Recent lattice results on topology
(in memory of Pierre van Baal)

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LATTICE '14, New York, Columbia University

Pierre van Baal

Naarden, June 9, 1955 – Leiden, December 29, 2013

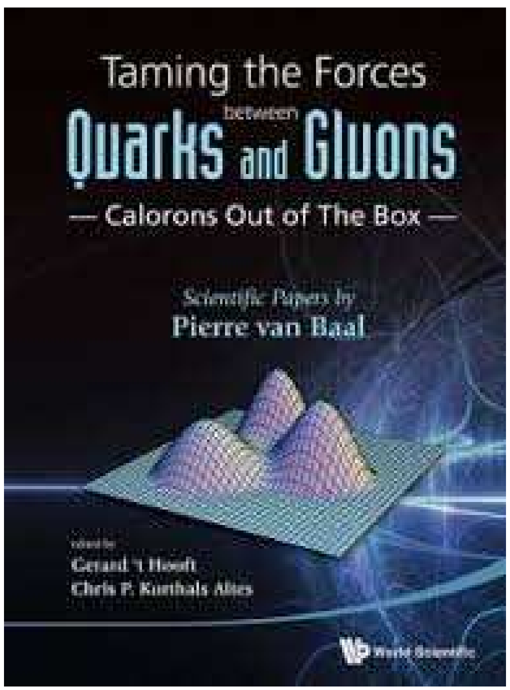
Pierre's CV

- B.Sc. in Physics 1977 and Mathematics 1978, Utrecht University.
- M.Sc. in Theoretical Physics 1980, Utrecht University.
- Ph.D. in Theoretical Physics 1984, advisor G. ‘t Hooft, Utrecht University.
- 1984 - 1987 Research Associate at ITP of Stony Brook and Fellow in Stony Brook's joint Math/Phys programme.
- 1987 - 1989 Fellow at CERN Theory Group.
- 1989 appointed as KNAW-fellow by Royal Academy of Sciences at University of Utrecht.
- 1992 appointed full professor in Field Theory and Particle Physics at Instituut-Lorentz for Theoretical Physics of the University of Leiden.

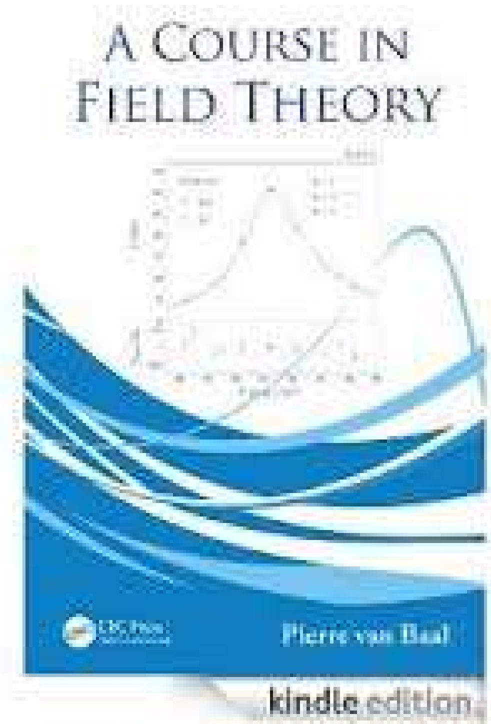
Pierre's scientific achievements

4 books + O(100) scientific papers

Collection of his scientific papers,
ed. by G. 't Hooft, C.P. Korthals Altes



His field theory book



His major topics

- SU(N) gauge fields on a torus, twisted b.c.'s - 1982 ff
  (with Jeffrey Koller)
- "Thoughts" on Gribov copies.- 1991 ff
- Instantons from over-improved cooling - 1993
  (with Margarita Garcia Perez, Antonio Gonzalez-Arroyo, Jeroen R. Snippe)
- Improved lattice actions - 1996
  (with Margarita Garcia Perez, Jeroen R. Snippe)
- Nahm transformation on a torus with twisted b.c.'2 - 1998 f
  (with Margarita Garcia Perez, Antonio Gonzalez-Arroyo, Carlos Pena)
- Periodic instantons (calorons) with nontrivial holonomy - 1998 ff
  (large series of papers with his student Thomas C. Kraan,
  lateron with Falk Bruckmann, Maxim Chernodub, Daniel Nogradi et al.)

[Courtesy to Jacobus Verbaarschot]

Pierre's stroke

"I had a stroke (bleeding in the head) on the evening of July 31, 2005. As a consequence of this I have accepted that since December 1, 2007 I am demoted to 20% and April 1, 2010 to 10% of a professorship. I could still teach (in a modified format), but since October 2008 I can not do it anymore. I can give seminars (twice as slow), but doing research (something new) is too difficult."

But now we miss him.

Outline:

1. Pierre van Baal
2. Topology, instantons, calorons - a 40 years old story
3. Measuring topology on the lattice
4. Status of $\eta' - \eta$ mixing †
5. $T > 0$: $U_A(1)$ symmetry restoration puzzle †
6. KvBLL calorons †
7. Miscellaneous
8. Summary

† not covered during the talk !

