$K \to \pi\pi$ decay amplitudes from improved Wilson fermion

at BNL 2015/02/05

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We present our results of $K \to \pi\pi$ decay amplitudes for both $\Delta I = 1/2$ and $3/2$.

- $N_f = 2+1$ improved Wilson fermion

  $a = 0.091\,\mathrm{fm}$, $La = 2.91\mathrm{fm}$

  $m_\pi = 280\mathrm{MeV}$ ($m_K \sim 2 \times m_\pi$)

  Decay process : $K(\mathbf{0}) \to \pi(\mathbf{0})\pi(\mathbf{0})$

1. Introduction

Long-standing problems of the Lattice QCD
in the neutral K meson system :

- Understanding $\Delta I = 1/2$ rule
- Calculation of ϵ'/ϵ from SM. → Precise verification of SM . Possibility of the beyond SM.

We need calculates two decay amplitudes :

$$A_I = \langle K|H|\pi\pi; I\rangle \quad (I = 0, 2)$$

issues :

(1) relation between the amplitude on the lattice (Euclid)
and that in the continuum (Minkowski).

: solved by Lellouch and Lüscher

Lellouch and Lüscher, Math.Phys.219(2001)31.

(2) large fluctuation from the dis-connected diagram.

Calculations has been unsuccessful for a long time.

First calculation of the K decay amplitudes decay for $\Delta I = 1/2$:

RBC-UKQCD

$N_f = 2+1$ Domain wall fermion , $a = 0.114\,\mathrm{fm}$

- $m_\pi = 422\,\mathrm{MeV}$, $La = 1.8\,\mathrm{fm}$ PRD84(2011)114503
- $m_\pi = 330\,\mathrm{MeV}$, $La = 2.7\,\mathrm{fm}$ LAT2011 [arXiv:1110.2143]

For the Wilson fermion, operator renormalization for parity odd part of $\Delta S = 1$ op. :

$$Q_i^{\overline{\mathrm{MS}}}(\mu) = \sum_j Z_{ij}(\mu)\,\overline{Q}_j^{\mathrm{Lat}} \quad (i, j = 1, 2, \cdots 10)$$

$$\overline{Q}_j^{\mathrm{Lat}} = Q_j - \alpha_j P \ , \ \underline{P = \bar{s}\gamma_5 d}$$

P does not give finite contributions in the continuum.
This is not true for the Wilson fermion.

$Z_{ij}(\mu)$: same form as for the chiral sym. preserved case

(from *CPS* symmetry)

The calculation of the amplitudes is also possible with Wilson fermion,
if we subtract the lower dimensional operator with a renormalization condition :

$$\alpha_j = \langle 0|Q_j|K\rangle / \langle 0|P|K\rangle$$

Calculation cost : Wilson fermion << Domain wall fermion

Statistical improvement is expected by using Wilson fermion.

2. Method

Parameter

$N_f = 2+1$ improved Wilson fermion + Iwasaki gauge action

$32^3 \times 64$, $a = 0.091\,\mathrm{fm}$, $La = 2.91\mathrm{fm}$

$m_\pi = 275.7(15)\,\mathrm{MeV}$, $m_K = 579.8(13)\,\mathrm{MeV}$

$m_K \sim 2 \times m_\pi \quad (m_K - 2 \cdot m_\pi = 28.3\,\mathrm{MeV})$

Decay process : $K(\mathbf{0}) \to \pi(\mathbf{0})\pi(\mathbf{0})$

Configurations :

PACS-CS (original) : 2,000 MD step

New : 10,000 MD step

# of conf. = 480 (every 25 MD step)

Our preliminary results
have been presented at Lat2013 and Lat2014.

arXiv:1311.0958, arXiv:1410.8237.

Time correlation function

$$G^I(Q_i)(t) = \frac{1}{T}\sum_{\delta=0}^{T-1} \langle 0|\, W_K(t_K+\delta)\, \overline{Q}_i(t+\delta)\, W^I_{\pi\pi}(t_\pi+\delta)\, |0\rangle$$

(: periodic BC. in time)

$$t_K = 24\ ,\ t_\pi = 0\ ,\ t = \mathrm{run}$$

$$\overline{Q}_i = Q_i - \alpha_i P$$

$$P = \bar{s}\gamma_5 d$$

$$\alpha_i = \langle 0|Q_i|K\rangle / \langle 0|P|K\rangle$$

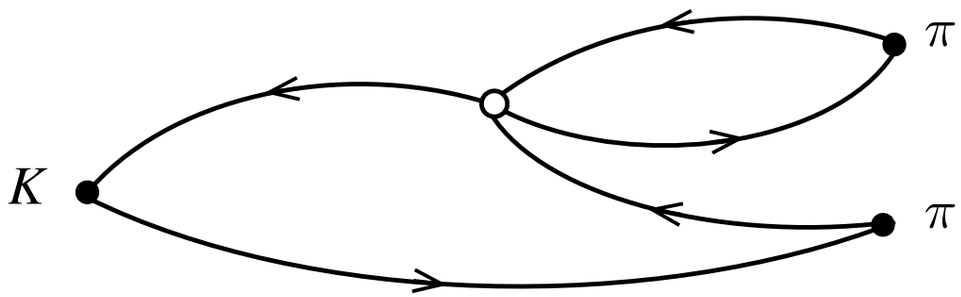
$W_K(t)\,,\, W^I_{\pi\pi}(t)$: Wall source for K and $\pi\pi$ with the iso-spoin I

(: used with Coulomb gauge fixing at time slice of the wall sources)

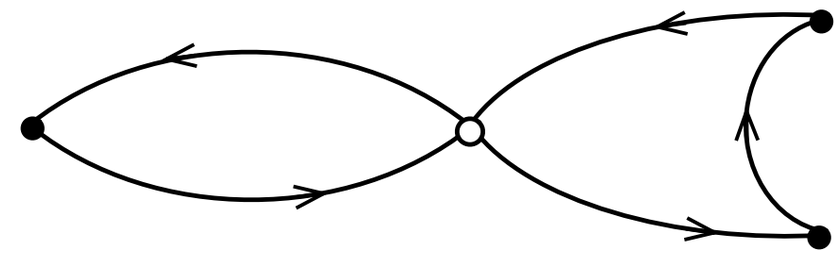
Note our convention : $K^0 = -\bar{s}\gamma_5 d$

Calculation of quark loop at weak operators

Quark contractions :



type1



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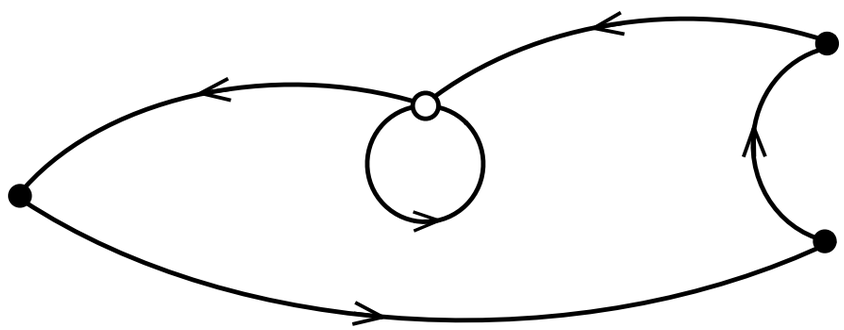

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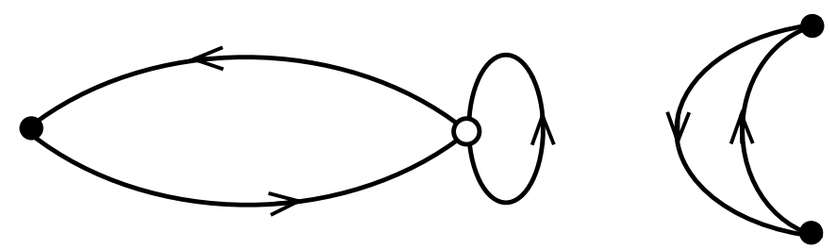


type4

Calculation of Quark loop :

Stochastic method

+ Hopping parameter expansion technique (HPE)

+ Truncated solver method (TSM)

(G.S.Bali et.al, Comp.Phys.Comm. 181(2010)1570.)

Hopping parameter expansion technique (HPE)

Wilson fermion :

$$S^W = \bar{\psi}\, W\, \psi = \bar{\psi}\,(M - D)\,\psi = \bar{\psi}\, M(1 - \overline{D})\,\psi \qquad \left(\overline{D} = M^{-1} D \right)$$

$$(M\psi)(x) = [\,1 - \kappa C_{SW}(\sigma \cdot F(x))/2\,]\,\psi(x)$$

$$(D\psi)(x) = \kappa \sum_{\mu} \left[\, (1-\gamma_\mu) U_\mu(x)\psi(x+\mu) \,+\, (1+\gamma_\mu) U_\mu^\dagger(x-\mu)\psi(x-\mu) \,\right]$$

Quark propagator :

$$Q = W^{-1} = (1-\overline{D})^{-1} M^{-1} = \sum_{n=0}^{\infty} \overline{D}^{\,n} M^{-1} = \sum_{n=0}^{k-1} \overline{D}^{\,n} M^{-1} + \overline{D}^{\,k} W^{-1} \qquad (\text{ for any } k\ )$$

$$= M^{-1} + \overline{D} M^{-1} + \overline{D}^{\,2} M^{-1} + \overline{D}^{\,3} M^{-1} + \overline{D}^{\,4} W^{-1} \qquad (\text{ for } k = 4\ )$$

