Discrete aspects of continuous symmetries in the tensorial formulation of Abelian gauge theories

Yannick Meurice

The University of Iowa

yannick-meurice@uiowa.edu

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- Why we need to learn Tensor Field Theory (TrFT)
- The Compact Abelian Higgs Model
- Discrete reformulation of Noether's theorem
- Transfer matrix and Hamiltonian
- Topological solutions (if time permits)
- Conclusions

References:

YM, arXiv:2003.10986, PRD 102 014506 and PRD 100 014506.

YM, R. Sakai, and J. Unmuth-Yockey, Tensor field theory with applications to quantum computing, arXiv:2010.06539; subm. to RMP.

YM, Quantum field theory: a quantum computation approach, IoP book submitted.
Contain refs. to work by Savit, Itzykson, Kogut, ...

TrFT: From compact to discrete (O(2) example)

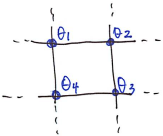
O(2) model

$$Z = \prod_x \int_{-\pi}^{\pi} d\theta_x \; e^{\beta \sum_{\langle xy\rangle} \cos(\theta_x-\theta_y)} = I_0^V(\beta)\, \mathrm{Tr}_{\{n,m\ldots\}} \prod_x T^{(x)}_{n_x n'_x m_x m'_x}$$

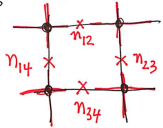
↳ nearest neighbor

$$e^{\beta\cos(\theta_x-\theta_y)} = \sum_{n_{xy}} e^{i n_{xy}(\theta_x-\theta_y)}\, I_{n_{xy}}(\beta)$$

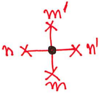
$$\prod_x \int_{-\pi}^{\pi} d\theta_x \longrightarrow \sum_{\{n\}}$$



continuous
compact



discrete
infinite



$$: \; T_{nn'mm'} = \sqrt{t_n t_{n'} t_m t_{m'}}\; \delta_{n+m,\,n'+m'}$$

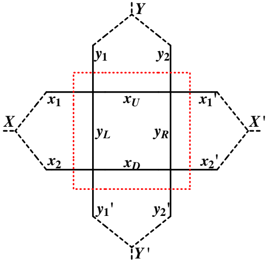
$$t_n = I_n(\beta)/I_0(\beta)$$

Classical tensor tools: from compact to discrete

- Naturally discrete for lattice models (Pontryagin, Peter-Weyl) .
- Fits the need of quantum computing.
- “Hard integrals" are done exactly.
- Connects smoothly Lagrangian and Hamiltonian.
- Checkings with importance sampling (MC and worm algorithms).
- Exact blocking but RG equations requires truncations (TRG).
- TRG: effective tensors are *local* (supersedes effective Hamiltonian).
- Symmetries are characterized by tensor selection rules and preserved by truncations (YM, PRD 100 014506).
- Noether theorem: for each symmetry, there is a corresponding tensor redundancy; noise-robust implementation of Gauss’s law (YM, arXiv:2003.10986, PRD 102 014506).

TRG blocking for 2D spin models (graphically)

Exact form of the partition function: $Z \propto \mathrm{Tr} \prod_i T^{(i)}_{xx'yy'}$.
Tr mean contractions over the indices attached to links.
Reproduces the closed paths ("worms") of the HT expansion.
TRG blocking separates the degrees of freedom inside the block which are integrated over, from those kept to communicate with the neighboring blocks. Graphically :



No sign problems.
Computing time goes like $\log(V)$ if we can control the truncations.

Checking with exact results for 2D Ising

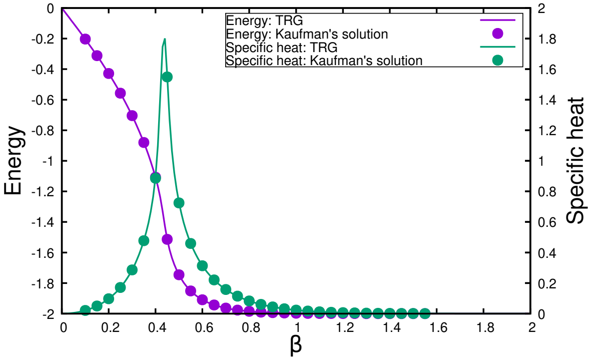


Figure: Energy and specific heat of the two dimensional Ising model on a 32×32 lattice. $D_{\text{cut}} = 32$. Graph by Ryo Sakai.

Compact Abelian Higgs Model (CAHM)

$$Z_{CAHM} = \prod_{x} \int_{-\pi}^{\pi} \frac{d\varphi_x}{2\pi} \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi} e^{-S_{gauge} - S_{matter}},$$

$$S_{gauge} = \beta_{pl.} \sum_{x,\mu<\nu} (1 - \cos(A_{x,\mu} + A_{x+\hat{\mu},\nu} - A_{x+\hat{\nu},\mu} - A_{x,\nu})),$$

$$S_{matter} = \beta_{l.} \sum_{x,\mu} (1 - \cos(\varphi_{x+\hat{\mu}} - \varphi_x + A_{x,\mu})).$$

Gauged version of the O(2) model: the global φ shift becomes local

$$\varphi'_x = \varphi_x + \alpha_x$$

Local changes in S_{matter} are compensated by

$$A'_{x,\mu} = A_{x,\mu} - (\alpha_{x+\hat{\mu}} - \alpha_x),$$

which leaves S_{gauge} invariant.
The matter fields can be decoupled by simply setting $\beta_{l.} = 0$ (we are left with the pure gauge $U(1)$ lattice model)