2. Topology, instantons, calorons - a 40 years old story

[Belavin, Polyakov, Schwarz, Tyupkin, '75; 't Hooft, '76; Callan, Dashen, Gross, '78 -'79]

Euclidean Yang-Mills action: $S[A] = -\frac{1}{2g^2} \int d^4x \; \mathrm{tr}\,(G_{\mu\nu} G_{\mu\nu})$

Topological charge:

$$Q_t[A] \equiv \int d^4x \; \rho_t(x), \;\; \rho_t(x) = -\frac{1}{16\pi^2} \; \mathrm{tr}\,(G_{\mu\nu} \tilde{G}_{\mu\nu}(x)), \;\; \tilde{G}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma} \, .$$

$$Q_t[A] \;\equiv\; \sum_{i=1}^{q} w_i \in \mathbf{Z} \, ,$$

w_i "windings" of continuous mappings $S^{(3)} \to SU(2)$ (homotopy classes), invariant w.r. to continuous deformations (but not on the lattice !!)

Example for topologically non-trivial field - “instanton”:

[Belavin, Polyakov, Schwarz, Tyupkin, ’75]

$$-\int d^4x \ \mathrm{tr}\,[(G_{\mu\nu} \pm \tilde{G}_{\mu\nu})^2] \geq 0 \quad \Longrightarrow \quad S[A] \ \geq \ \frac{8\pi^2}{g^2}|Q_t[A]|$$

$$\text{iff} \quad S[A] = \frac{8\pi^2}{g^2}\ |Q_t[A]|, \qquad \text{then} \quad G_{\mu\nu} = \pm\tilde{G}_{\mu\nu} \quad \text{(anti) selfduality}\,.$$

$|Q_t| = 1$: BPST one-(anti)instanton solution (singular gauge) for $SU(2)$:

$$\mathcal{A}^{(\pm)}_{a,\mu}(x-z,\rho,R) = R^{a\alpha}\eta^{(\pm)}_{\alpha\mu\nu}\frac{2\ \rho^2\ (x-z)_\nu}{(x-z)^2\ ((x-z)^2+\rho^2)},$$

For $SU(N_c)$ embedding of $SU(2)$ solutions required.

Dilute instanton gas (DIG) $\longrightarrow$ instanton liquid (IL):

path integral “approximated” by superpositions of (anti-) instantons and represented as partition function in the modular space of instanton parameters.

[Callan, Dashen, Gross, ’78 -’79; Ilgenfritz, M.-P., ’81; Shuryak, ’81 - ’82; Diakonov, Petrov, ’84]

$\Longrightarrow$ may explain chiral symmetry breaking, but fails to explain confinement.

Axial anomaly

[Adler, '69; Bell, Jackiw, '69; Bardeen, '74]

$$\partial_\mu j^{\mu 5}(x) = D(x) + 2N_f\ \rho_t(x)$$

$$\text{with}\quad j^{\mu 5}(x) = \sum_f^{N_f} \bar\psi_f(x)\gamma^\mu\gamma^5\psi_f(x),\quad D(x) = 2im\sum_{f=1}^{N_f}\bar\psi_f(x)\gamma^5\psi_f(x)$$

$\rho_t \neq 0$ due to non-trivial topology $\Longrightarrow$ solution of the $U_A(1)$ problem:

η'-meson (pseudoscalar singlet) for $m \to 0$ not a Goldstone boson, $m_{\eta'} \gg m_\pi$.

Related Ward identity:

$$\begin{aligned}
4N_f^2\int d^4x\ \langle\rho_t(x)\rho_t(0)\rangle &= 2iN_f\langle -2m\bar\psi_f\psi_f\rangle + \int d^4x\ \langle D(x)D(0)\rangle \\
&= 2iN_f m_\pi^2 F_\pi^2 + O(m^2) \\
\chi_t \equiv \frac{1}{V}\langle Q_t^2\rangle\Big|_{N_f} &= \frac{i}{2N_f}m_\pi^2 F_\pi^2 + O(m_\pi^4) \quad \to 0 \text{ for } m_\pi \to 0.
\end{aligned}$$

$1/N_c$-expansion, i.e. fermion loops suppressed ("quenched approximation")

[Witten, '79, Veneziano '79]

$$\chi_t^q = \frac{1}{V}\langle Q_t^2\rangle\Big|_{N_f=0} = \frac{1}{2N_f}F_\pi^2\ [m_{\eta'}^2 + m_\eta^2 - 2m_K^2] \simeq (180\text{MeV})^4.$$

Integrating axial anomaly we get Atiyah-Singer index theorem

$$Q_t[A] = n_+ - n_- \in \mathbf{Z}$$

$n_\pm$ number of zero modes $f_r(x)$ of Dirac operator $i\gamma^\mu \mathcal{D}_\mu[A]$
with chirality $\gamma_5 f_r = \pm f_r$.

$\Longrightarrow$ For lattice computations employ a chiral operator $i\gamma^\mu \mathcal{D}_\mu$.
$\Longrightarrow$ Not free of lattice artifacts, use improved gauge action.

Topology becomes unique only for lattice fields smooth enough.

Sufficient (!) bound to plaquette values can be given. [Lüscher, '82].