Quark loop :

$$Q(x,x) = \left[\, M^{-1} + \cancel{\overline{D} M^{-1}} + \overline{D}^{\,2} M^{-1} + \cancel{\overline{D}^{\,3} M^{-1}} + \overline{D}^{\,4} W^{-1} \right] (x,x)$$

$$= \left[\, M^{-1} + \overline{D}^{\,2} M^{-1} + \overline{D}^{\,4} W^{-1} \right] (x,x)$$

Calculation of the quark loop by the stochastic method

$$Q(\mathbf{x},t;\mathbf{x},t) = \frac{1}{N_R} \sum_{i=1}^{N_R} \xi_i^*(\mathbf{x},t)\, S_i(\mathbf{x},t) \qquad \left(\delta^3(\mathbf{x}-\mathbf{y}) = \lim_{N_R \to \infty} \frac{1}{N_R} \sum_{i=1}^{N_R} \xi_i^*(\mathbf{x},t)\, \xi_i(\mathbf{y},t) \right)$$

$$S_i(\mathbf{x},t) = \sum_{\mathbf{y}} \left[\, M^{-1} + \overline{D}^{\,2} M^{-1} + \overline{D}^{\,4} W^{-1} \right] (\mathbf{x},t;\mathbf{y},t)\ \xi_i(\mathbf{y},t)$$

Truncated solver method (TSM)

$$Q(\mathbf{x},t;\mathbf{x},t) = \frac{1}{N_R}\sum_{i=1}^{N_R} \xi_i^*(\mathbf{x},t)\ \left[S_i(\mathbf{x},t) - S_i^T(\mathbf{x},t)\right] + \frac{1}{N_T}\sum_{i=N_R+1}^{N_T+N_R} \xi_i^*(\mathbf{x},t)\, S_i^T(\mathbf{x},t)$$

$$S_i(\mathbf{x},t) = \sum_{\mathbf{y}} \left[\ M^{-1} + \overline{D}^{\,2} M^{-1} + \overline{D}^{\,4} W^{-1}\right](\mathbf{x},t;\mathbf{y},t)\ \xi_i(\mathbf{y},t)$$

$S_i^T(\mathbf{x},t)$: with W^{-1} calculated with a loose stopping condition

$N_R = 1$ tor. $< 10^{-14}$ (tor. $= |WW^{-1} - \xi|/|\xi|$)

$N_T = 5$ tor. $< 1.2 \times 10^{-6}$

2. Results

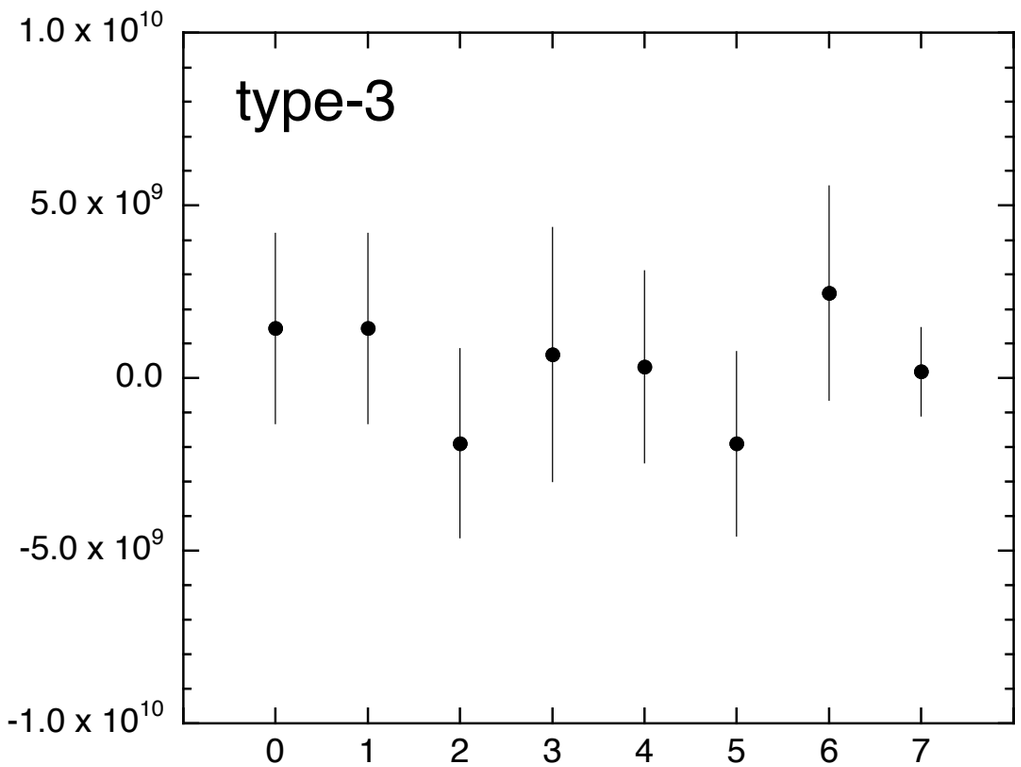
Effect of TSM

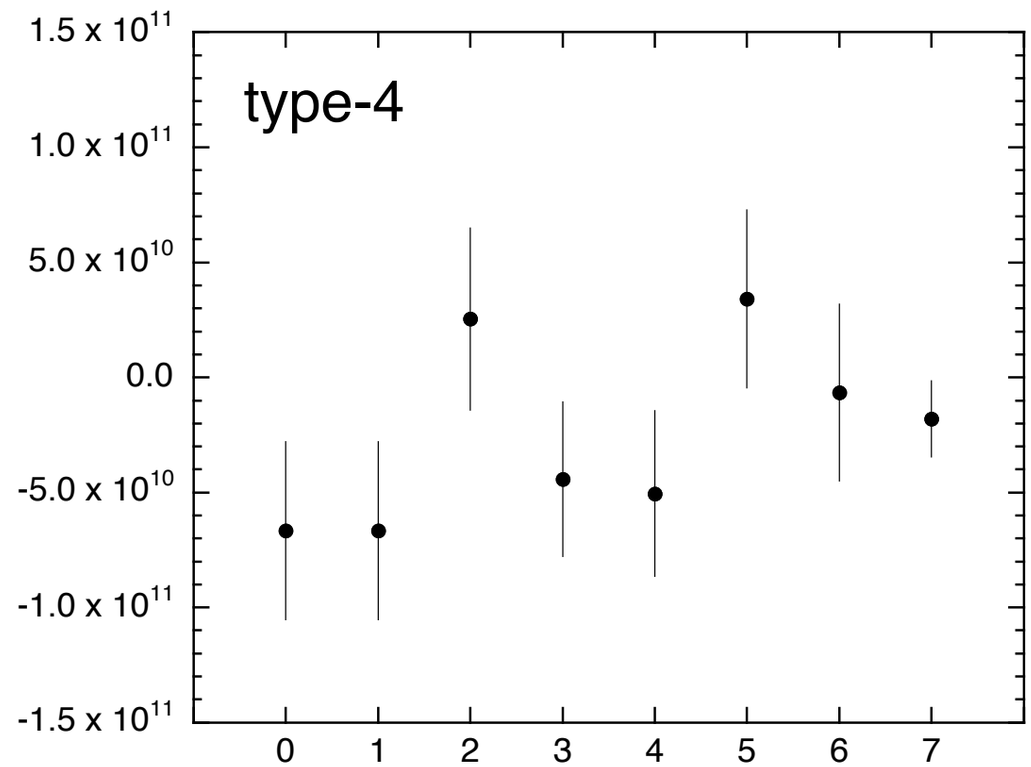
$$Q(\mathbf{x},t;\mathbf{x},t) = \frac{1}{N_R}\sum_{i=1}^{N_R} \xi_i^*(\mathbf{x},t)\ \left[S_i(\mathbf{x},t) - S_i^T(\mathbf{x},t)\right] + \frac{1}{N_T}\sum_{i=N_R+1}^{N_T+N_R} \xi_i^*(\mathbf{x},t)\, S_i^T(\mathbf{x},t)$$

$$(\ N_R = 1\ ,\ N_T = 5\ )$$

$$S_i(\mathbf{x},t) = \sum_{\mathbf{y}} \left[\ M^{-1} + \overline{D}^{\,2} M^{-1} + \overline{D}^{\,4} W^{-1}\right](\mathbf{x},t;\mathbf{y},t)\ \xi_i(\mathbf{y},t)$$

$G^{I=0}(Q_2)$ at $t = 9$





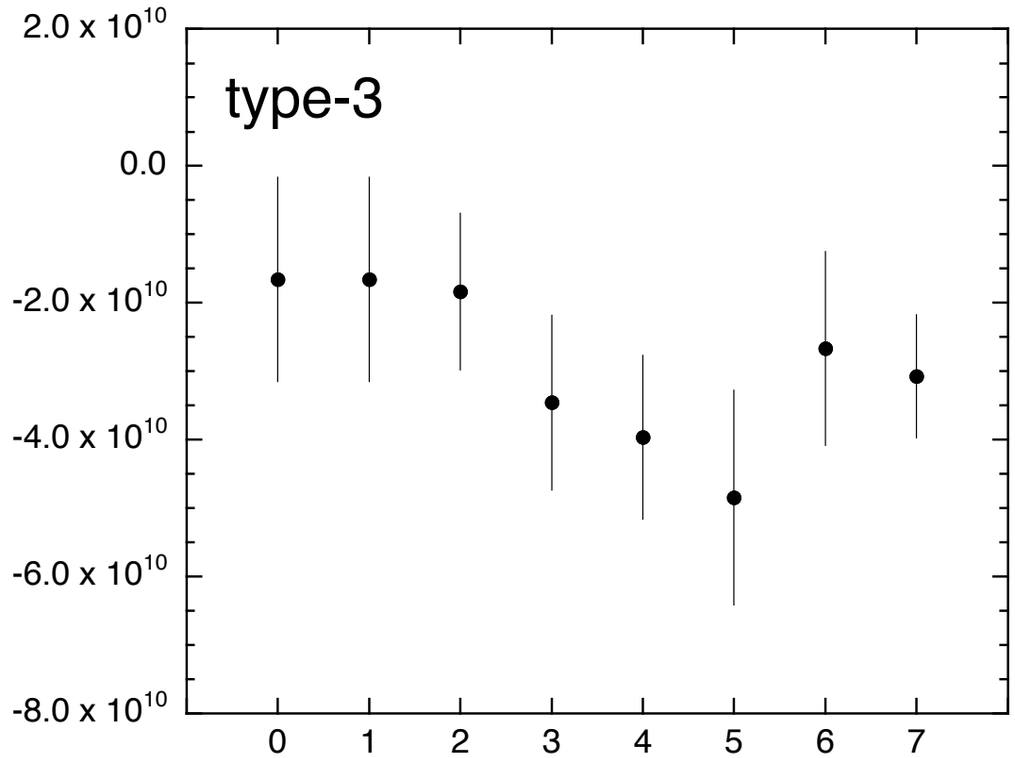
$$x = 0\ :\quad Q(\mathbf{x},t;\mathbf{x},t) = \xi_i^*(\mathbf{x},t) S_i(\mathbf{x},t) \qquad \text{for } i = 1$$

