\rm snk}, \mathbf{x}_{\rm src}} e^{-i\mathbf{q}/2\cdot\mathbf{x}_{\rm src}} e^{-i\mathbf{q}/2\cdot\mathbf{x}_{\rm snk}} (-)(-ie)^6 \\
\cdot & \left\langle \sum_q \mathrm{tr}(\gamma_\mu S_q(x_{\rm op}, x)\gamma_\rho S_q(x, z)\gamma_\nu S_q(z, y)\gamma_\sigma S_q(y, x_{\rm op})) \right\rangle_{\rm QCD} \\
\cdot & \sum_{x', y', z'} G_{\rho\rho'}(x, x') G_{\sigma\sigma'}(y, y') G_{\nu\nu'}(z, z') \\
\cdot & [\, S(x_{\rm snk}, x')\gamma_{\rho'} S(x', z')\gamma_{\nu'} S(z', y')\gamma_{\sigma'} S(y', x_{\rm src}) \\
& + S(x_{\rm snk}, z')\gamma_{\nu'} S(z', x')\gamma_{\rho'} S(x', y')\gamma_{\sigma'} S(y', x_{\rm src}) \\
& + \text{other 4 permutations}]
\end{aligned}
\tag{16}
$$

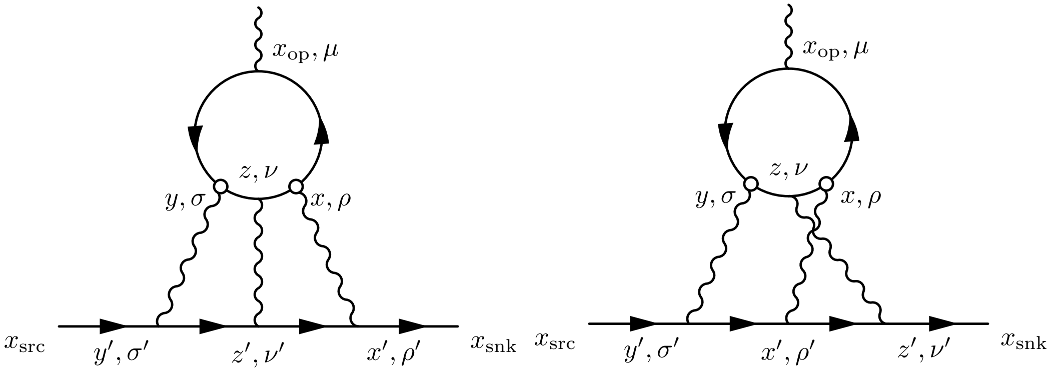
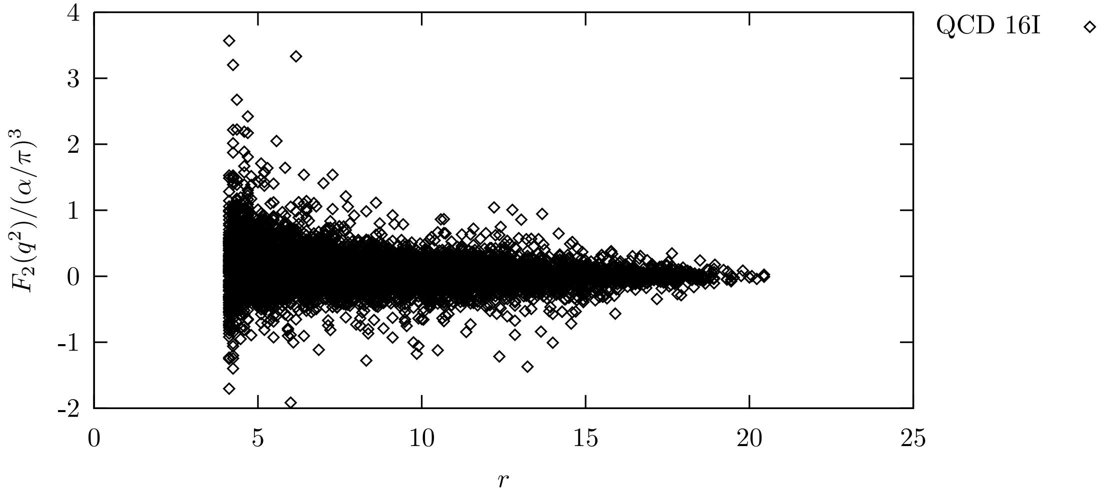


Figure 19. Light by Light diagrams. There are 4 other possible permutations.