Case $T > 0$: x_4-periodic instantons - "calorons"

Semiclassical treatment of the partition function [Gross, Pisarski, Yaffe, '81]
with "caloron" solution $\equiv$ x_4-periodic instanton chain ($1/T = b$)
[Harrington, Shepard, '77]

$$A_\mu^{a\,\mathrm{HS}}(x) = \eta^{(\pm)}_{a\mu\nu}\ \partial_\nu \log(\Phi(x))$$

$$\Phi(x) - 1 = \sum_{k \in \mathbf{Z}} \frac{\rho^2}{(\vec{x}-\vec{z})^2 + (x_4 - z_4 - kb)^2} = \frac{\pi\rho^2}{b|\vec{x}-\vec{z}|}\ \frac{\sinh\left(\frac{2\pi}{b}|\vec{x}-\vec{z}|\right)}{\cosh\left(\frac{2\pi}{b}|\vec{x}-\vec{z}|\right) - \cos\left(\frac{2\pi}{b}(x_4 - z_4)\right)}$$

- $Q_t = -\frac{1}{16\pi^2}\int_0^b\ dx_4\ \int\ d^3x\ \rho_t(x) = \pm 1\,.$

- (as for instantons) it exhibits trivial holonomy, i.e. Polyakov loop behaves as:

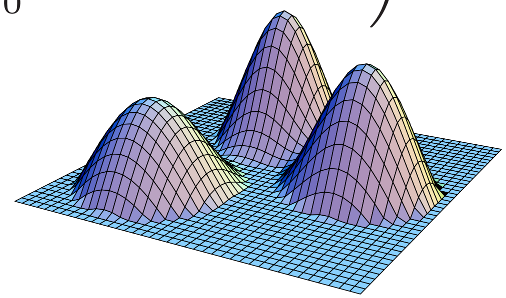
$$\frac{1}{2}\mathrm{tr}\,\mathbf{P}\exp\left(i\int_0^b A_4(\vec{x},t)\ dt\right) \overset{|\vec{x}|\to\infty}{\Longrightarrow} \pm 1$$

Kraan - van Baal solutions

= (anti-) selfdual caloron solutions with non-trivial holonomy

[K. Lee, Lu, '98, Kraan, van Baal, '98 - '99, Garcia-Perez et al. '99]

$$P(\vec{x}) = \mathbf{P} \exp\left(i \int_0^{b=1/T} A_4(\vec{x},t)\ dt \right) \stackrel{|\vec{x}|\to\infty}{\Longrightarrow} \quad \mathcal{P}_\infty \notin \mathbf{Z}(N_c)$$



Action density of a single (but dissolved) $SU(3)$ caloron with $Q_t = 1$ (van Baal, '99)
$\Longrightarrow$ not a simple $SU(2)$ embedding into $SU(3)$!!

Dissociation into caloron constituents (BPS monopoles or "dyons") gives hope for modelling confinement for $T < T_c$ as well as the deconfinement transition.

[Gerhold, Ilgenfritz, M.-P., '07; Diakonov, Petrov, et al., '07 - '12; Bruckmann, Dinter, Ilgenfritz, Maier, M.-P., Wagner, '12; Shuryak, Sulejmanpasic, '12-'13; Faccioli, Shuryak, '13; cf. talk by E.Shuryak]

Systematic development of the semiclassical approach + perturbation theory
"resurgent trans-series expansions" ...

[Dunne, Ünsal and collaborators, '12-'14; cf. talk by M. Ünsal]

3. Measuring topology on the lattice:

Gauge field approaches:

- Field theoretic with (improved) loop discretization of $G_{\mu\nu}$
  [Fabricius, Di Vecchia, G.C. Rossi, Veneziano, '81; Makhaldiani, M.-P., '83]
  in combination with cooling, 4d APE smearing, HYP smearing, (inverse) blocking or cycling, gradient flow,... = *smoothing*.
  $\Longrightarrow$ approximate integer Q_t.
  $\Longrightarrow$ allows to reveal large-scale topological structures (instantons, calorons, dyons,..)

- Geometric definitions [Lüscher, '82; Woit, '83; Phillips, Stone, '86],
  (used with and without *smoothing*).

Fermionic approaches:

- Index of Ginsparg-Wilson fermion operators: $Q_t = n_+ - n_-$

[Hasenfratz, Laliena, Niedermayer, '98; Neuberger, '01;...]

- From corresponding spectral representation of ρ_t

$$\rho_t(x) = \mathrm{tr}\,\gamma_5(\frac{1}{2}D_{x,x} - 1) = \sum_{n=1}^{N}(\frac{\lambda_n}{2} - 1)\psi_n^\dagger(x)\gamma_5\psi_n(x)$$

- Index from spectral flow of Hermitian Wilson-Dirac operator

[Edwards, Heller, Narayanan, '98]

- Fermionic representation: [Smit, Vink, '87]

$$N_f Q_t = \kappa\ \mathrm{Tr}\frac{m\ \gamma_5}{D+m}, \qquad \kappa\ \text{ renorm. factor}$$

- Topological susceptibility from higher moments and spectral projectors

[Lüscher, '04; Giusti, Lüscher, '08; Lüscher, Palombi, '10; Cichy, Garcia Ramos, Jansen, '13-'14

(A) Cooling versus gradient (Wilson) flow:

Cooling:

Old days lattice search for multi-instanton solutions

[Berg,'81; Iwasaki, et al., '83; Teper, '85; Ilgenfritz, Laursen, M.-P., Schierholz, '86],

lateron, for non-trivial holonomy KvBLL calorons

[Garcia Perez, Gonzalez-Arroyo, Montero, van Baal, '99; Ilgenfritz, Martemyanov, M.-P., Shcheredin, Veselov, '02; Ilgenfritz, M.-P., Peschka, '05]

- Solve the lattice field equation locally (for a given link variable),
- replace old by new link variable,
- step through the lattice (order not unique),
- find plateau values for the topological charge and action.