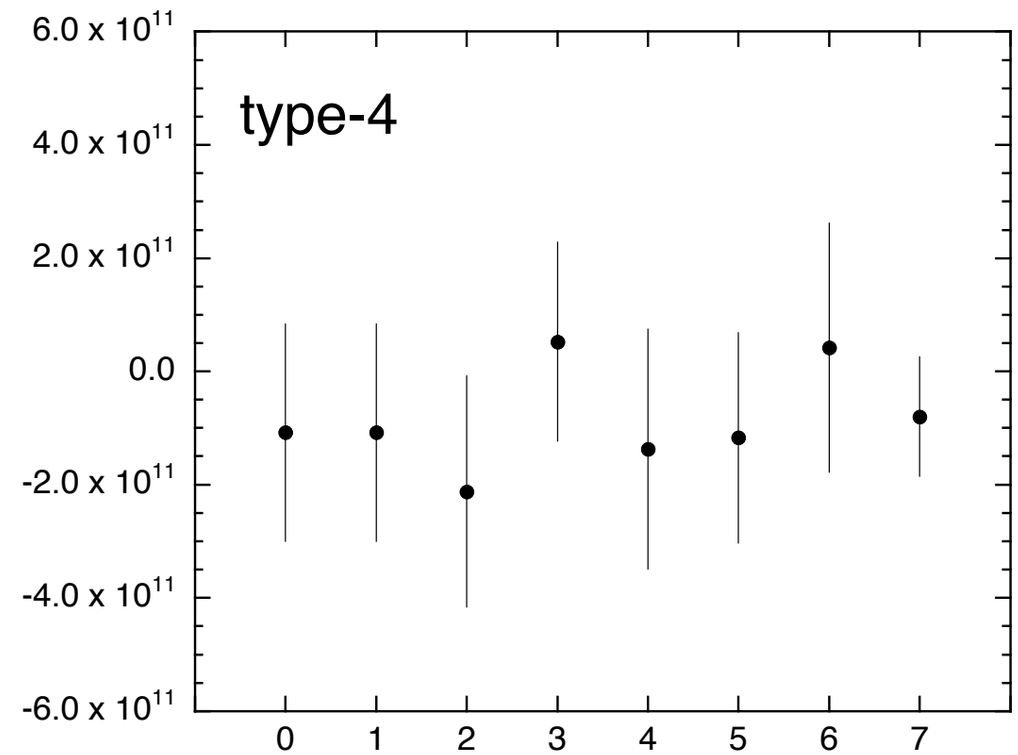
$$x = 1\ :\quad Q(\mathbf{x},t;\mathbf{x},t) = \xi_i^*(\mathbf{x},t) S_i^T(\mathbf{x},t) \qquad \text{for } i = 1$$

$$x = 2\cdots 6\ :\quad Q(\mathbf{x},t;\mathbf{x},t) = \xi_i^*(\mathbf{x},t) S_i^T(\mathbf{x},t) \qquad \text{for } i = 2\cdots 6$$

$$x = 7\ :\quad Q(\mathbf{x},t;\mathbf{x},t) = \frac{1}{6}\sum_{i=1}^{6} \xi_i^*(\mathbf{x},t) S_i^T(\mathbf{x},t)$$

$G^{I=0}(Q_6)$ at $t = 9$



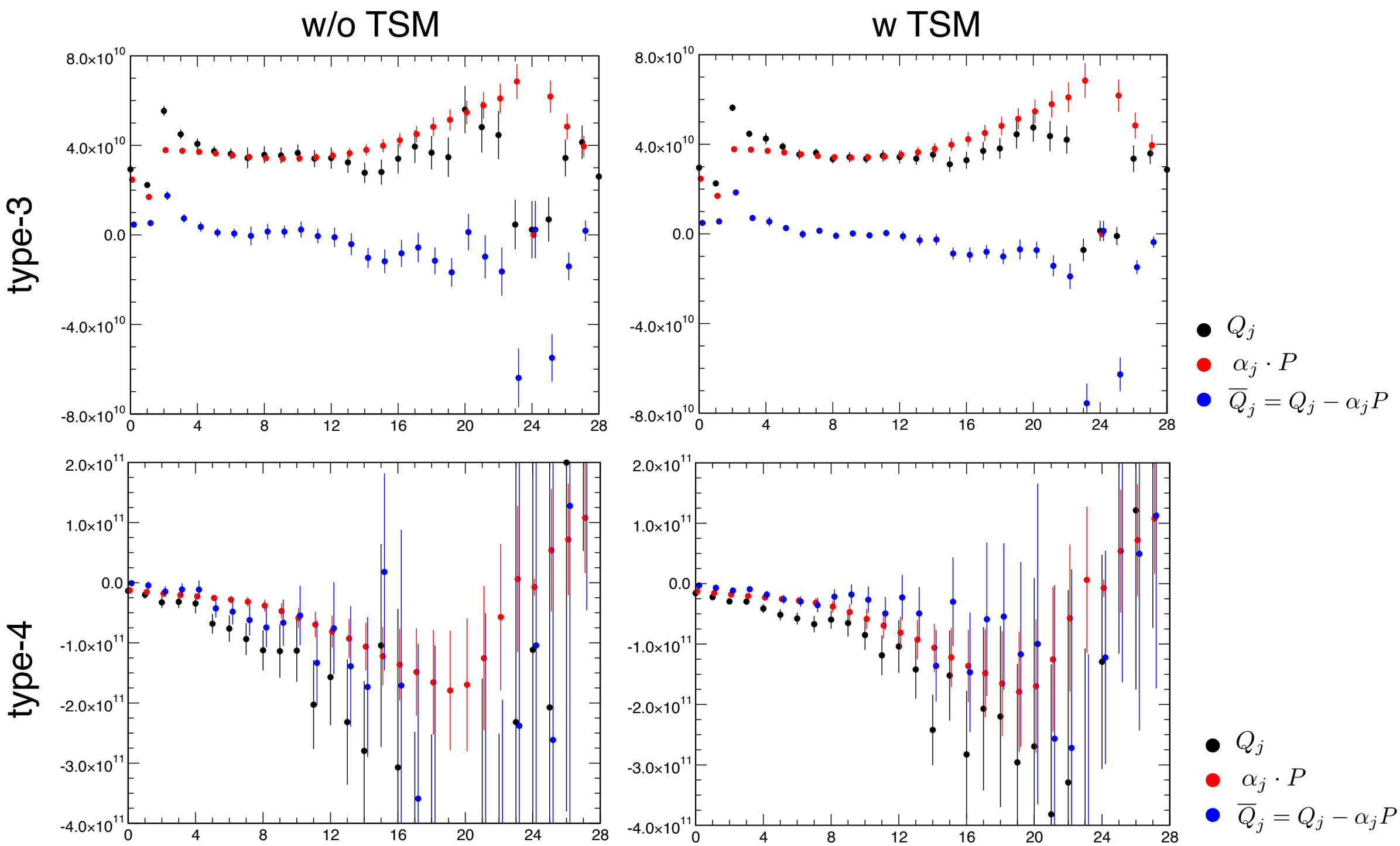


$$
\begin{aligned}
x = 0 \;&:\quad Q(\mathbf{x},t;\mathbf{x},t) = \xi_i^*(\mathbf{x},t) S_i(\mathbf{x},t) \qquad \text{for } i = 1 \\
x = 1 \;&:\quad Q(\mathbf{x},t;\mathbf{x},t) = \xi_i^*(\mathbf{x},t) S_i^T(\mathbf{x},t) \qquad \text{for } i = 1 \\
x = 2\cdots 6 \;&:\quad Q(\mathbf{x},t;\mathbf{x},t) = \xi_i^*(\mathbf{x},t) S_i^T(\mathbf{x},t) \qquad \text{for } i = 2\cdots 6 \\
x = 7 \;&:\quad Q(\mathbf{x},t;\mathbf{x},t) = \frac{1}{6}\sum_{i=1}^{6} \xi_i^*(\mathbf{x},t) S_i^T(\mathbf{x},t)
\end{aligned}
$$

The correction terms are negligible.