Fourier expansions and field integrations

Links: $e^{\beta_{l.}\cos(\varphi_{x+\hat{\mu}}-\varphi_x+A_{x,\mu})} = \sum_{n_{x,\mu}=-\infty}^{+\infty} e^{in_{x,\mu}(\varphi_{x+\hat{\mu}}-\varphi_x+A_{x,\mu})} I_{n_{x,\mu}}(\beta_{l.})$,

φ integration provides the O(2) selection rule:

$$\sum_{\mu}[-n_{x,\mu}+n_{x-\hat{\mu},\mu}]=0. \quad (1)$$

Plaquettes: $e^{\beta_{pl.}\cos(A_{x,\mu}+A_{x+\hat{\mu},\nu}-A_{x+\hat{\nu},\mu}-A_{x,\nu})} =$
$\sum_{m_{x,\mu,\nu}=-\infty}^{+\infty} e^{im_{x,\mu,\nu}(A_{x,\mu}+A_{x+\hat{\mu},\nu}-A_{x+\hat{\nu},\mu}-A_{x,\nu})} I_{m_{x,\mu,\nu}}(\beta_{pl.})$,

$A_{x,\mu}$ integration provides the selection rule:

$$\sum_{\nu>\mu}[m_{x,\mu,\nu}-m_{x-\hat{\nu},\mu,\nu}]+\sum_{\nu<\mu}[-m_{x,\nu,\mu}+m_{x-\hat{\nu},\nu,\mu}]+n_{x,\mu}=0. \quad (2)$$

Note 1: (2) implies (1) (discrete version of $\partial_\mu\partial_\nu F^{\mu\nu}=\partial_\mu J^\mu=0$.)
Note 2: Gauge quant. numbers m determine matter $n_{x,\mu}(\{m\})$
Note 3: In the unitary gauge, φ disappears

Selection rules are discrete Maxwell equations

The integers associated with time plaquette are discrete electric fields $e_{x,j} \equiv m_{x,j,D}$, with $j = 1, \ldots, D-1$. In the pure gauge limit, the selection rule for $\mu = D$ reads

$$\sum_{j=1}^{D-1} (e_{x,j} - e_{x-\hat{j},j}) = 0.$$

This is a discrete form of Gauss's law in the pure gauge limit $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$.

For $D = 3$, $b_x \equiv m_{x,1,2}$ and the pure gauge selection rule for for $\mu = 1$ and 2 are

$$\begin{aligned} e_{x,1} - e_{x-\hat{\tau},1} &= -(b_x - b_{x-\hat{2}}), \\ e_{x,2} - e_{x-\hat{\tau},2} &= (b_x - b_{x-\hat{1}}). \end{aligned} \tag{1}$$

These are a discrete version of the $D = 3$ Euclidean pure gauge Maxwell's equations with $B = F^{12}$

$$\partial_1 B = \partial_\tau E_2,\ \partial_2 B = -\partial_\tau E_1,$$

No discrete homogeneous Maxwell's equations

$D = 3$: no discrete version of

$$\partial_\mu \epsilon^{\mu\nu\sigma} F_{\nu\sigma} = 0,$$

which can also be written $\dot{B} = -\boldsymbol{\nabla} \times \mathbf{E}$. Examples of legal configurations violating the discrete version can be constructed.
For $D = 4$, we can introduce

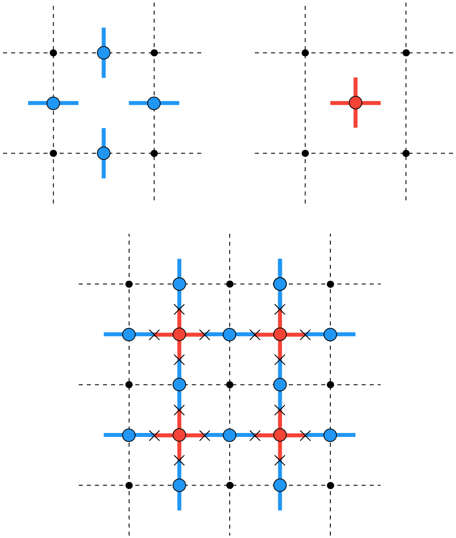
$$b_{x,j} \equiv \epsilon_{jkl} m_{x,k,l},$$

and obtain a discrete version of

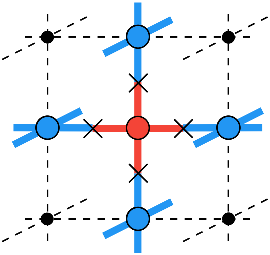
$$\partial_\tau \mathbf{E} = -\boldsymbol{\nabla} \times \mathbf{B}, \ (3)$$

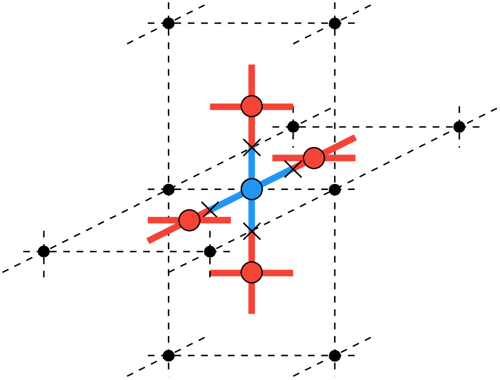
Again there is no discrete version of the homogeneous equations for the dual field strength $\dot{\mathbf{B}} = -\boldsymbol{\nabla} \times \mathbf{E}$ and $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. Note that the sign in (3) is different in Euclidean and Minkowskian spaces. It can be traced to the minus sign in the Minkowskian Klein-Gordon equation.