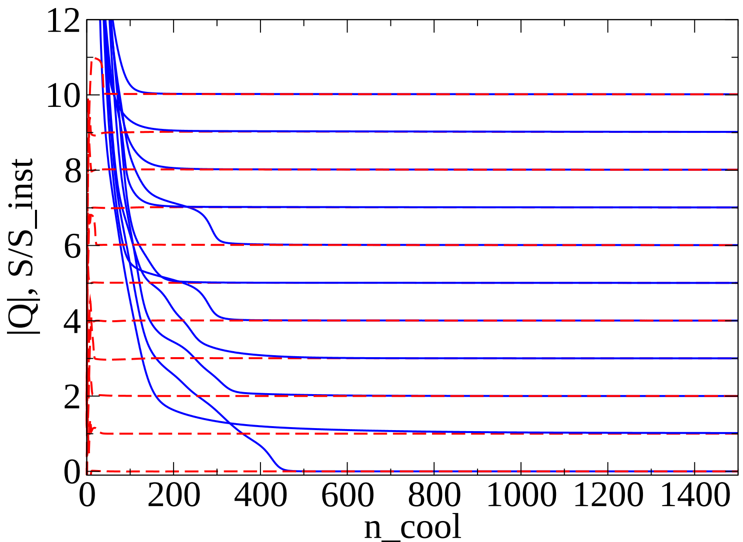
- Over-improved cooling and improved $G_{\mu\nu}$

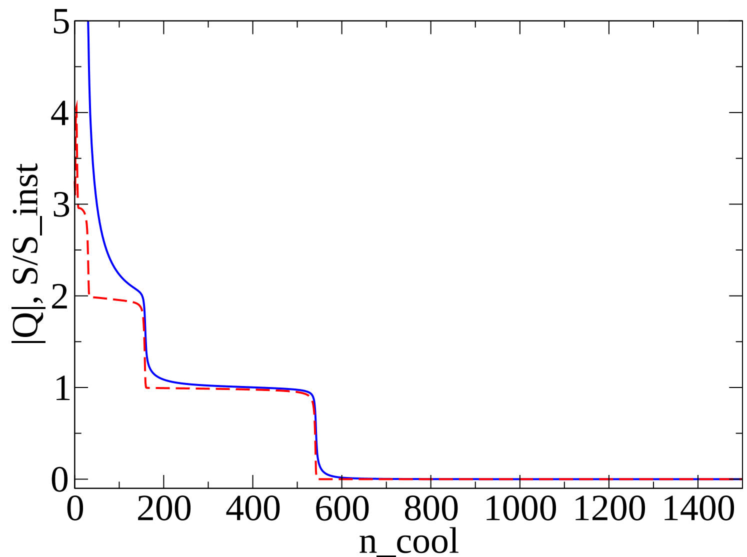
$\Longrightarrow$ for $T < T_c$ early and extremely stable plateaus at nearly integer Q_t.

[Garcia Perez, Gonzalez Arroyo, Snippe, van Baal, '94; de Forcrand, Garcia Perez, Stamatescu, '96; Bruckmann, Ilgenfritz, Martemyanov, van Baal, '04]

Typical examples of gluodynamics for $T > 0$ (Q_t, S).

$T = 0.88 T_c$ $\qquad$ $T = 1.12 T_c$





[Bornyakov, Ilgenfritz, Martemyanov, M.-P., Mitrjushkin, '13]

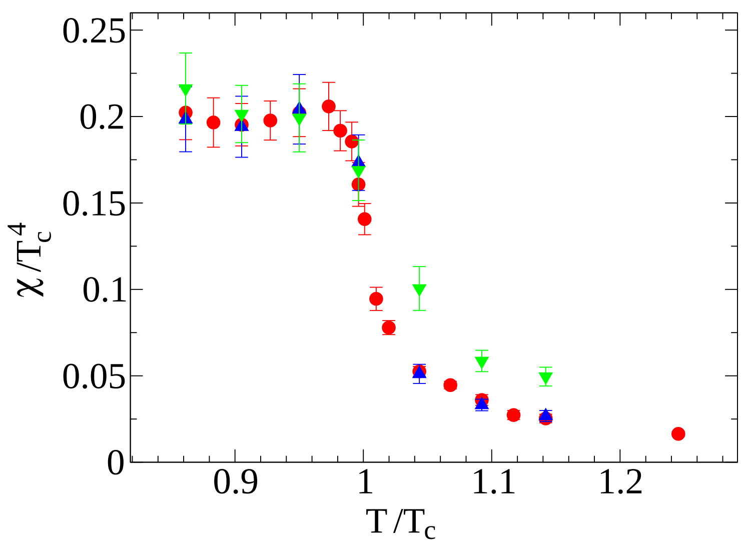
Stability (decay) of plateaus for $T < T_c$ ($T > T_c$) related to KvBLL caloron structure and non-trivial (trivial) holonomy (dyon mass symmetry / asymmetry).