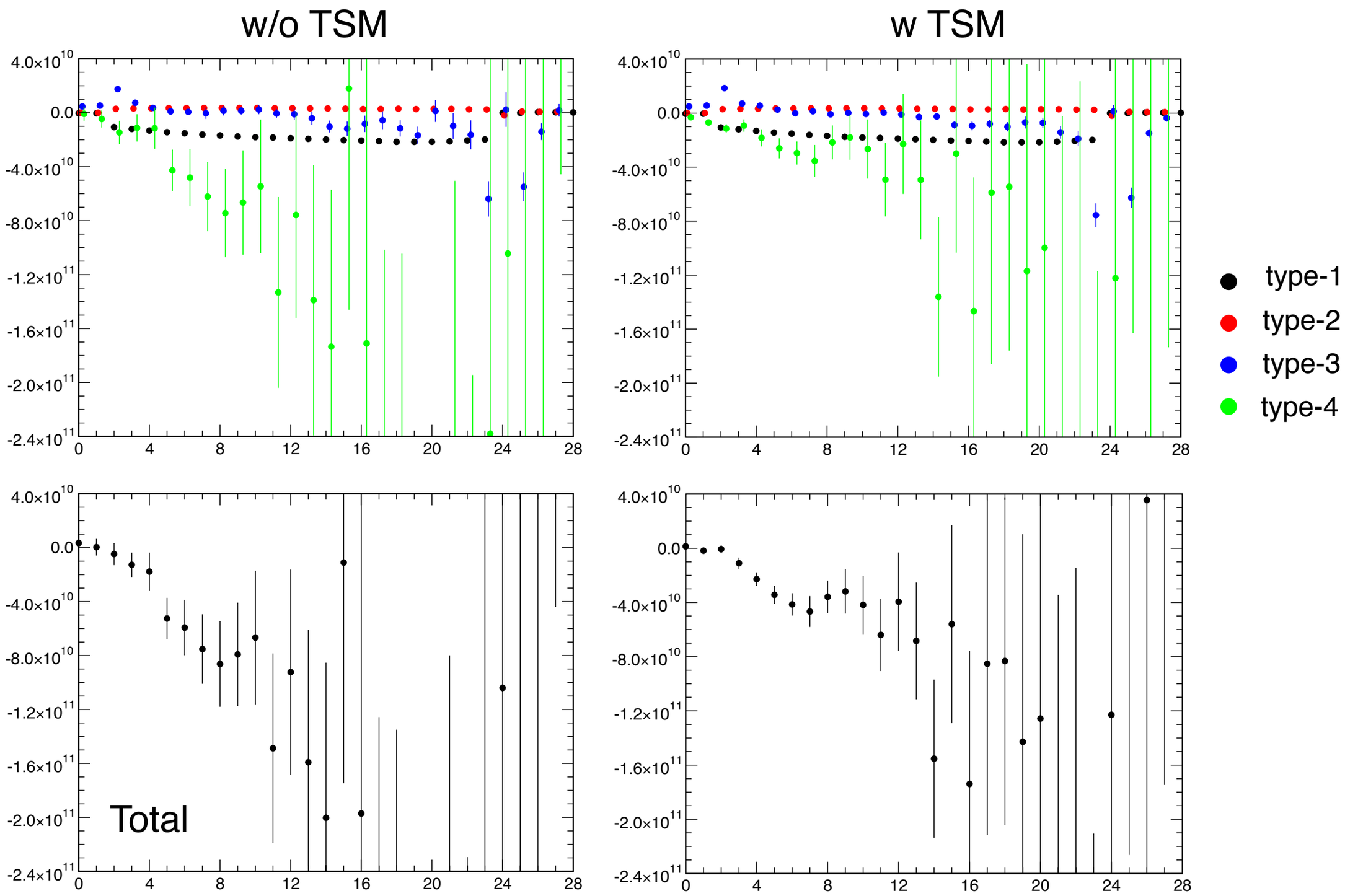
We use x=7 for calculations of the decay amplitudes.

$G^{I=0}(Q_2)$ from type-3 and type-4 $(\, t_K = 24\,,\, t_\pi = 0\,,\, Q(t) : \mathrm{run}\,)$



TSM reduces the statistical error

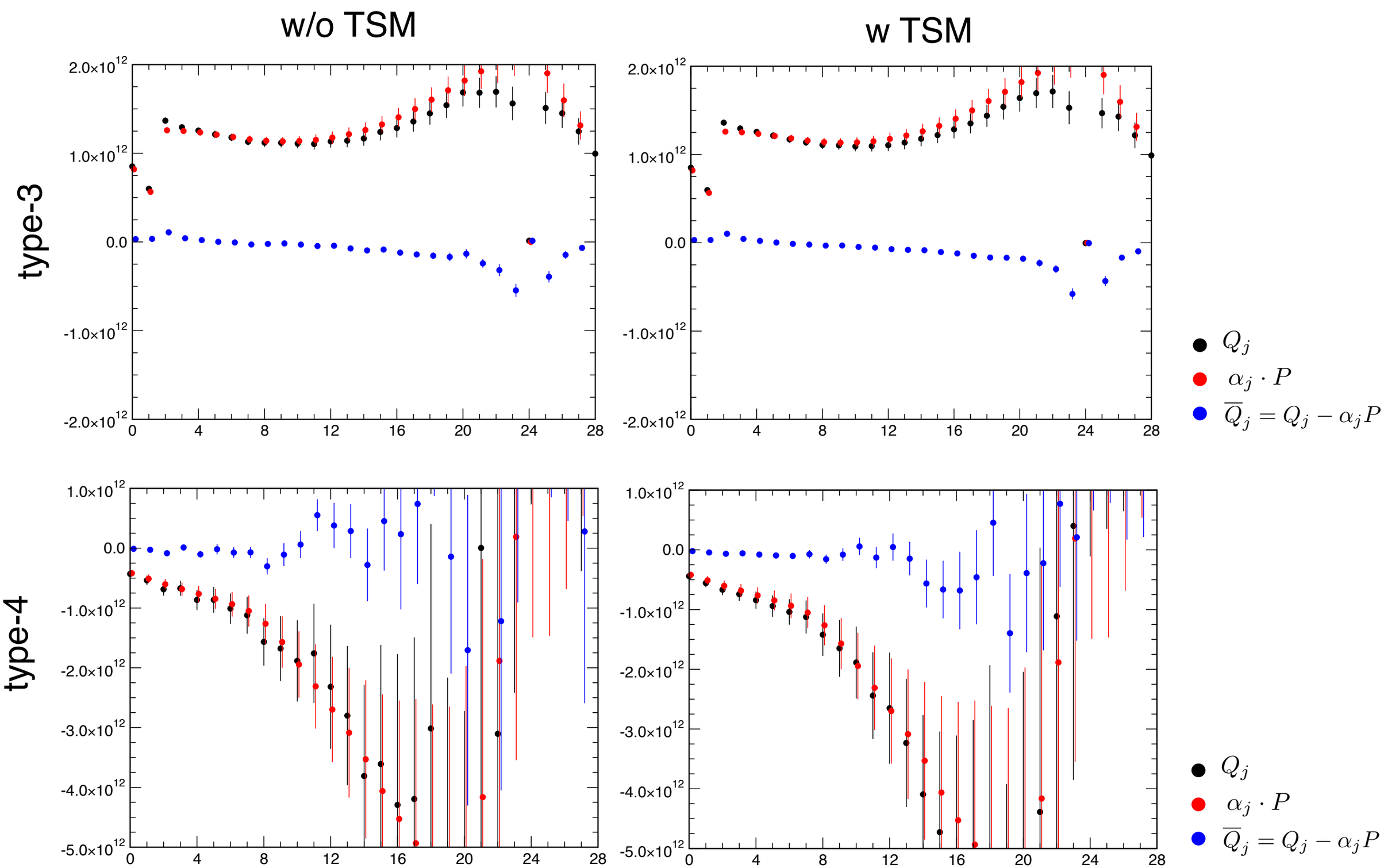
$G^{I=0}(Q_2)$ $(t_K = 24 , t_\pi = 0 , Q(t) : \mathrm{run})$



