Entanglement, partial set of measurements, and diagonality of the density matrix in the parton model

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arXiv.2001.01726, with Alex Kovner and Vladimir V. Skokov

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Motivation

- Proton:
  - $\tau \sim \infty \Rightarrow H\,|\text{Proton}\rangle = M\,|\text{Proton}\rangle$, (pure) energy eigenstate.
  - Parton model treats proton as collection of **nearly** free particles
  - Suggested resolution of this apparent paradox: quantum entanglement (arXiv.1702.03489,Kharzeev & Levin)
  - Postulation: reduced density matrix for observed parton is diagonal in particle number basis

- Color Glass Condensate:
  - Hamiltonian is non-perturbative and unknown, so is the wavefunction
  - A model for proton wavefunction

$$|\text{proton}\rangle = \sum_{\rho_a} |v;\rho_a\rangle \otimes |s;\rho_a, A_b\rangle$$

- Is CGC reduced density matrix diagonal in gluon number basis?

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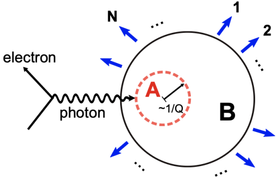
Reduced density matrix

Density matrix $\hat{\rho}(A,B) \to$ reduced density matrix

$$\hat{\rho}_A = \mathrm{Tr}_B\, \hat{\rho}(A,B)$$

- Here A can be probed in DIS partons of the parton model; B is the unobserved part of the parton wavefunction.
- The property of interest: if $\hat{\rho}(A,B)$ is pure, non-pure reduced density matrix $\hat{\rho}_A \to$ entanglement



from arXiv.1904.11974

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Quantum entropies

Common entropies in quantum information theory:

- Renyi entropy $S_R^N = \frac{1}{1-N} \ln \mathrm{Tr}\{\hat{\rho}^N\}$
- von Neumann entropy $S_V = \lim_{N\to 1} S_R^N = -\,\mathrm{Tr}\{\hat{\rho} \ln \hat{\rho}\}$

Entropy of entanglement: entropy of the reduced density matrix

Entropy of ignorance

- Any experimental measurement is limited: one can study only part of the full $\hat{\rho}$.
- Parton model: most (if not all) observables probe diagonal components of the density matrix in the number of parton representation.
- Ignorance density matrix: replace the off-diagonal elements of the density matrix with zeros. The Ignorance density matrix is positive-definite and is **definitely** not pure.
- **Entropy of ignorance**: entropy of the ignorance density matrix

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Example

Given a pure state $|\phi_{AB}\rangle = \frac{\sqrt{2}}{2}|0_A\rangle \otimes |0_B\rangle + \frac{1}{2}|0_A\rangle \otimes (|0_B\rangle + |1_B\rangle)$

$$\hat{\rho}_A = \frac{1}{4}\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \qquad \hat{\rho}^I_A = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The ignorance density matrix ρ^I_A is defined in particle number basis.

$$S_V(\hat{\rho}_A) \sim 0.426 \quad S_V(\hat{\rho}^I_A) \sim 0.693$$

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Proton wave function

CGC model for Proton wavefunction,

$$|\text{proton}\rangle = \sum_{\rho_a} |v; \rho_a\rangle \otimes |s; \rho_a, A_b\rangle$$

where

- $|v\rangle$ describes the valance dof
- $|s\rangle$ stands for soft gluons
- $\rho_a(x)$ is the color charge density of the valance modes
- A_b is the gluon field generated from the source ρ_a

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Reduced density matrix for soft gluons in MV model

Our goal is the reduced density matrix for soft gluons

$$\hat{\rho}_s = \mathrm{Tr}_v(|v\rangle\langle v| \otimes |s\rangle\langle s|)$$