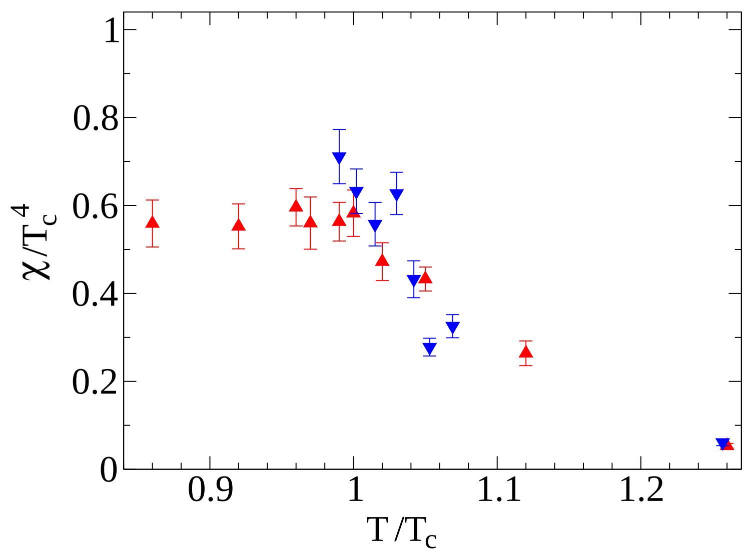
Topological susceptibility χ_t (for two lattice sizes and spacings):

gluodynamics

full QCD

(clover-impr. $N_f = 2\,, \quad m_\pi \simeq 1$ GeV)





- χ_t smoother in full QCD (crossover) than in gluodynamics (1st order).
  Should show up also in the $U_A(1)$ restoration.
- What about chiral limit ?

Gradient flow:

Proposed and thoroughly investigated by M. Lüscher
(perturbatively with P. Weisz) since 2009 (cf. talks at LATTICE 2010 and 2013).

Flow time evolution uniquely defined for arbitrary lattice field $\{U_\mu(x)\}$ by

$$\dot{V}_\mu(x,\tau) = -g_0^2 \Big[\partial_{x,\mu} S(V(\tau))\Big] V_\mu(x,\tau), \quad V_\mu(x,0) = U_\mu(x) \ .$$

- Diffusion process continuously minimizing action, scale $\lambda_s \simeq \sqrt{8t}, \quad t = a^2\tau$.
- Allows efficient scale-setting (t_0, t_1)
  by demanding e.g. $t^2 \langle -\frac{1}{2}\mathrm{tr}\ G_{\mu\nu}G_{\mu\nu}\rangle|_{t=t_0,t_1} = 0.3, \quad \frac{2}{3}$.
- Emergence of topological sectors at sufficient large length scale becomes clear.
- Renormalization becomes simple (in particular in the fermionic sector).

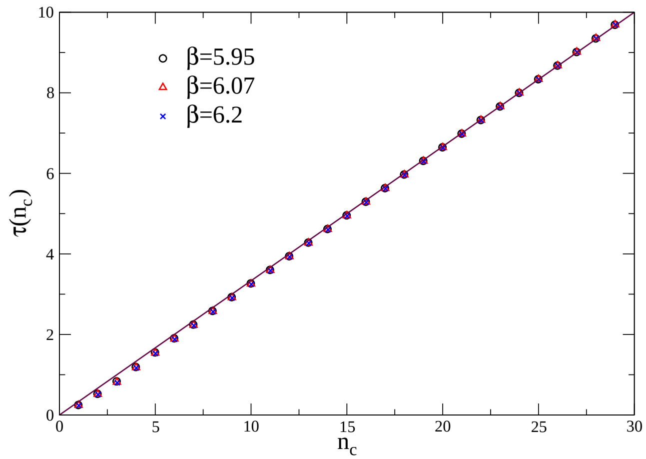
$\Longrightarrow$ Easy to handle, theoretically sound prescription !!

Comparison gradient flow with cooling: [C. Bonati, M. D'Elia, 1401.2441]

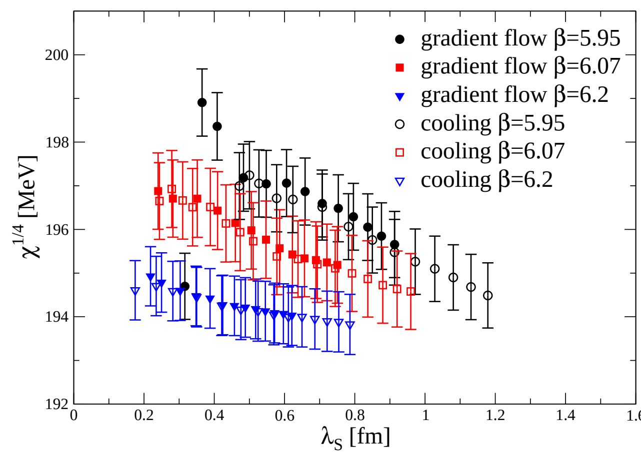
Pure gluodynamics:

- For given number of cooling sweeps n_c find gradient flow time τ yielding same Wilson plaquette action value.
- Perturbation theory: $\tau = n_c/3$

τ/n_c scaling



$\chi_t^{\frac{1}{4}}$ vs. λ_s



- Lattice spacing dependence at fixed λ_s clearly visible.
- Moreover: cooling and gradient flow show same spatial topological structure. Holds also for $\rho_t(x)$ filtered with adjusted # ferm. (overlap) modes [Solbrig et al., '07; Ilgenfritz et al., '08].
- Comparison smearing and gradient flow for Wilson loops [cf. talk by M. Okawa]

(B) Full QCD case: mass dependence of χ_t ?