Assembly of the A (links, blue) and B (plaquette, red) tensors for $D = 2$ (Figures by Ryo Sakai)



Assembly of the A (links, blue) and B (plaquette, red) tensors for $D = 3$ (Figures by Ryo Sakai)





Explicit form of the *A* and *B* tensors

The four legs attached to a *B*-tensor on a given plaquette (x, μ, ν) carry a single value *m*.

$$B^{(x,\mu,\nu)}_{m_1 m_2 m_3 m_4} = \begin{cases} t_{m_1}(\beta_{pl.}), & \text{if all } m_i = m \\ 0, & \text{otherwise.} \end{cases}$$

$$t_n(\beta) \equiv \frac{I_n(\beta)}{I_0(\beta)} \simeq \begin{cases} 1 - \frac{n^2}{2\beta} + \mathcal{O}(1/\beta^2), \text{ for } \beta \to \infty \\ \frac{\beta^n}{2^n n!} + \mathcal{O}(\beta^{n+2}), \text{ for } \beta \to 0 \end{cases} .$$

These are assembled (traced) together with "*A*-tensors" attached to links with $2(D-1)$ legs orthogonal to the link

$$A^{(x,\mu)}_{m_1 \dots m_{2(D-1)}} = t_{n_{x,\mu}}(\beta_{l.}) \delta_{n_{x,\mu}, n_{x,\mu}(\{m\})}.$$

Partition function with PBC:

$$\begin{aligned} Z &= (e^{-\beta_{pl.}} I_0(\beta_{pl.}))^{VD(D-1)/2} (e^{-\beta_{l.}} I_0(\beta_{l.}))^{VD} \\ &\times \mathrm{Tr} \prod_{l.} A^{(l.)}_{m_1, \dots m_{2(D-1)}} \prod_{pl.} B^{(pl.)}_{m_1 m_2 m_3 m_4}, \end{aligned} \tag{2}$$

Gauge redundancy

In presence of (bosonic) matter, the matter quantum numbers are completely determined by the gauge quantum numbers $n_{x,\mu}(\{m\})$ and the gauge selection rules are then satisfied as expressed in

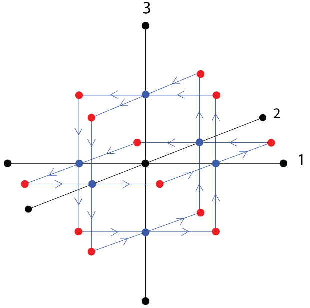
$$A^{(x,\mu)}_{m_1 \dots m_{2(D-1)}} = t_{n_{x,\mu}}(\beta_{l.})\delta_{n_{x,\mu}, n_{x,\mu}(\{m\})}$$

The O(2) selection rule can be omitted because it is automatically satisfied. Equivalently, φ can be gauged away.

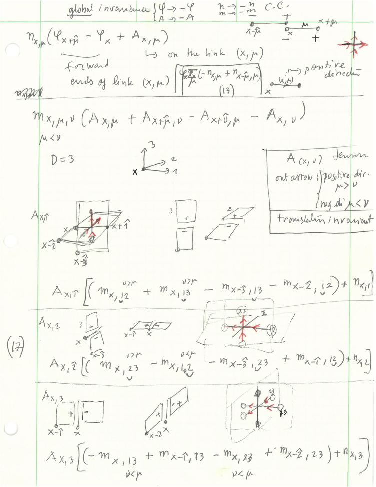
In the pure gauge limit, divergenceless conditions equivalent to discrete Maxwell's equations need to be imposed at each link. However they are not completely independent. Out of the $2D$ Kronecker delta functions associated to the A tensors attached to the links coming out of a site, one of them is a consequence of the $2D-1$ others (see YM, PRD 102). Equivalently, we can gauge away one of the links coming out of the site.

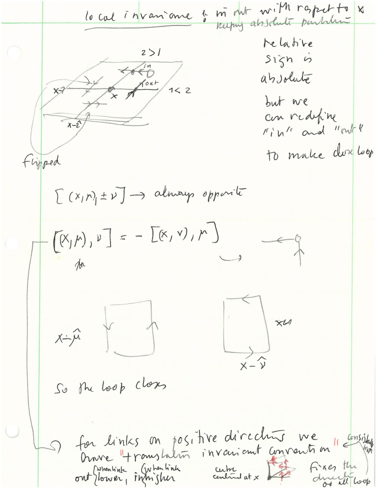
Gauge redundancy (pure gauge)

if Gauss's law is satisfied for a A-tensor attached to the $((\mathbf{x}, \tau), D)$ time link which is assembled with the divergenceless A-tensors attached to the $2(D-1)$ spatial links $((\mathbf{x}, \tau+1), j)$ and $((\mathbf{x}-\hat{j}, \tau+1), j)$ with $j = 1, \ldots, D-1$, then the A-tensors attached to the time link $((\mathbf{x}, \tau+1), D)$ is forced to obey Gauss's law because of a discrete version of $\partial_\tau(\boldsymbol{\nabla} \cdot \mathbf{E}) = 0$. This is illustrated for $D = 3$ for in-out assignments discussed in YM, PRD 102.