In MV model

$$\langle v|v\rangle = \exp\left\{-\int_k \frac{\rho_a(k)\rho_a^*(k)}{2\mu^2}\right\}$$

$$\mathrm{Tr}_v \Rightarrow \int D[\rho_a]$$

$$|s\rangle = \mathcal{C}\,|0\rangle\,; \quad \mathcal{C} = \exp\left\{i\int_k b_a^i(k)\phi_a^{*i}\right\}$$

$$b_a^i(k) = \frac{igk^i}{k^2}\rho_a(k) + \mathcal{O}\left(\rho_a^2\right)$$

$$\phi_a^{*i}(k) = a_a^{i\dagger}(k) + a_a^i(-k)$$

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Entropy of entanglement

$$\hat{\rho}(\phi,\Phi) = \mathcal{N}\int D[\rho_a] e^{-\int_k \frac{\rho_a(k)\rho_a^*(k)}{2\mu^2}} \langle\phi|\,\mathcal{C}\,|0\rangle\langle 0|\,\mathcal{C}^\dagger\,|\Phi\rangle$$

To compute the entanglement entropy, one recall

$$-\,\mathrm{Tr}\{\hat{\rho}\ln(\hat{\rho})\} = \lim_{N\to 1}\frac{1}{1-N}\ln\big(\mathrm{Tr}\{\hat{\rho}^N\}\big)$$

and in terms of functional integrals

$$\mathrm{Tr}\{\hat{\rho}^N\} = \int D[\phi_1,\phi_2,..\phi_N]\rho(\phi_1,\phi_2)\rho(\phi_2,\phi_3)...\rho(\phi_N,\phi_1)$$

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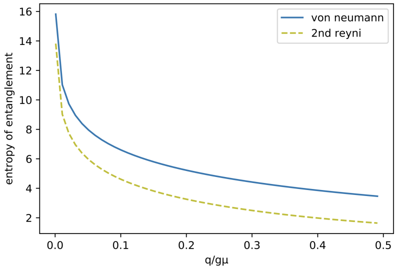
Analytic results for entropy of entanglement

$$S_R^2 = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2} \ln\left(1 + 4\frac{g^2\mu^2}{q^2}\right).$$

$$S_V = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2} \left[\ln\left(\frac{g^2\mu^2}{q^2}\right) + \sqrt{1 + 4\frac{g^2\mu^2}{q^2}} \ln\left(1 + \frac{q^2}{2g^2\mu^2} + \frac{q^2}{2g^2\mu^2}\sqrt{1 + 4\frac{g^2\mu^2}{q^2}}\right)\right].$$

- Extensive in terms of transverse area $S_\perp$
- $\lim_{q\to\infty} S(q) = 0$

ArXiv.1506.05394 by Alex Kovner, Michael Lublinsky



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Soft gluon in particle number basis

Recall the definition of soft gluon state

$$|s\rangle = e^{i\int_k b_a(k)(a_a^\dagger(k)+a_a(-k))}\,|0\rangle$$

- Discretize the momentum $\int \frac{dk^2}{(2\pi)^2} \to \sum \frac{\Delta^2}{(2\pi)^2}$
- The coherent operator can be rewritten as

$$\mathcal{C}\,|0\rangle = e^{i\int_k b_a(k)a_a^\dagger(k)+b_a^*(k)a_a(k)}\,|0\rangle = \boldsymbol{e^{i\int_k b_a(k)a_a^\dagger(k)}}\,e^{-\frac{1}{2}\int_k \frac{g^2}{k^2}|\rho_a|^2}\,|0\rangle$$

- Expanding $\boldsymbol{e^{i\frac{\Delta^2}{(2\pi)^2}b_a(k)a_a^\dagger(k)}}$ will allow us to do calculation in particle number basis.