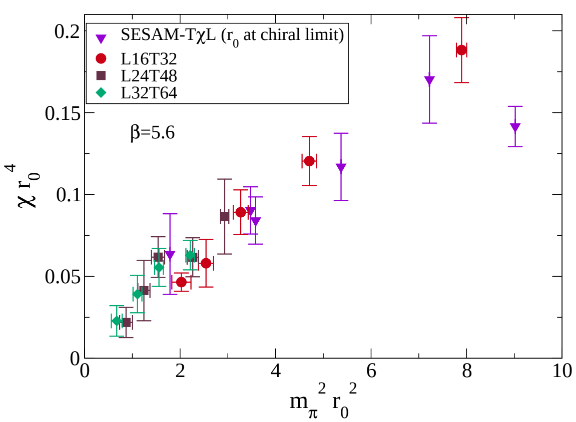
Only quite recently the expected chiral behavior $\chi_t \sim F_\pi^2 m_\pi^2 \sim m_q \langle \bar{q}q \rangle$ becomes clearly established.

SINP Kolkata group [A. Chowdhury et al., '11-'12]:

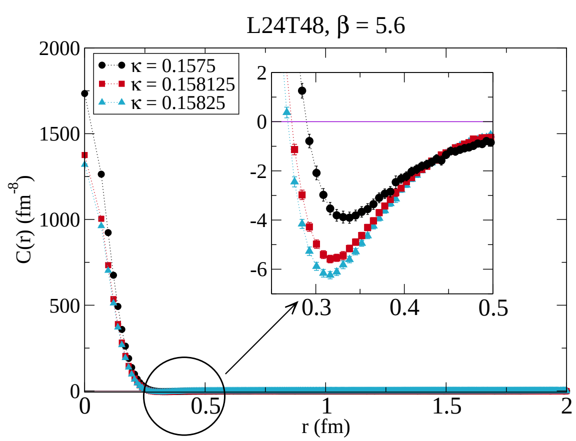
- standard Wilson gauge and fermion action ($N_f = 2$), $m_\pi \geq 300$ MeV
- Q_t measured after blocking-inverse blocking (*smoothing*) with improved ρ_t

  [DeGrand, A. Hasenfratz, Kovacs, '97; A. Hasenfratz, Nieter, '98]
- top. correlation function for varying volume and quark mass studied,
- strong lattice spacing effect seen !!

χ_t vs. m_π^2



ρ_t corr. at diff. m_q



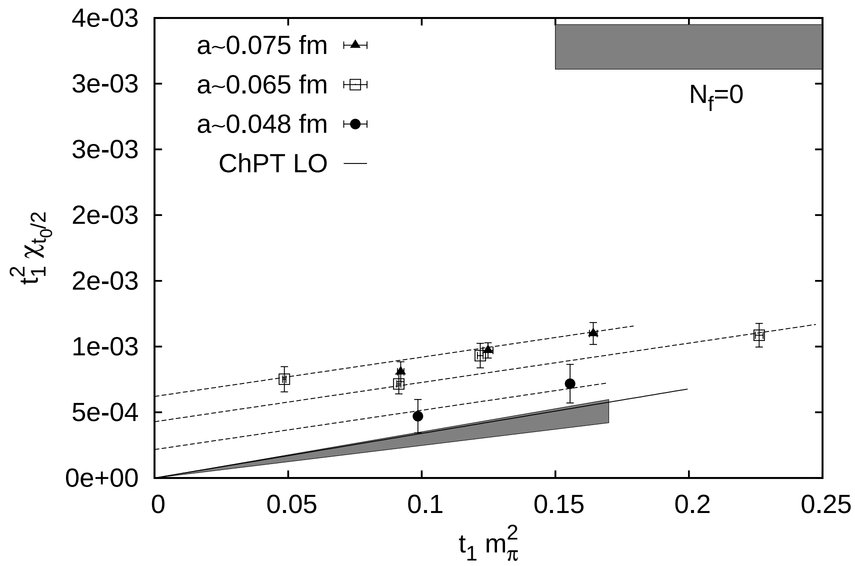
Gradient flow analysis by ALPHA collaboration

[Bruno, Schäfer, Sommer; cf. talk by M. Bruno].

$N_f = 2$ LQCD with O(a)-improved Wilson fermions and Wilson gauge action.

CLS ensembles for 3 lattice spacings and $m_\pi \in [190, 630]$ MeV with $Lm_\pi > 4$.

- Study periodic as well as open b.c.'s.
- Surprise: Q_t autocorrelations weaker with decreasing pion mass.
- Overall fit with χPT ansatz: $t_1 \chi_t = c\ t_1 m_\pi^2 + b\ \frac{a^2}{t_1}$.