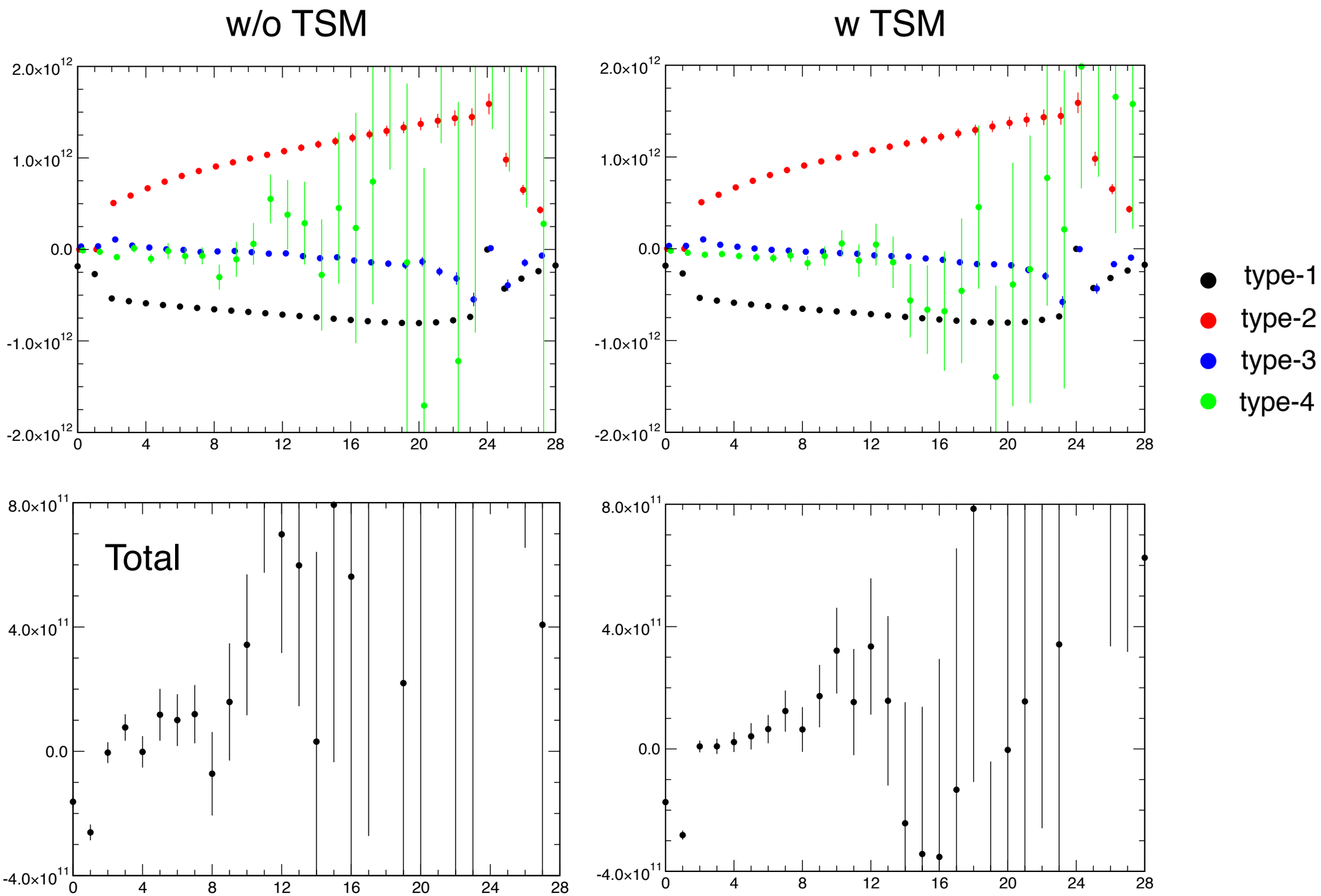
type-4 is large !!

type-4 ~ type-1

$G^{I=0}(Q_6)$ from type-3 and type-4 $(\, t_K = 24\,,\, t_\pi = 0\,,\, Q(t) : \mathrm{run}\,)$

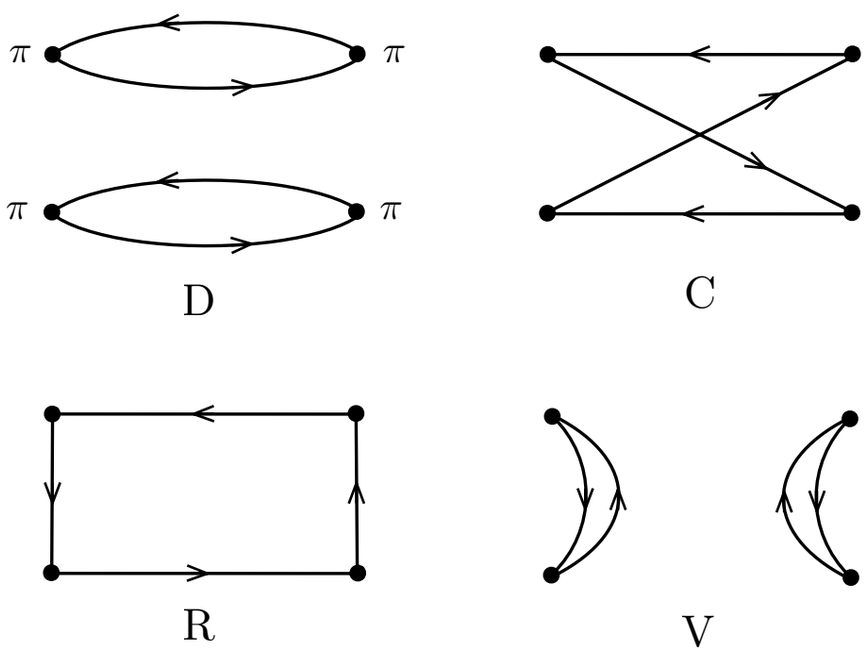


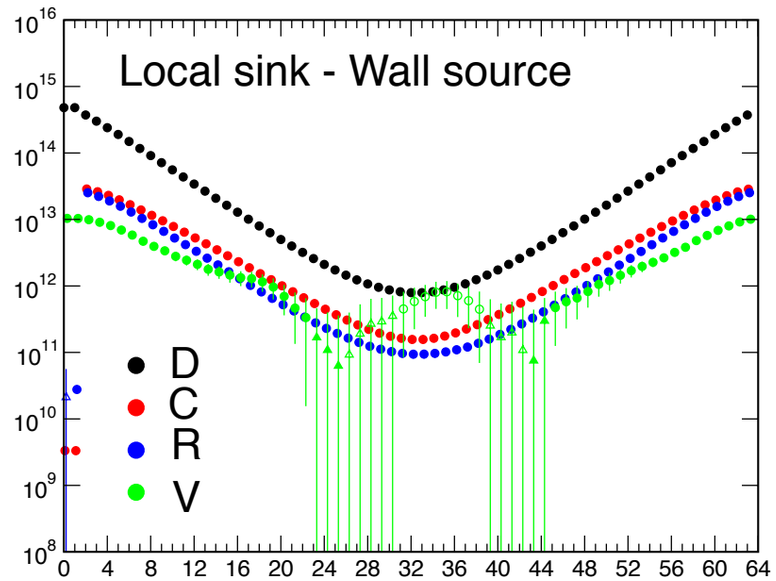
$G^{I=0}(Q_6)$ ($t_K = 24$, $t_\pi = 0$, $Q(t)$: run)

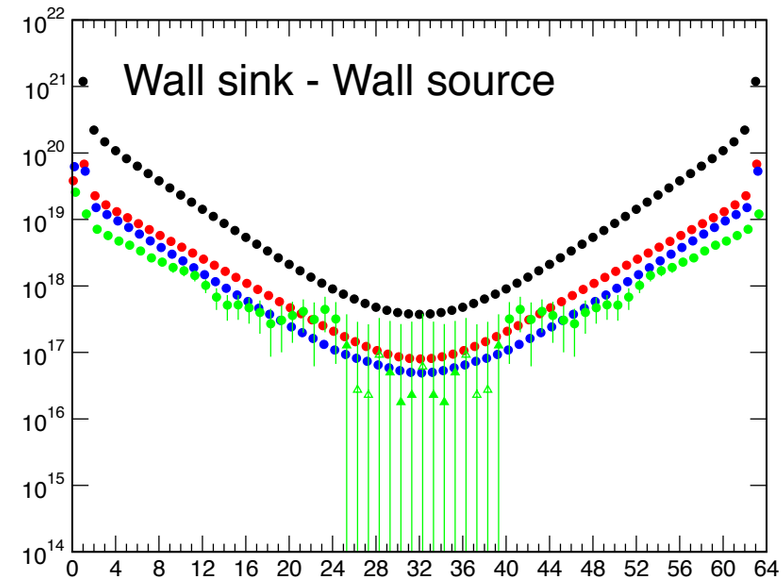


Large cancelation between type-1 and type-2.

Time correlation function of $\pi\pi \to \pi\pi$



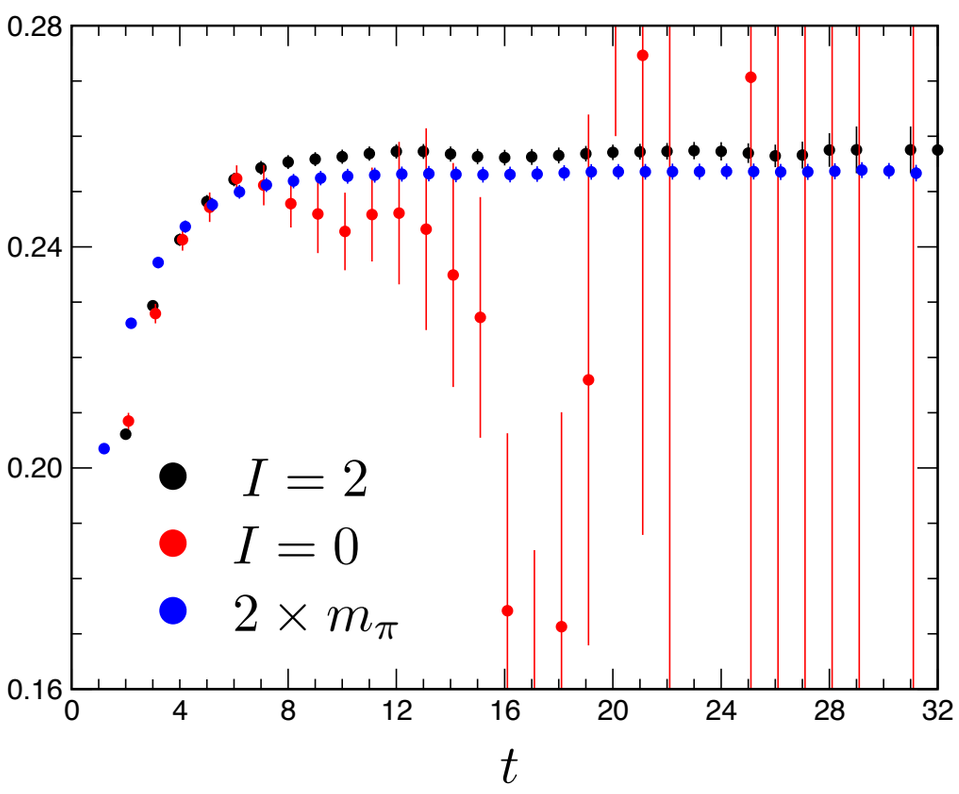




$$G(t) = A^I \cdot \left(\mathrm{e}^{-E^I_{\pi\pi} t} + \mathrm{e}^{-E^T_{\pi\pi}(T-t)} \right) + C^I$$