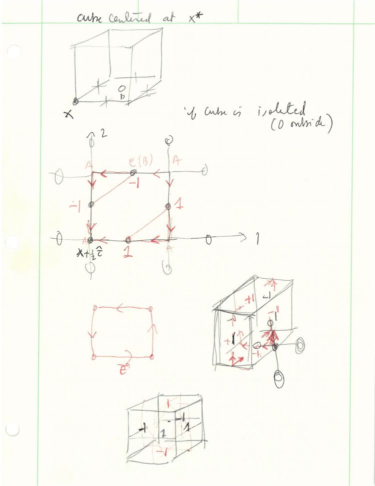


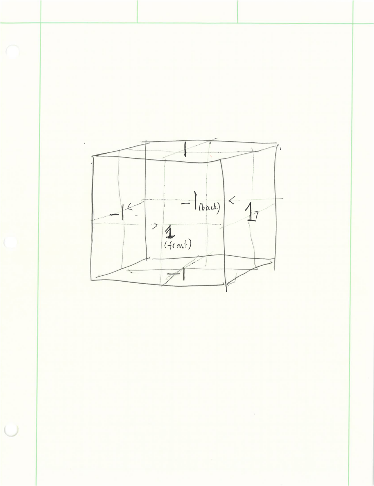
Backup “blackboard" (selection rules and redundancy)





Backup “blackboard" (no homogeneous Maxwell)





Discrete Gauss theorem (global in-out conventions)

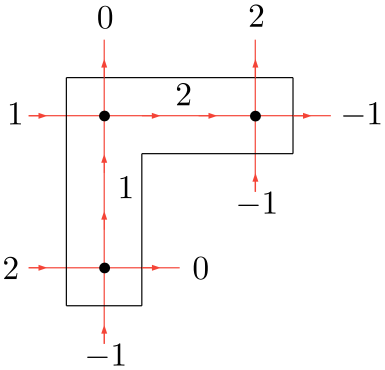


Figure: Example of flux cancelations in $D = 2$. The total flux in and out the upside-down L-shaped domain is +1.

Boundary conditions (graph by Ryo Sakai)

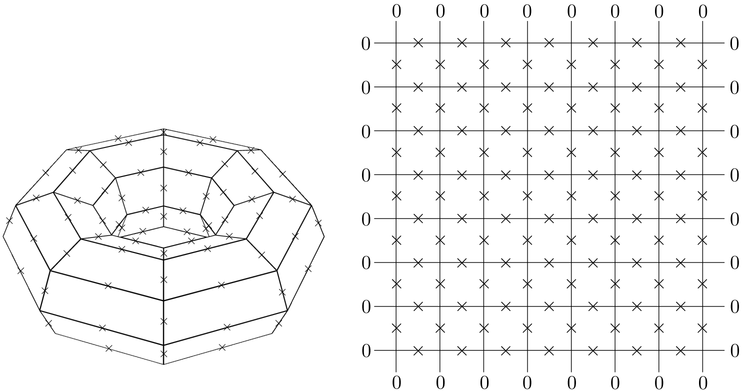


Figure: Assembling the translation invariant tensor with PBC (left), or using new tensors at the boundary for OBC (right). Tensors are assumed to be put on each lattice site.

Reformulation of Noether's theorem

- Redundant selection rules are in one-to-one correspondence with irrelevant integrations.
- We can skip the integrations that produce redundant selection rules and replace these integrated fields by arbitrary values. This is exactly what gauge-fixing does.
- The argument can be extended to global symmetries. In the case of the O(2) model in-out assignments for the $2D$ legs of the divergenceless tensor attached to sites imply that *one* of the divergenceless conditions is a consequence of all the other ones. This requires the whole tensor network to be isolated. The O(2) symmetry allows us to fix *one* of the φ fields to an arbitrary value.
- The redundancy argument extends to discrete $\mathbb{Z}_q$ subgroups of $U(1)$ where the divergenceless condition is expressed modulo q
- Noether's theorem can be expressed in the tensor formulation context as: for each symmetry, there is a corresponding tensor redundancy. This applies to global, local, continuous and discrete Abelian symmetries.

Transfer Matrix

We can organize the trace in the expression of Z by assembling “time layers" corresponding to “magnetic" time slices and “electric" slices half-way between the magnetic time slices. This construction singles out a time direction as for the Hamiltonian treatment.

$$Z = \mathrm{Tr}\, \mathbb{T}^{N_\tau},$$

with N_τ the number of sites in the temporal direction.

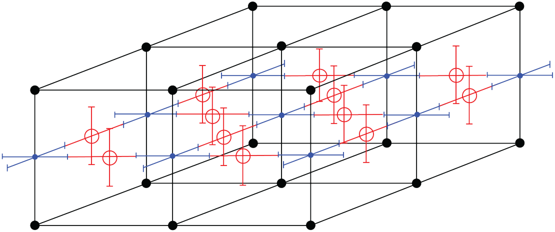
$$\begin{aligned} \mathbb{T} \;\equiv\; & (e^{-\beta_{pl.}} I_0(\beta_{pl.}))^{(V/N_\tau)D(D-1)/2}(e^{-\beta_{l.}} I_0(\beta_{l.}))^{(V/N_\tau)D} \\ & \times \mathbb{T}_E^{1/2} \mathbb{T}_M \mathbb{T}_E^{1/2}, \end{aligned} \tag{3}$$

For $D = 3$, this construction can be visualized as a “lasagna".

Electric layer

B-tensors on space-time plaquettes labelled by $e_{(\mathbf{x},\tau),j}$ with a fixed τ

A-tensors attached to their time links with $2(D-1)$ legs all in spatial directions. This is illustrated for $D = 3$ below.