Lattice artifacts quite strong.

Chiral limit requires continuum limit.

$\chi_t|_{N_f=2}$ significantly smaller than $\chi_t|_{N_f=0}$.

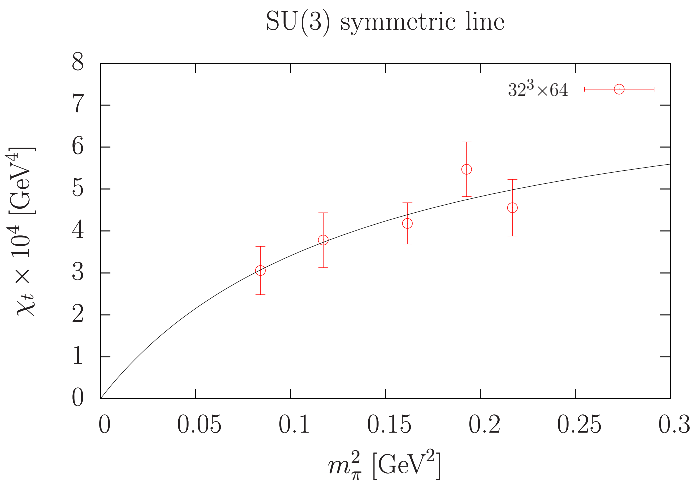
Cont. limit for grad. flow can be improved

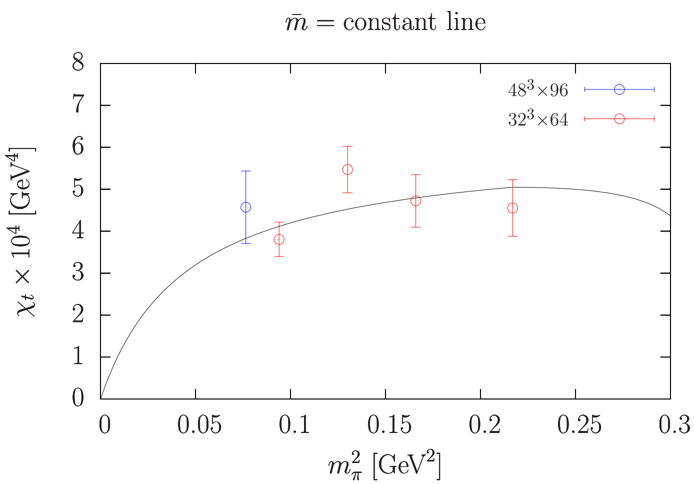
[cf. Talks by S. Sint; D. Nogradi]

Similar gradient flow analysis by QCDSF for $N_f = 2 + 1$ [Horsley et al., '14]

- (tree-level) Symanzik improved gauge action
- (stout smeared) clover-improved Wilson fermions
- two chiral limit strategies:

$$m_u = m_d = m_s \qquad\qquad m_u + m_d + m_s = \bar{(m)} = \text{const.}.$$





- lines correspond to chiral fits based on flavor-singlet and flavor-octet Gell-Mann-Oakes-Renner relations and

(C) Spectral projectors applied to twisted mass fermions (ETMC):

[Cichy, Garcia Ramos, Jansen, '13; cf. talks by E. Garcia Ramos, K. Cichy]

- Represent χ_t via singularity-free density chain correlators [Lüscher, '04]
- treated with spectral projectors $\mathbf{P}_M$ [Giusti, Lüscher, '09] projecting onto subspace of $D^\dagger D$ eigenmodes below threshold M^2
- $\mathbf{P}_M$ approx. by rational function $\mathbf{R}_M$ [Lüscher, Palumbo, '10]

$$\begin{aligned}\chi_t &= \frac{\langle \mathrm{Tr}\,\{\mathbf{R}_M^4\}\rangle}{\langle \mathrm{Tr}\,\{\gamma_5\mathbf{R}_M^2\gamma_5\mathbf{R}_M^2\}\rangle}\frac{\langle \mathrm{Tr}\,\{\gamma_5\mathbf{R}_M^2\}\mathrm{Tr}\,\{\gamma_5\mathbf{R}_M^2\}\rangle}{V} \\ &= \frac{Z_S^2}{Z_P^2}\,\frac{\langle \mathrm{Tr}\,\{\gamma_5\mathbf{R}_M^2\}\mathrm{Tr}\,\{\gamma_5\mathbf{R}_M^2\}\rangle}{V} = \frac{Z_S^2}{Z_P^2}\,\frac{\langle \mathcal{C}^2\rangle - \frac{\langle\mathcal{B}\rangle}{N}}{V}\end{aligned}$$