- M^2 statistics: We use two point source propagators to calculate the QCD four point function. We can calculate M point source propagators and calculate the QCD four point function with $M(M-1)/2$ combinations.
- Importance sampling: We use importance sampling when choosing the M points. So we calculate the region that contribute to the noise the most with the highest frequency but lowest weight.
- Exact short distance: We can calculate all the short distance part of the sum without using the Monte Carlo technique.
- Conserved current: In order to make the loop integral converge, the external photon need to couple to conserved current, while all the three internal photons may couple to local currents.



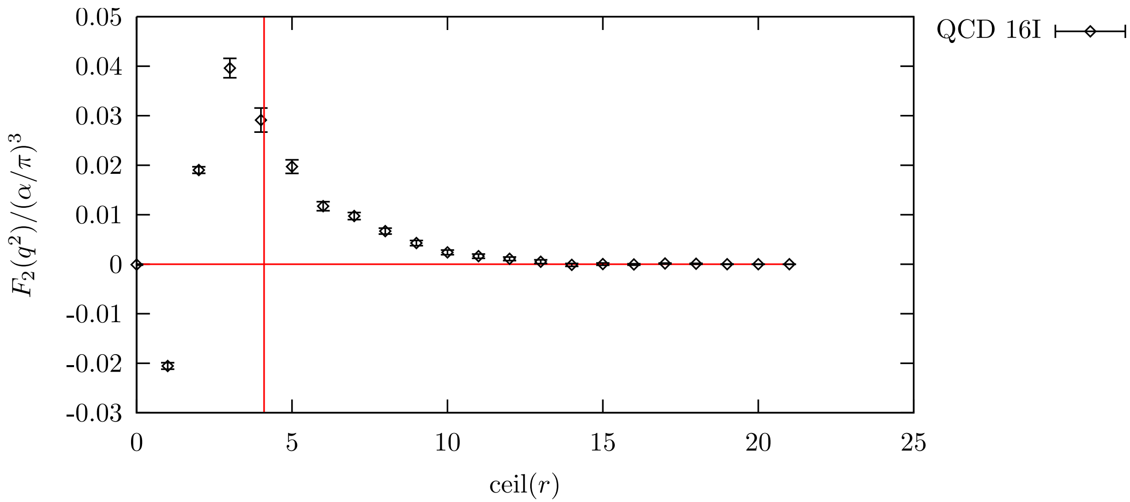
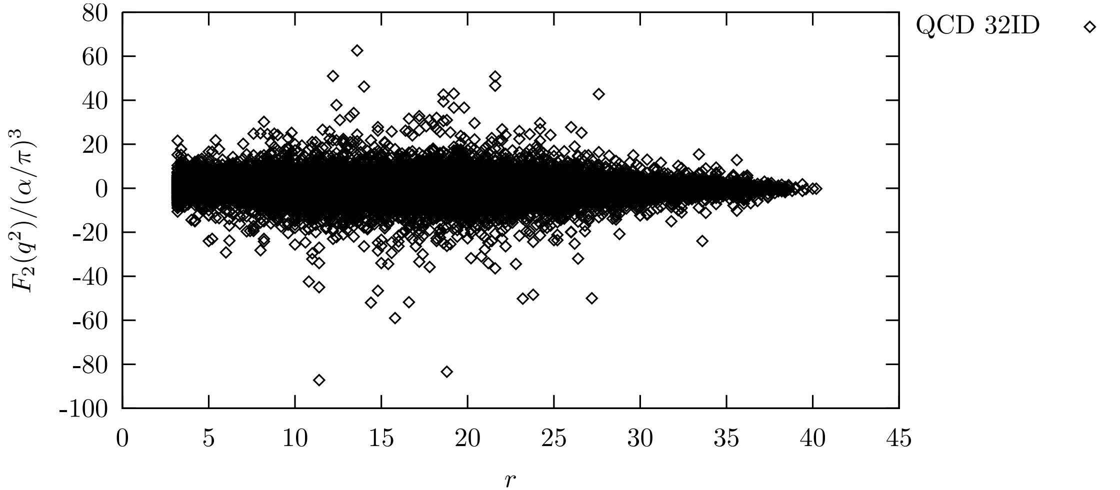