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Matrix elements in particle number basis

For a single momemtum mode q, including normalization

$$\langle n_c(q), m_c(-q)|\hat{\rho}_s(q)|\alpha_c(q), \beta_c(-q)\rangle$$
$$= (1-R)\frac{(n+\beta)!}{\sqrt{n!m!\alpha!\beta!}}\left(\frac{R}{2}\right)^{n+\beta}\delta_{(n+\beta),(m+\alpha)}\,; \quad R = (1+\frac{q^2}{2g^2\mu^2})^{-1}$$

Nonzero off-diagonal elements, eg, $\langle 0,0|\hat{\rho}_s(q)|1,1\rangle = \frac{(1-R)R}{2}$
The delta function is from the gaussian integral of $\rho_a(q)$, the diagonal matrix elements are

$$\langle n_c(q), m_c(-q)|\hat{\rho}_s(q)|n_c(q), m_c(-q)\rangle$$
$$= (1-R)\frac{(n_c+m_c)!}{n_c!m_c!}\left(\frac{R}{2}\right)^{n_c+m_c}$$

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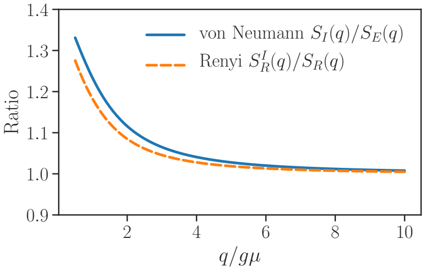
Ratio between entropies of entanglement and ignorance

- $\rho_{nm} \propto (1-R)R^{m_c+n_c}$ at momentum q

$$R = (1 + \frac{q^2}{2g^2\mu^2})^{-1}$$

- At large q, $S_I \simeq S_E$
  - $R \simeq 0$
  - vacuum contribution dominates
- At small q, $S_I > S_E$
  - $R \sim \mathcal{O}(1)$
  - higher states and **interference terms** are also important



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Experimental observation and the entropy of ignorance

- Observation was done in particle number basis in experiment
  $S_{\text{entanglement}} \Rightarrow S_{\text{hadron}} = -\sum P(N_h)\ln(P(N_h))$
- Kharzeev & Levin
  - The reduced density matrix $\hat{\rho}_r = \sum_{N_p} P_{N_p} |N_p\rangle\langle N_p|$
  - $S = -\sum P_{N_p} \ln\big(P_{N_p}\big)$
  - At small x, entropy of gluon $\Rightarrow$ entropy of hadron

$$S \simeq \ln\big(xG(x, Q^2)\big)$$

- $\ln\big(xG(x, Q^2)\big)$ corresponds to entropy of ignorance in our calculation

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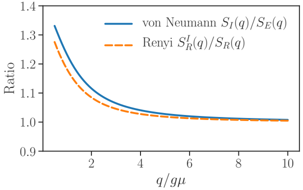
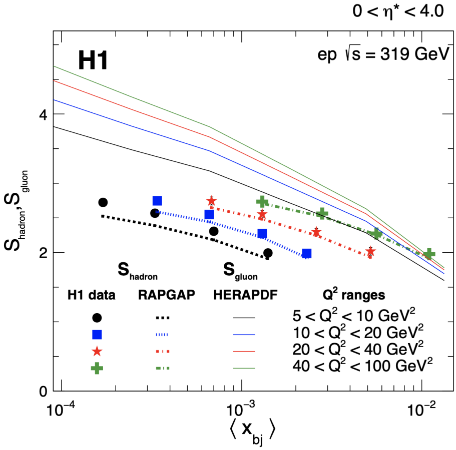
DIS data from HERA

- $\ln\left(xG(x, Q^2)\right)$ overestimates hadron entropy
- Difference becomes small at large Q^2 which is consistent with our analysis



arXiv 2011.01812, H1 Collaboration



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In what basis $S_I = S_E$?