Electric layer (Figures by Ryo Sakai)

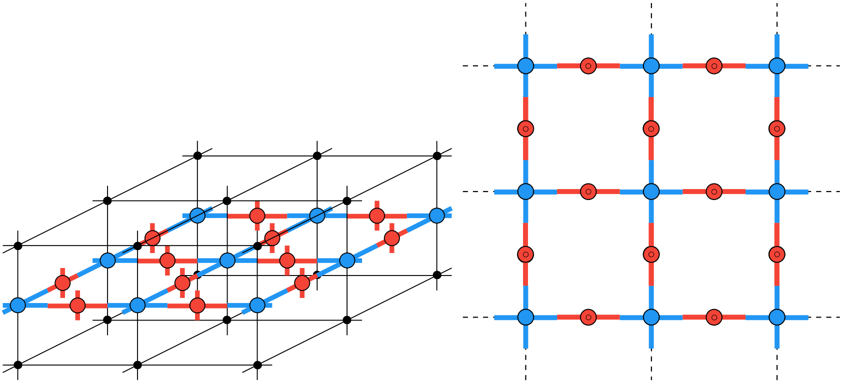


Figure: Electric layer of the transfer matrix for $D = 3$ between two time slices (left) and “from above" (right).

Magnetic layer (Figures by Ryo Sakai)

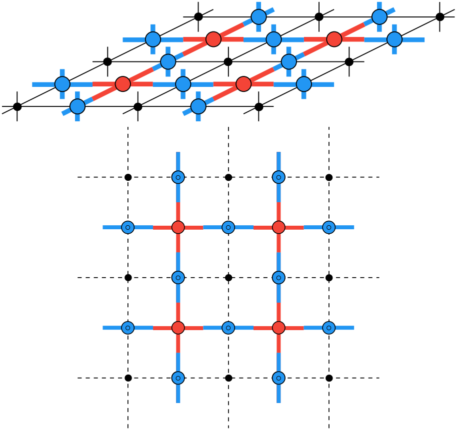


Figure: magnetic layer of the transfer matrix for $D = 3$ in a time slice (top) and "from above" (bottom).

Electric matrix elements

We can think of these two types of layers as matrices connecting electric states

$$|\{\mathbf{e}\}\rangle = \otimes_{\mathbf{x},j}|e_{\mathbf{x},j}\rangle.$$

This is a natural choice because the B-tensors on the space-time plaquettes have two legs in the time direction. In this basis, the electric layer can be expressed as a diagonal matrix $\mathbb{T}_E$ with matrix elements

$$\langle\{\mathbf{e}'\}|\mathbb{T}_E|\{\mathbf{e}\}\rangle = \delta_{\{\mathbf{e}\},\{\mathbf{e}'\}} T_E(\{\mathbf{e}\}),$$

where $T_E(\{\mathbf{e}\})$ can be written with some implicit notations as a traced product of A tensors on time links with B tensors on space-time plaquettes

$$T_E(\{\mathbf{e}\}) = \mathrm{Tr} \prod_{time\ l.} A^{(l.)}_{m_1,\dots m_{2(D-1)}} \prod_{sp.-time\ pl.} B^{(pl.)}(\mathbf{e}).$$

Magnetic matrix elements

Similarly, we can define a magnetic matrix $\mathbb{T}_M$ with matrix elements

$$\langle\{\mathbf{e}\}|\mathbb{T}_M|\{\mathbf{e}'\}\rangle$$

with the indices **e** and **e**′ carried by the time legs of the *A*-tensors.

$$\langle\{\mathbf{e}'\}|\mathbb{T}_M|\{\mathbf{e}\}\rangle = \mathrm{Tr}\prod_{sp.\ l.} A^{(l.)}_{m_1,\dots m_{2(D-1)}}(\mathbf{e},\mathbf{e}')\prod_{sp.-sp.\ pl.} B^{(pl.)}.$$

All the traces are over the spatial legs of the tensors, while the time legs carry the the indices **e** and **e**′.

Hamiltonian limit

The crucial features of $\mathbb{T}_E$ is that it only involves time links and plaquettes having one direction in time. We introduce separate β_τ couplings for $\mathbb{T}_E$ and use redefinitions in terms of the time lattice spacing a_τ:

$$\beta_{\tau pl.} = \frac{1}{a_\tau g^2_{pl.}}, \text{ and } \beta_{\tau l.} = \frac{1}{a_\tau g^2_{l.}}.$$

Given the weak coupling (large β) behavior of $t_n(\beta)$ given in Eq. (13), at first order in a_τ, we get “rotor" energies $(1/2)g^2_{pl.}m^2$ for the plaquettes and $(1/2)g^2_{l.}n^2$ for the links.
On the other hand, $\mathbb{T}_M$ only involves space links and space-space plaquettes and we redefine

$$\beta_{s\ pl.} = a_\tau J_{pl.}, \text{ and } \beta_{s\ l.} = a_\tau h_{l.}.$$

$$\mathbb{T} = \mathbb{I} - a_\tau \mathbb{H} + \mathcal{O}(a_\tau^2)$$

Following Banks, Kogut, Susskind 76, we define $\hat{e}_{\mathbf{x},j}$ and $\hat{U}_{\mathbf{x},j}$ such that