Figure 20. $16^3 \times 32$ lattice, with $a^{-1} = 1.747\mathrm{GeV}$, $m_\pi = 424\mathrm{MeV}$, $m_\mu = 332\mathrm{MeV}$.



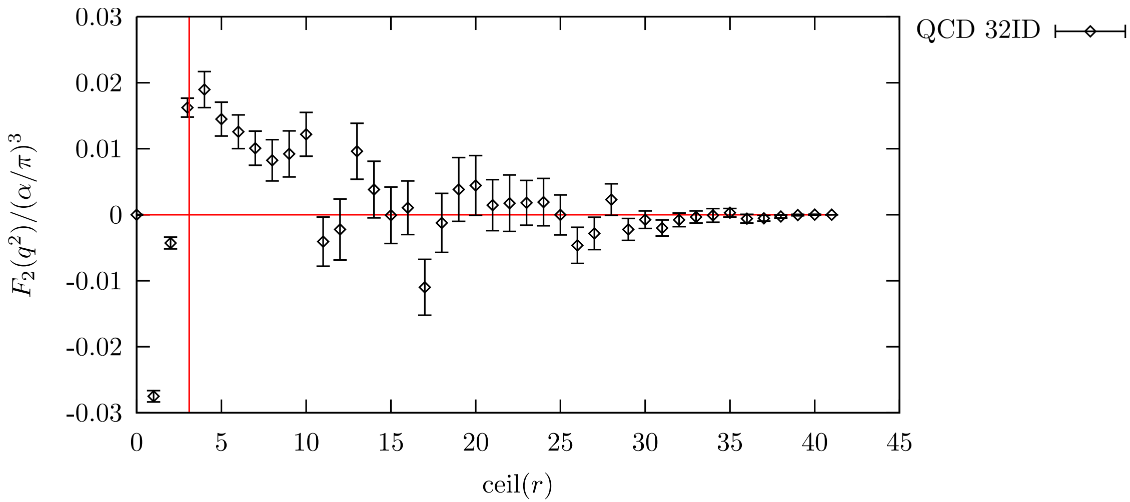


Figure 21. $32^3 \times 64$ lattice, with $a^{-1} = 1.371\mathrm{GeV}$, $m_\pi = 171\mathrm{MeV}$, $m_\mu = 134\mathrm{MeV}$.

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Label	size	$\frac{2\pi}{m_\mu L}$	m_π/GeV	#qcdtraj	t_{sep}	$\frac{F_2 \pm \mathrm{Err}}{(\alpha/\pi)^3}$	$\frac{\mathrm{Cost}}{\mathrm{BG/Q\ rack\ days}}$
16I	$16^3 \times 32$	2.07	0.423	8	16	0.1248 ± 0.0047	0.31
24I	$24^3 \times 64$	1.38	0.423	5	32	0.2436 ± 0.0181	0.85
24I-L	$24^3 \times 64$	1.74	0.333	5	32	0.1676 ± 0.0091	0.85
32ID	$32^3 \times 64$	2.00	0.171	47	32	0.0693 ± 0.0218	10

Table 3. Central values and errors. $a^{-1} = 1.747\mathrm{GeV}$. Muon mass and pion mass ratio is fixed at physical value. #prop is the number of propagator calculated for the loop.

- We plan to follow this with a calculation with a physical pion mass and a $6\mathrm{fm}$ volume with either $a = 0.18\mathrm{fm}$ or $a = 0.12\mathrm{fm}$ depending on the computer time needed.
- Possible strategies for the calculation of all disconnected diagrams are being developed and we hope to begin numerical experiments this year.

Thank You!