C^I : around-the-world effect

Effective mass of LW



$2 \times m_\pi = 0.2535(14)$

$E^{I=2}_{\pi\pi} = 0.2567(14)$ (from LW)

$N^{I=2}_{\pi\pi} = \langle 0 | W^{I=2}_{\pi\pi} | \pi\pi; I=2 \rangle = 1.5852(85) \times 10^{10}$

$t = [9, 32]$ (from WW)

$E^{I=0}_{\pi\pi} = 0.2499(83)$ (from LW)

$N^{I=0}_{\pi\pi} = \langle 0 | W^{I=0}_{\pi\pi} | \pi\pi; I=0 \rangle = 1.552(42) \times 10^{10}$

$t = [9, 12]$ (from WW)

Extraction of the amplitudes

Effective amplitude :

$$M^I(Q_i)(t) = G^I(Q_i)(t) \cdot F^I_{LL}/(N_K N^I_{\pi\pi}) \cdot \mathrm{e}^{m_K(t_K - t) + E^I_{\pi\pi}(t - t_\pi)} \times \underline{(-1)}$$

(for our convention of $K^0 = -\bar{s}\gamma_5 d$)

$$\longrightarrow \quad M^I(Q_i) = \langle K | \overline{Q}_i | \pi\pi; I \rangle \quad \text{for } t_K \gg t \gg t_\pi$$

$$(\ t_K = 24 \, , \, t_\pi = 0 \, , \, t : \text{run} \ )$$

$$N_K = \langle 0 | W_K | K \rangle \qquad E^I_{\pi\pi} \ : \ \text{energy of } |\pi\pi; I\rangle$$

$$N^I_{\pi\pi} = \langle 0 | W^I_{\pi\pi} | \pi\pi; I \rangle$$ (from K and $\pi\pi$ correlation function)

Lellouch - Lüscher factor :

$$(F^I_{LL})^2 = \langle K | K \rangle \cdot \langle \pi\pi; I | \pi\pi; I \rangle / V^2$$

$$= (4\pi) \left(\frac{(E^I_{\pi\pi})^2 m_K}{p^3} \right) \left(p \frac{\partial \delta^I(p)}{\partial p} + q \frac{\partial \phi(q)}{\partial q} \right)$$

$$E^I_{\pi\pi} = 2\sqrt{p^2 + m_\pi^2}$$

$$q = pL/(2\pi)$$

$$\tan \phi(q) = -\pi^{3/2} q / Z_{00}(1 : q^2)$$

$$Z_{00}(s : q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n} \in \mathbb{Z}^3} (n^2 - q^2)^{-s}$$

for *I=0* stat. of SC. phase is not enough.

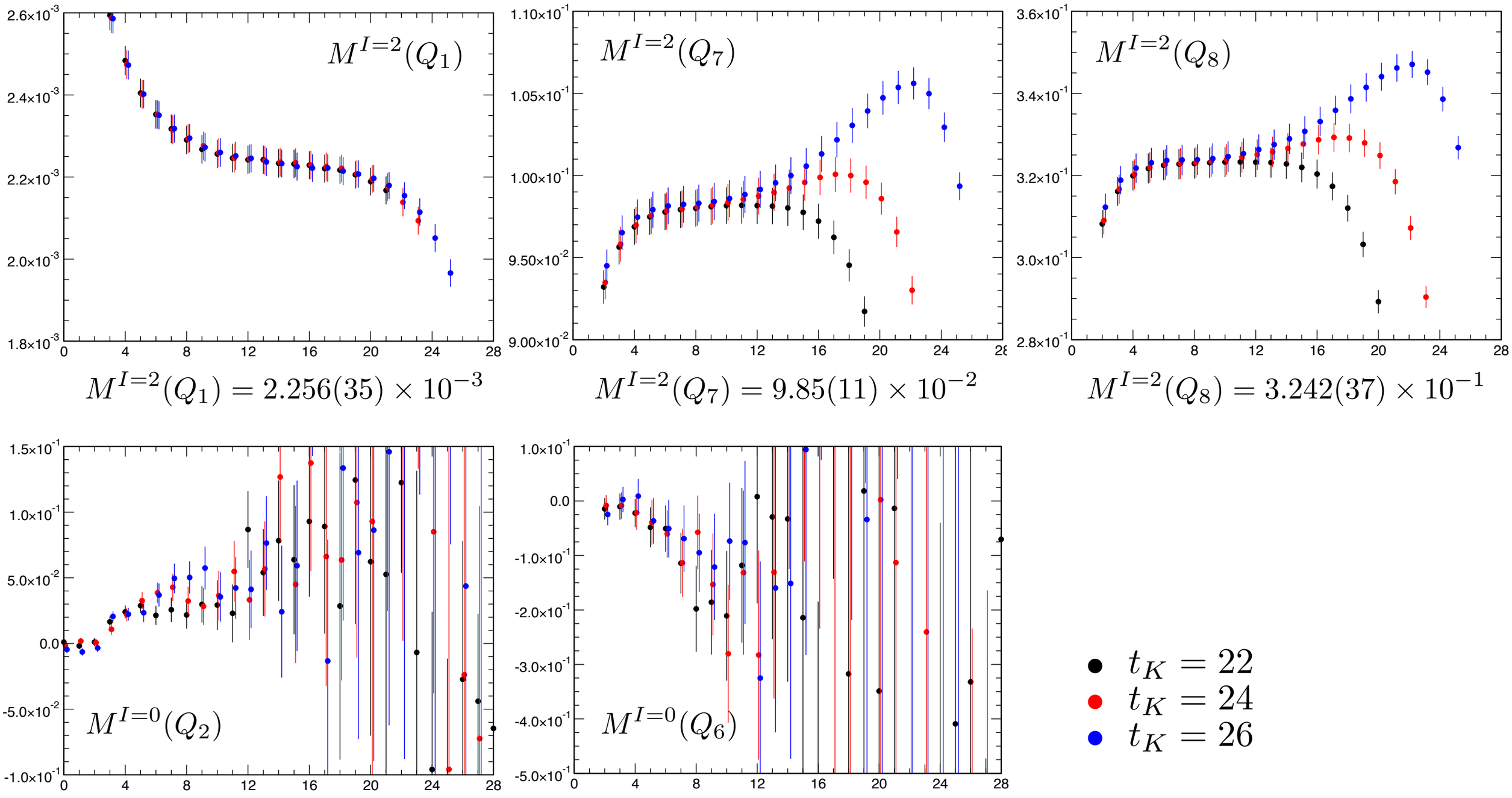
the factor for non-interacting case is used.

for *I=2* the factor is estimated, neglecting the cubic term of

$$\delta^I(p) = a \cdot p + \mathrm{O}(p^3)$$

$$F^{I=2}_{LL} / F^{\text{free}}_{LL} = 0.9254(64)$$

Effective amplitudes $(t_K = 22, 24, 26 \, , \, t_\pi = 0 \, , \, Q(t) : \mathrm{run} \,)$



$M^{I=2}(Q_1) = 2.256(35) \times 10^{-3}$ $\quad$ $M^{I=2}(Q_7) = 9.85(11) \times 10^{-2}$ $\quad$ $M^{I=2}(Q_8) = 3.242(37) \times 10^{-1}$

$M^{I=0}(Q_2) = 3.55(1.43) \times 10^{-2}$ $\quad$ $M^{I=0}(Q_6) = -1.96(1.06) \times 10^{-1}$

The around-the-world effect for two pion state can be avoided for the time range t=[9,12].

Physical decay amplitudes

From the lattice to the continuum :

$$Q_i^{\overline{\rm MS}}(\mu) = \sum_j Z_{ij}(\mu)\, \overline{Q}_j^{\rm Lat} \qquad (\ i,j = 1,2,\cdots 10\ )$$

with perturbative renormalization factor (1 loop).

Y. Taniguchi, JHEP04(2012)143.

matching point : $\mu = 1/a$

(also $\mu = \pi/a$ to estimate higher order effect)

Coefficient function : G. Bychalla, A.J.Buras, M.E. Lautenbacher, RMP 68(1996)125.

$$H = \sum_i C_i(\mu) Q_i^{\overline{\rm MS}}(\mu) = \frac{G_F}{\sqrt{2}} V_{ud}^* V_{us} \sum_i \left(z_i(\mu) + \tau y_i(\mu)\right) Q_i^{\overline{\rm MS}}(\mu)$$

Physical decay amplitudes :

$$A_I = \langle K|H|\pi\pi; I\rangle = \sum_{ij} C_i(\mu) Z_{ij}(\mu) M^I(Q_j)$$

$M^I(Q_i)$: matrix element on the lattice

Physical decay amplitudes

	$\mu = 1/a$	$\mu = \pi/a$	RBC-UKQCD		Exp
$a\,(\mathrm{fm})$	0.091		0.114	0.114	
$m_\pi\,(\mathrm{MeV})$	280		330	422	140
$\mathrm{ReA}_2\,(\times 10^{-8}\,\mathrm{GeV})$	2.426(38)	2.460(38)	2.668(14)	4.911(31)	1.479(4)
$\mathrm{ReA}_0\,(\times 10^{-8}\,\mathrm{GeV})$	60(36)	56(32)	31.1(45)	38.0(82)	33.2(2)
$\mathrm{Re}A_0/\mathrm{Re}A_2$	25(15)	23(13)	12.0(17)	7.7(17)	22.45(6)
$\mathrm{ImA}_2\,(\times 10^{-12}\,\mathrm{GeV})$	−1.14(13)	−0.7457(83)	−0.6509(34)	−0.5502(40)	
$\mathrm{ImA}_0\,(\times 10^{-12}\,\mathrm{GeV})$	−67(56)	−52(48)	−33(15)	−25(22)	
$\mathrm{Re}(\epsilon'/\epsilon)\,(\times 10^{-3})$	0.8(25)	0.9(25)	2.0(17)	2.7(26)	1.66(23)