$$
\begin{aligned}
\hat{e}_{\mathbf{x},j}|e_{\mathbf{x},j}\rangle &= e_{\mathbf{x},j}|e_{\mathbf{x},j}\rangle \\
\hat{U}_{\mathbf{x},j}|e_{\mathbf{x},j}\rangle &= |e_{\mathbf{x},j}+1\rangle \\
\hat{U}^{\dagger}_{\mathbf{x},j}|e_{\mathbf{x},j}\rangle &= |e_{\mathbf{x},j}-1\rangle,
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
\mathbb{H} = \quad & \frac{1}{2}g^2_{pl.}\sum_{\mathbf{x},j}(\hat{e}_{\mathbf{x},j})^2 \\
+ \quad & \frac{1}{2}g^2_{l.}(\sum_{\mathbf{x}}(\sum_{j}(\hat{e}_{\mathbf{x},j}-\hat{e}_{\mathbf{x}-\hat{j},j})))^2 \\
- \quad & h_{l.}\sum_{\mathbf{x},j}(\hat{U}_{\mathbf{x},j}+h.c.) \\
- \quad & J_{pl.}\sum_{\mathbf{x},j<k}(\hat{U}_{\mathbf{x},j}\hat{U}_{\mathbf{x}+\hat{j},k}\hat{U}^{\dagger}_{\mathbf{x}+\hat{k},j}\hat{U}^{\dagger}_{\mathbf{x},k}+h.c.).
\end{aligned}
\tag{5}
$$

We have used $\sum_{j=1}^{D-1}(e_{x,j}-e_{x-\hat{j},j}) = n_{x,D}$ to eliminate $n_{x,D}$.

Pure gauge: a robust way to implement Gauss's law

We introduce a new set of quantum numbers $c_{\mathbf{x},j,k}$, associated with the plaquettes of a $D-1$ lattice, and unrelated to the existing gauge quantum numbers.

$$\begin{aligned} e_{x,j} &= \sum_{k>j}[-c_{x,j,k} + c_{x-\hat{k},j,k}] \\ &+ \sum_{k<j}[c_{x,k,j} - c_{x-\hat{k},k,j}], \end{aligned} \quad (6)$$

and Gauss's law is automatically satisfied. This is a discrete version of

$$E^k = \partial_j C^{jk},$$

For an arbitrary antisymmetric tensor C^{jk} with indices $j,\ k$ running from 1 to $D-1$. It is possible to introduce dimension-dependent "magnetic" notations such as $G = \epsilon^{kl}C^{kl}$ for $D=3$ and $G^j = \epsilon^{jkl}C^{kl}$ for $D=4$.

For a $D = 3$ pure gauge theory we can visualize the electric Hilbert space as a $D = 2$ O(2) model being on a plane between two time slices. We can further imagine the auxilliary variables located in the middle of the plaquettes of this “horizontal" plane, which means in the center of the $D = 3$ cubes of the original lattice. This is equivalent to the dual formulation discussed by J. Unmuth-Yockey PRD 99, 074502 (2019).

For $D = 4$, this reparametrization is a discrete equivalent of setting

$$\mathbf{E} = \boldsymbol{\nabla} \times \mathbf{G}.$$

This guarantees Gauss’s law, but $\boldsymbol{\nabla} \times \mathbf{E}$ is in general non-zero so we don’t use this trick for conventional electrostatics because one of the homogeneous Maxwell’s equation ($\dot{\mathbf{B}} = -\boldsymbol{\nabla} \times \mathbf{E}$) implies that the magnetic field changes with time.

This method is optimal for $D = 3$, because it reduces the dimensionality of the Hilbert space to one index per site ($c_{x,1,2}$) rather than 2 ($e_{\mathbf{x},1}$ and $e_{\mathbf{x},2}$). For $D = 4$, there are 3 indices per sites in both case, because $c_{x,j,k}$ is only defined up to a gradient. When the Hilbert space is parametrized with the new quantum numbers, the relation between the $e_{x,j}$ and $c_{x,j,k}$ is linear. We can study the effect of changing one of the $c_{x,j,k}$ by ± 1. For instance, $\Delta c_{x,1,2} = 1$ generates the following changes:

$$\Delta e_{x,1} = -1,\ \Delta e_{x+\hat{2},1} = 1,\ \Delta e_{x,2} = 1,\ \Delta e_{x+\hat{1},2} = -1.$$

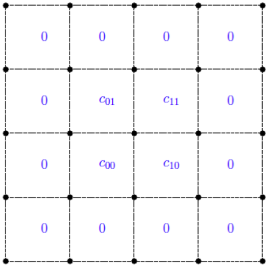
This change can be visualized as an electric field circulating clockwise on a plaquette in the 1-2 plane and it clearly satisfies Gauss's law. The changes correspond to the $U^\dagger U^\dagger UU$ term in the Kogut-Susskind Hamiltonian. For $D = 3$, we can efficiently replace the term with two raising and two lowering operators by a term with a single raising or lowering operator (J. Unmuth-Yockey PRD 99). The construction can be repeated for any pair of directions in higher dimensions, but the $c_{x,j,k}$ have some redundancy. For $D = 4$, see YM PRD 102

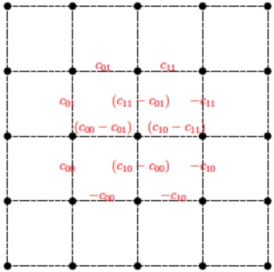
Exercise

Consider a $U(1)$ pure gauge theory in 2+1 dimensions on a 3 by 3 spatial lattice (4 plaquettes) with OBC. Calculate the 12 electric quantum numbers on the links as a function of the 4 $c_{\mathbf{x},1,2} \equiv c_{\mathbf{x}}$.
Check that Gauss's law is satisfied at the 9 sites.
Solution: Use

$$e_{\mathbf{x},1} = -c_{\mathbf{x}} + c_{\mathbf{x}-\hat{2}}, \text{and } e_{\mathbf{x},2} = c_{\mathbf{x}} - c_{\mathbf{x}-\hat{1}},$$

with $c_{\mathbf{x}}$ is non-zero inside the four plaquettes and zero outside.