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Thermal density matrix

A different perspective, consider the following **reduced** density matrix

$$\hat{\rho}_r = (1 - e^{-\beta\omega_0}) \sum_{n=0} e^{-n\beta\omega_0} |n\rangle\langle n|$$

where n is the energy level, and define $f = \frac{1}{e^{\beta\omega_0 - 1}}$. The corresponding von Neumann entropy is

$$S_V = (1+f)\ln(1+f) - f\ln(f)$$

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A further examination of CGC S_V

$$S_V = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2} \left[\ln\left(\frac{g^2\mu^2}{q^2}\right) + \sqrt{1 + 4\frac{g^2\mu^2}{q^2}} \ln\left(1 + \frac{q^2}{2g^2\mu^2} + \frac{q^2}{2g^2\mu^2}\sqrt{1 + 4\frac{g^2\mu^2}{q^2}}\right)\right]$$

If set $\beta\omega_0 = 2\ln\left(\frac{q}{2g\mu} + \sqrt{1 + \frac{q^2}{4g^2\mu^2}}\right)$, we recover the same structure

$$S_V = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2} \left[(1+f)\ln(1+f) - f\ln(f)\right]$$

Which indicates the leading order CGC density matrix describe a thermal system of $quasi - particles$

$$c(q) = \frac{1}{2}(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}})\ a(q) + \frac{1}{2}(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}})\ a^\dagger(-q)$$

where $\alpha = \sqrt{1 + \frac{4g^2\mu^2}{q^2}}$

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Conclusion

- CGC provides a calculable model for proton wave function
- In CGC, density matrix for soft gluons is not diagonal in particle number basis; this contradicts to Kharzeev and Levin's assumption
- Entropy of ignorance overestimates entropy of hadrons
- CGC reduced density matrix can be diagonalized into thermal form

Through diagonalization of $\hat{\rho}_r$

In field basis

$$\hat{\rho}_r = \int D[\phi, \Phi] \;\; \rho_r(\phi, \Phi)|\phi\rangle\langle\Phi|$$

To diagonalize it, we construct a wave functional

$$|\Psi\rangle = \int D[\psi] f(\psi)|\psi\rangle$$

and we then have the eigen-equation

$$\hat{\rho}_r|\Psi\rangle = \lambda|\Psi\rangle$$

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Through diagonalization of $\hat{\rho}_r$

In terms of field explicitly

$$\int D[\Phi] \;\; \rho_r(\phi, \Phi) f(\Phi) = \lambda f(\phi)$$

Our assumption is based on quantum harmonic oscillator such that the ground state is given by

$$f(\phi) = \exp\{-\alpha \phi \phi^*\}$$

One can build higher excited states use ladder operators.

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Thermal eigenvalues

It turns out, the reduced density matrix can be exactly diagonalized in the "quantum harmonic oscillator" basis, with Boltzmann weight eigenvalues.

$$\lambda_n = \exp\left\{-(\frac{1}{2} + n)\omega\beta\right\}$$

where n=0, 1, 2,.....

$$\beta\omega = 2\ln\left(\frac{q}{2g\mu} + \sqrt{1 + \frac{q^2}{4g^2\mu^2}}\right)$$

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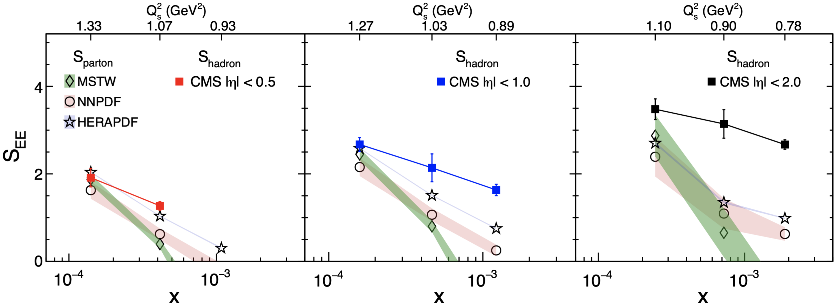
PP collision data from LHC

ArXiv.1904.11974,by Zhoudunming Tu, Dmitri E. Kharzeev, Thomas Ullrich



$$S_{Parton} = \ln\big(xG(x, Q^2)\big)$$
$$S_{hadron} = -\sum P(N) \ln P(N)$$

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