(used $|\epsilon^{\mathrm{EXP}}| = 2.22 \times 10^{-3}$)

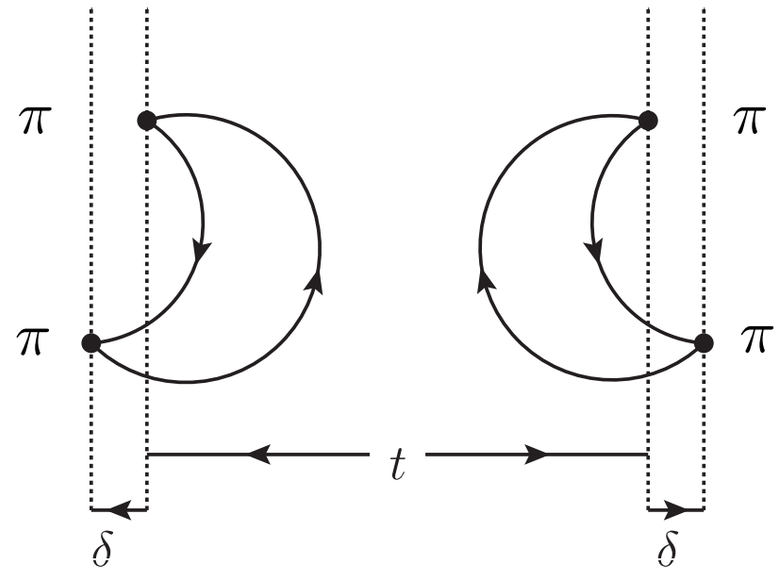
- Stat. error of our A_0 is much larger than those of RBC-UKQCD at $m_\pi = 330\,\mathrm{MeV}$

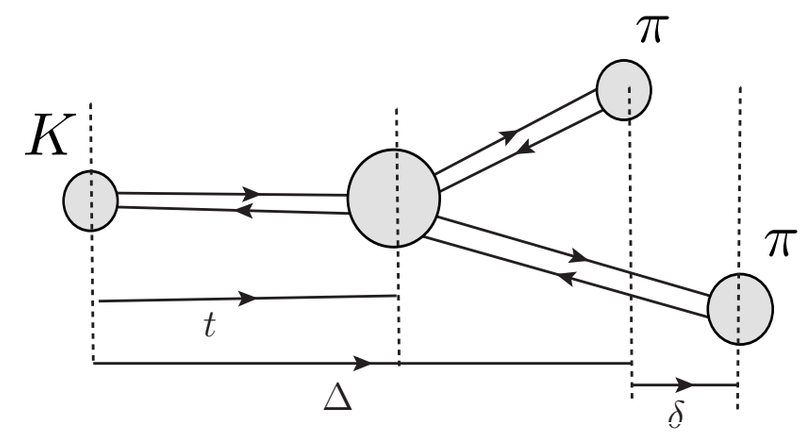
  They used a different two-pion operator from ours
  and set the fitting range closer to the operator.

- Matching point dependence is very large for ImA_2 .
- Enhancement of $\Delta I = 1/2$ process is seen.
- Further improvement of statistics is necessary for ϵ'/ϵ .

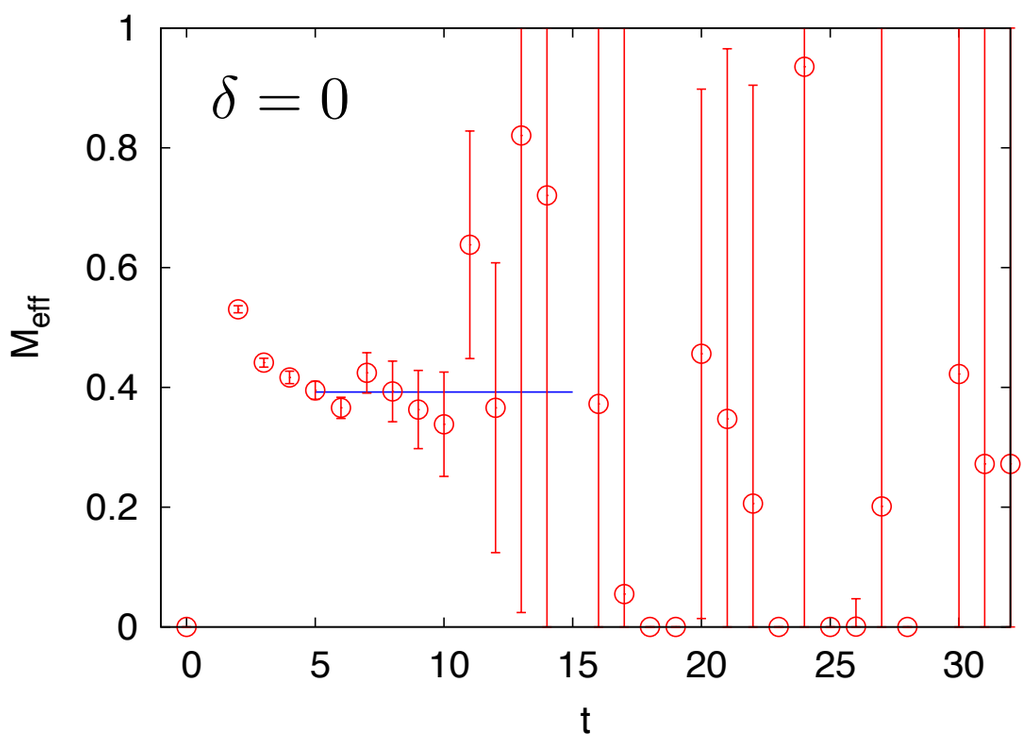
arXiv:1110.2143 (RBC-UKQCD)

$m_\pi = 330\,\mathrm{MeV}$

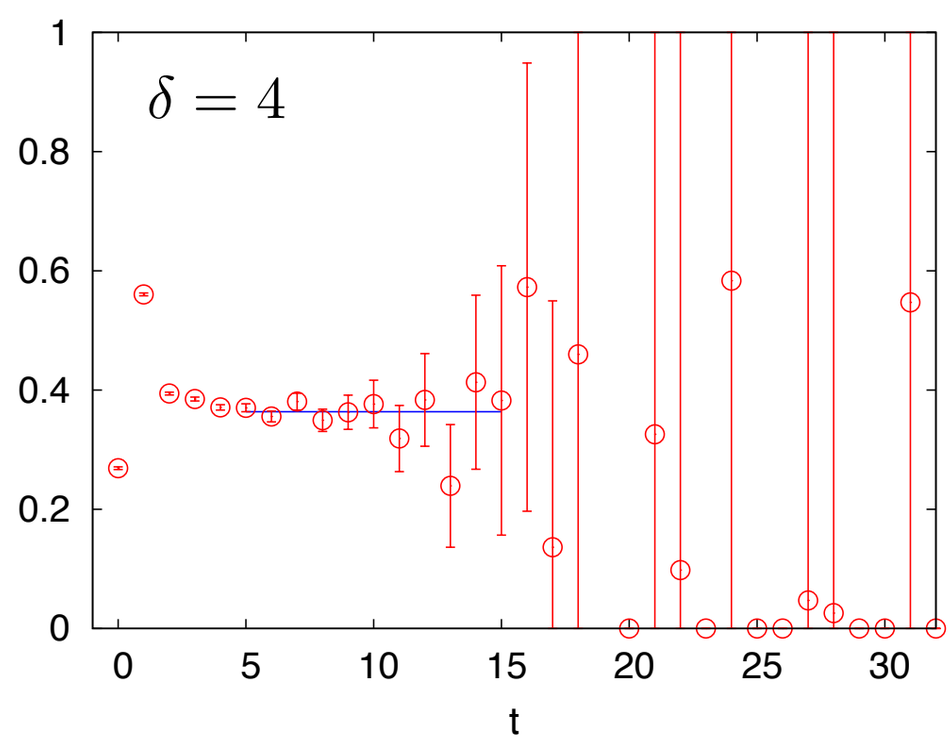




effective mass of $\pi\pi \to \pi\pi \ (I=0)$



$E = 0.3922(126)$



$E = 0.3639(55)$

Error is reduced by taking non-zero δ .

Reason ?