Equations of motion for φ

For the scalar fields, we first introduce the notation

$$d_{x,\mu} \equiv \varphi_{x+\hat{\mu}} - \varphi_x + A_{x,\mu}$$

which approximates the covariant derivative of φ. The equation of motion

$$\begin{aligned} \partial S/\partial\varphi_x &= \beta_{l.} \sum_\mu [-\sin(d_{x,\mu}) + \sin(d_{x-\hat{\mu},\mu})] \\ &= 0. \end{aligned} \tag{7}$$

On the other hand the integration with respect to φ_x implies

$$\sum_\mu [-n_{x,\mu} + n_{x-\hat{\mu},\mu}] = 0.$$

Equations of motion for the gauge fields

$$f_{x,\mu,\nu} \equiv A_{x,\mu} + A_{x+\hat{\mu},\nu} - A_{x+\hat{\nu},\mu} - A_{x,\nu}.$$

As in the continuum they are gauge invariant.
With these notations,

$$\begin{aligned}
\partial S/\partial A_{x,\mu} &= \beta_{pl.} \sum_{\nu>\mu} [\sin(f_{x,\mu,\nu}) - \sin(f_{x-\hat{\nu},\mu,\nu})] \\
&\quad + \beta_{pl.} \sum_{\nu<\mu} [-\sin(f_{x,\nu,\mu}) + \sin(f_{x-\hat{\nu},\nu,\mu})] \\
&\quad + \beta_{l.} \sin(d_{x,\mu}) \\
&= 0. \qquad (8)
\end{aligned}$$

On the other hand, the integration over $A_{x,\mu}$ yields the selection rule

$$\sum_{\nu>\mu} [m_{x,\mu,\nu} - m_{x-\hat{\nu},\mu,\nu}] + \sum_{\nu<\mu} [-m_{x,\nu,\mu} + m_{x-\hat{\nu},\nu,\mu}] + n_{x,\mu} = 0.$$

Topological solutions and semi-classical approximations

solvable cases: the $D = 1$ O(2) spin model and the $D = 2$ pure gauge $U(1)$ model.
For the $D = 1$ O(2) spin model with PBC and N_τ sites, the equations of motion: $\sin(\varphi_{x+\hat{1}} - \varphi_x)$ takes the same value on every link. These equations have many solutions and we will focus our attention on the ones that can be interpreted as continuous topological solutions in the continuum limit for PBC. If we impose that $\varphi_{x+\hat{1}} - \varphi_x$ is a small constant, we can obtain a solution that meets this requirement. Given any choice for the constant, we can then “integrate" the equations: starting with some φ_0, we obtain φ_1, and so on until, due to PBC, we get an independent value for φ_0 which should be consistent with the initial value modulo an integer multiple of 2π.

This approximately corresponds to a smooth mapping of the circle into itself provided that the successive changes can be made arbitrarily small. This can be accomplished by requiring that for all links

$$\varphi_{x+\hat{1}} - \varphi_x = \frac{2\pi}{N_\tau}\ell,$$

for a given integer ℓ. By taking, N_τ large with fixed ℓ we obtain a solution which can be interpreted as a topological solution with winding number ℓ. In the limit $\ell \ll N_\tau$, these solutions have classical action

$$S_\ell \simeq \frac{\beta}{2}(\frac{2\pi}{N_\tau}\ell)^2 N_\tau.$$

We can calculate the quadratic fluctuations with respect to this solution. We can first use the global O(2) symmetry to set $\varphi_0 = 0$. Other values of φ_0 are taken into account by performing the integration over φ_0 which with our normalization of the measure yields a factor 1. By construction, the linear fluctuations vanish because the first derivatives are zero and all we need to calculate are the quadratic fluctuations

$$\Delta = \prod_{x=1}^{N_\tau - 1} \int_{-\pi}^{\pi} \frac{d\varphi_x}{2\pi} e^{-S_\ell^{quad.}},$$

with

$$S_\ell^{quad.} = \frac{\beta}{2} \cos(\frac{2\pi}{N_\tau}\ell)(\varphi_1^2 + (\varphi_2 - \varphi_1)^2 + \cdots + \varphi_{N_\tau - 1}^2)$$

Following the standard quadratic path integral procedure, we find

$$\Delta = N_\tau^{-1/2}(2\pi\beta \cos(\frac{2\pi}{N_\tau}\ell))^{-(N_\tau - 1)/2}.$$

We can now attempt to re-sum the topological contributions. This is delicate because we have assumed $\ell \ll N_\tau$, however if β is large enough, the terms with large ℓ are exponentially suppressed. In the same spirit, we will ignore the ℓ dependence of Δ and use the Poisson summation formula

$$\sum_{\ell=-\infty}^{\infty} e^{-\frac{B}{2}\ell^2} = \sqrt{\frac{2\pi}{B}} \sum_{n=-\infty}^{\infty} e^{-\frac{(2\pi)^2}{2B}n^2},$$

with $B = \beta(2\pi)^2/N_\tau$. Putting everything together, we get a semi-classical approximation of the partition function in the large β limit

$$Z \simeq (2\pi\beta)^{-N_\tau/2} \sum_{n=-\infty}^{\infty} (e^{-\frac{n^2}{2\beta}})^{N_\tau}.$$

We now consider the solutions of the discrete current conservation. The solution is that $n_{x,1}$ should be constant. With PBC, this implies the exact expression:

$$Z = \sum_{n=-\infty}^{\infty} (e^{-\beta} I_n(\beta))^{N_\tau},$$

which can be compared to the semi-classical expression. Using the large β approximations

$$e^{-\beta} I_0(\beta) \simeq \frac{1}{\sqrt{2\pi\beta}}(1 + \mathcal{O}(1/\beta)),$$

and the Bessel function approximate behavior in the same limit, we see the approximate correspondence between the two expressions.