Contribution of the matrix element on the lattice

$\mu = 1/a$

i	$\mathrm{Re}A_2$ (GeV)	$\mathrm{Im}e A_2$ (GeV)
1	$-1.887(29) \times 10^{-08}$	0
2	$4.330(68) \times 10^{-08}$	0
7	$1.053(12) \times 10^{-10}$	$2.772(32) \times 10^{-13}$
8	$-2.722(31) \times 10^{-10}$	$-1.670(19) \times 10^{-12}$
9	$-1.140(18) \times 10^{-12}$	$3.762(59) \times 10^{-13}$
10	$3.771(59) \times 10^{-10}$	$-1.756(27) \times 10^{-13}$
Total	$2.426(38) \times 10^{-08}$	$-1.192(14) \times 10^{-12}$

i	$\mathrm{Re}A_0$ (GeV)	$\mathrm{Im}e A_0$ (GeV)
1	$-4(11) \times 10^{-08}$	0
2	$6.8(28) \times 10^{-07}$	0
3	$-1.25(65) \times 10^{-08}$	$-2.5(13) \times 10^{-11}$
4	$5.3(20) \times 10^{-08}$	$6.6(25) \times 10^{-11}$
5	$1.5(59) \times 10^{-09}$	$1.7(68) \times 10^{-12}$
6	$-8.4(46) \times 10^{-08}$	$-1.03(56) \times 10^{-10}$
7	$2.58(19) \times 10^{-10}$	$6.81(50) \times 10^{-13}$
8	$-6.26(45) \times 10^{-10}$	$-3.84(28) \times 10^{-12}$
9	$1.02(48) \times 10^{-11}$	$-3.4(16) \times 10^{-12}$
10	$0.0(14) \times 10^{-11}$	$-0.1(64) \times 10^{-13}$
Total	$6.0(36) \times 10^{-07}$	$-6.7(56) \times 10^{-11}$

$$A_I = \langle K|H|\pi\pi; I\rangle \ = \sum_i \bar{A}_i$$

$$\bar{A}_i = \sum_j C_j(\mu) Z_{ji}(\mu) M^I(Q_i)$$

$M^I(Q_i)$: bare matrix element on the lattice

5. Summary

We calculate $K \to \pi\pi$ decay amplitudes
for the process $K(\mathbf{0}) \to \pi(\mathbf{0})\pi(\mathbf{0})$ at $m_\pi = 280\mathrm{MeV}$ ($m_K \sim 2 \times m_\pi$)

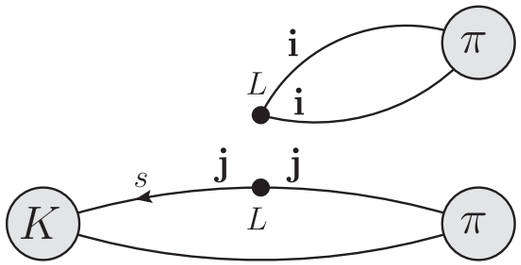
- $N_f = 2+1$ improved Wilson fermion with
  Non-perturbative subtraction of the lower dimensional operator .
- Calculation of quark loop by
  Stochastic method with HPE and TSM

We found :

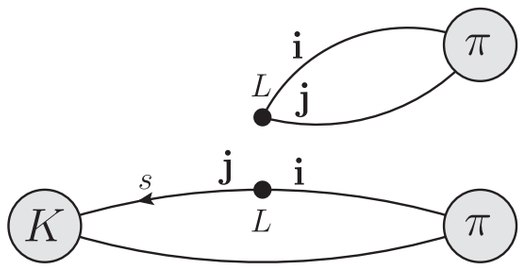
- TSM is an efficient method.
- For Q_2 , the contribution of type-4 (OZI-suppression diag.) is large.
  type-4 ~ type-1 (: Wilson fermion ?)
- Stat. error of our A_0 is much larger than those of RBC-UKQCD.
  Improvement of the *K* and *ππ* operator is necessary.
- Matching point dependence is very large for ImA_2 .
  Non-perturbative renormalization factor is needed.
- Enhancement of $\Delta I = 1/2$ process is seen.
  $\mathrm{Re}A_0/\mathrm{Re}A_2 = 25 \pm 15$
- Further improvement of statistics is necessary for ϵ'/ϵ .
  Improvement of the *K* and *ππ* operator is necessary.

Back-up

Comparison with PRL 110,152001(2013) by RBC-UKQCD



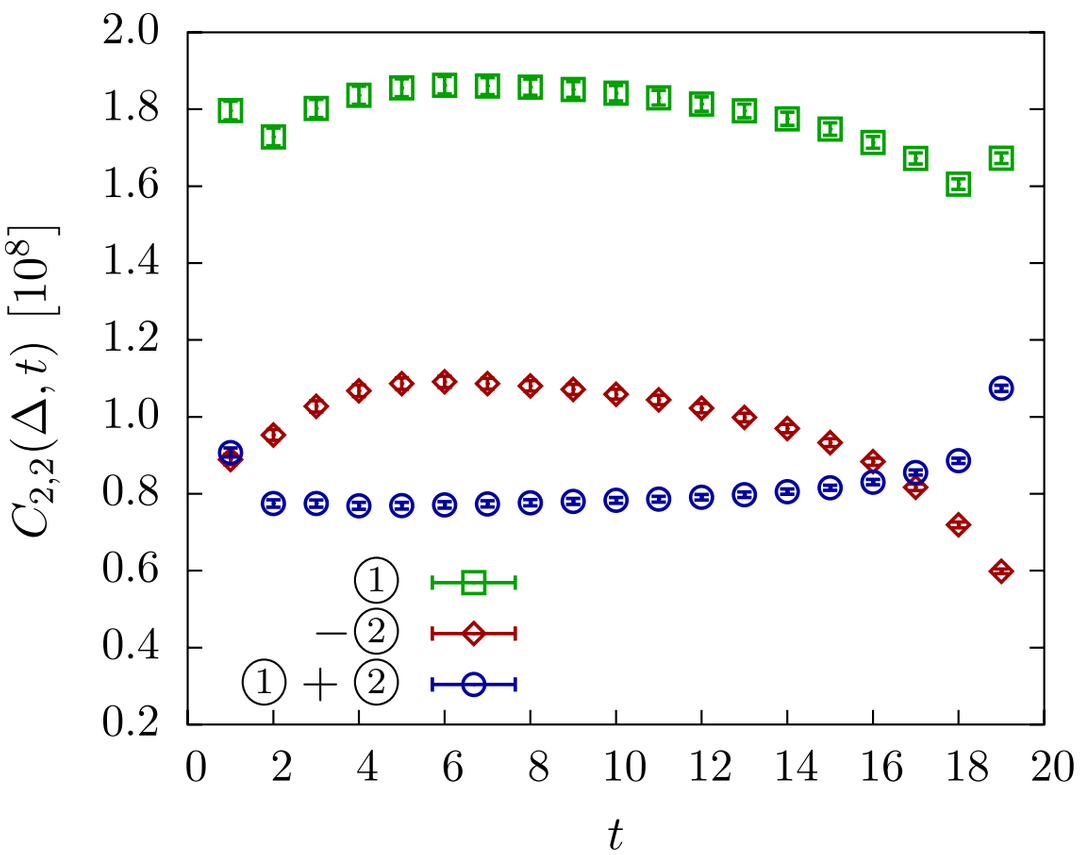
Contraction ①.



Contraction ②.

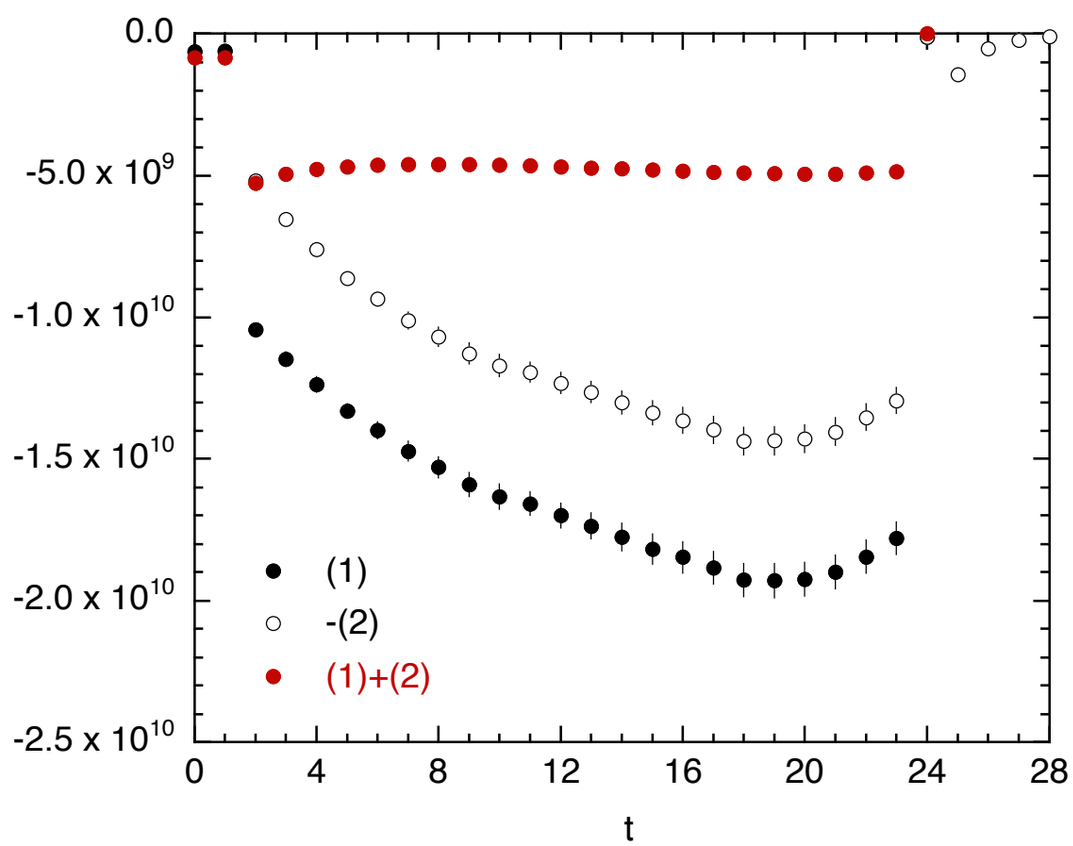
$$\mathrm{Re}A_2 = (\ (1)\ +\ (2)\ )$$
$$\mathrm{Re}A_0 = (\ 2\times(1)\ -\ (2)\ )$$

RBC-UKQCD



$m_\pi = 330\,\mathrm{MeV}$
$t_K = 0\,,\ t_\pi = 20$

Ours



$m_\pi = 280\,\mathrm{MeV}$
$t_\pi = 0\,,\ t_K = 24$

Note our convention : $K^0 = -\bar{s}\gamma_5 d$