A similar construction can be carried for the $D = 2$ pure gauge $U(1)$ model with PBC. We consider a rectangular $N_s \times N_\tau$ lattice. The equation of motion requires that $\sin(f_{x,1,2})$ is constant. Following the analogy with the O(2) case, we start with

$$f_{x,1,2} \equiv A_{x,1} + A_{x+\hat{1},2} - A_{x+\hat{2},1} - A_{x,2} = \delta,$$

with δ a constant to be determined with PBC. We can gauge fix the temporal links with a given spatial coordinate x_1 to the identity with the exception of one time layer. For definiteness, we take this layer of nontrivial time links to be between $\tau = N_\tau - 1$ and N_τ which is identified with 0 due to PBC. The space links with a given spatial coordinate, which can be visualized as a vertical ladder can be treated as the indices of a $D = 1$ O(2) model changing by $-\delta$ at each step until we get to the “last" rung and temporal links are present. The constancy of the “last" plaquette requires that

$$A_{(x_1+1,N_\tau-1),2} - A_{(x_1,N_\tau-1),2} = N_\tau\delta.$$

Iterating in the spatial direction, we obtain PBC in the spatial direction provided that

$$\delta = \frac{2\pi}{N_s N_\tau}\ell.$$

The action for this topological solution is

$$S_\ell^{U(1)} \simeq \frac{\beta}{2}\left(\frac{2\pi}{N_s N_\tau}\ell\right)^2 N_s N_\tau .$$

Note that we could have obtained another periodic solution by setting *all* the time links to 1 and imposing PBC in time for N_s independent $D = 1$ O(2) models, however, the action for these configurations is larger by a factor N_s^2.

The quadratic fluctuations can be calculated as in the O(2) case but with extra complications due to the special time layer. Keeping track of all the 2π factors and using Poisson summation for the winding numbers, we obtain the semi-classical approximation

$$Z^{U(1)} \simeq (2\pi\beta)^{-N_s N_\tau/2} \sum_{n=-\infty}^{\infty} (e^{-\frac{n^2}{2\beta}})^{N_s N_\tau},$$

which agrees with the exact expression at leading order.

As a test of the semi-classical picture we can calculate the topological susceptibility. For this purpose we first calculate

$$Z(\beta,\theta) = \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi} e^{-S_{gauge} - i\theta Q},$$

with the topological charge Q defined as

$$Q = \frac{1}{2\pi} \sum_{x} \sin(A_{x,1} + A_{x+\hat{1},2} - A_{x+\hat{2},1} - A_{x,2}).$$

The topological susceptibility is defined as

$$\chi = -\frac{d^2}{d\theta^2} \ln(Z)|_{\theta=0}.$$

Exact resummation (Gattringer et al. PRD 92, 114508, 2015)

$$Z(\beta,\theta) = \sum_{n=-\infty}^{\infty} \left[\quad e^{-\beta} I_n\left(\sqrt{\beta^2 - (\frac{\theta}{2\pi})^2}\right) \right.$$
$$\left. \times \left(\frac{\beta - \frac{\theta}{2\pi}}{\beta + \frac{\theta}{2\pi}}\right)^{n/2} \right]^{N_s N_\tau}. \tag{9}$$

If χ is dominated by configurations corresponding to winding number ± 1 where $|Q| \simeq 1$ in the continuum limit, we have the large-β estimate

$$\begin{aligned} \chi \simeq & \quad (0)^2 1 + (1)^2 \exp(-\frac{\beta}{2}\frac{(2\pi)^2(1)^2}{N_s N_\tau}) \\ + & \quad (-1)^2 \exp(-\frac{\beta}{2}\frac{(2\pi)^2}{N_s N_\tau}(-1)^2). \end{aligned} \tag{10}$$

Fig. 7 shows that this estimate is reasonably good when β is large enough.

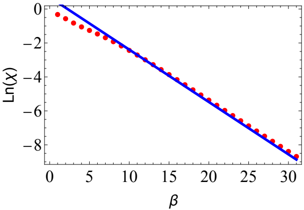


Figure: Logarithm of the topological susceptibility using the exact formula for $N_s = N_\tau = 8$ expanded up to order 5 (dots) and the semi-classical approximation Eq. (10) (continuous line).

Conclusions

- QC/QIS in HEP and NP: we need big goals with many intermediate steps.
- Tensor Field Theory is a generic tool to discretize path integral formulations of lattice model with compact variables.
- TRG: exact blocking, a friendly competitor to QC.
- Truncations respect symmetries.
- TRG: gauge-invariant approach for the quantum simulation of gauge models.
- Noether theorem: for each symmetry, there is a corresponding tensor redundancy.
- Noise-robust economical implementation of Gauss's law for pure gauge models.
- Need for quantum simulations and computations dedicated to theoretical physics
- Thanks for listening!