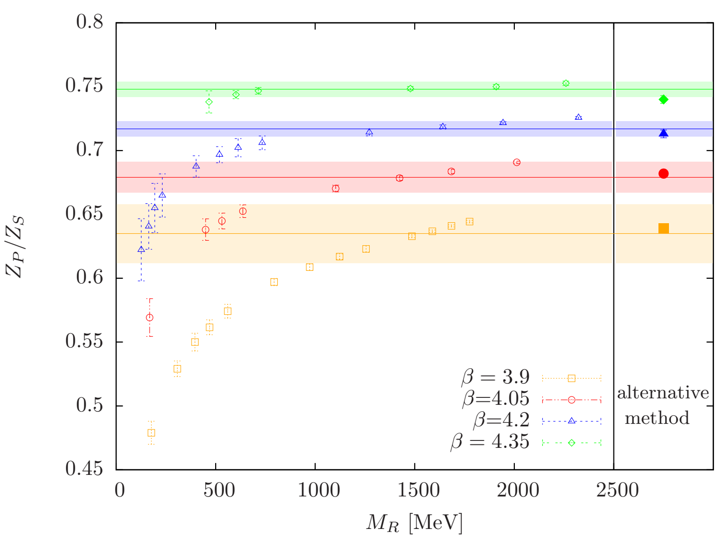
with $Z(2)$ random estimators for $\mathcal{B}$ and $\mathcal{C}$: $\mathcal{C} = \frac{1}{N}\sum_{k=1}^{N}(\mathbf{R}_M\eta_k, \gamma_5\mathbf{R}_M\eta_k)$.

- Note: renorm. constants $\frac{Z_S}{Z_P} = 1$ and $\mathcal{C} \equiv Q_t \in \mathbb{Z}$ for $N \to \infty$ for Ginsparg-Wilson operators D (e.g. overlap).

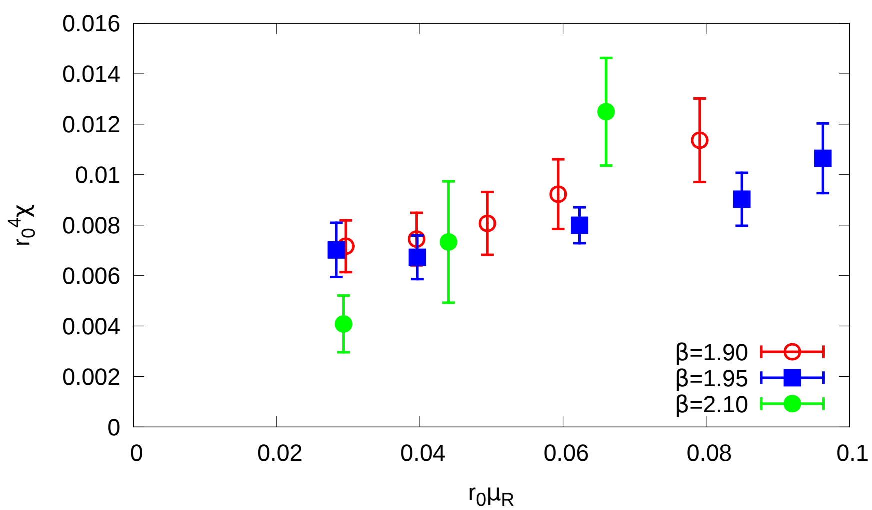
- Authors study: $N_f = 2$, $N_f = 2+1+1$ w. r. to a, m_q dependence
- improved gauge actions and Wilson twisted-mass fermions used
  automatical $O(a)$ improvement $\Longrightarrow$ weak a-dependence (?)
- renormalization constants Z_S, Z_P with projector method computed,
  find consistency with other methods
- $C \sim Q_t$ values turn out nicely gaussian distributed
- from χ_t knowing μ_R the light quark condensate can be estimated: works well.
- $\chi_t|$quenched $\Longrightarrow$ consistent with Witten-Veneziano.

  [cf. Garcia Ramos' talk]

renorm. constants vs. M_R ($N_f = 2$)



$r_0^4\chi_t$ vs. renorm. quark mass $r_0\mu_R$ ($N_f = 2+1+1$)



(D) Joint ETMC effort: compare various methods for Q_t and χ_t:

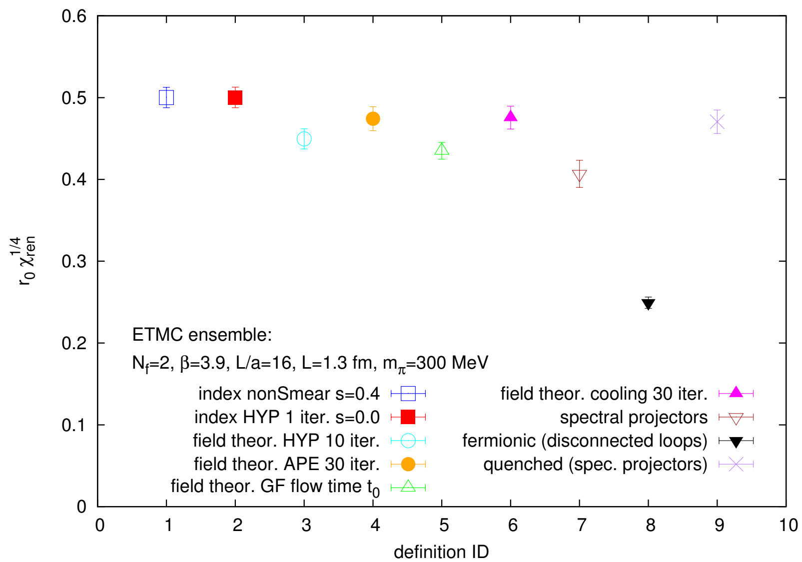
[cf. talk by K. Cichy]

- $N_f = 2$ twisted mass fermions, tree level Symanzik improved gauge action
- all computations on one set of configs.:

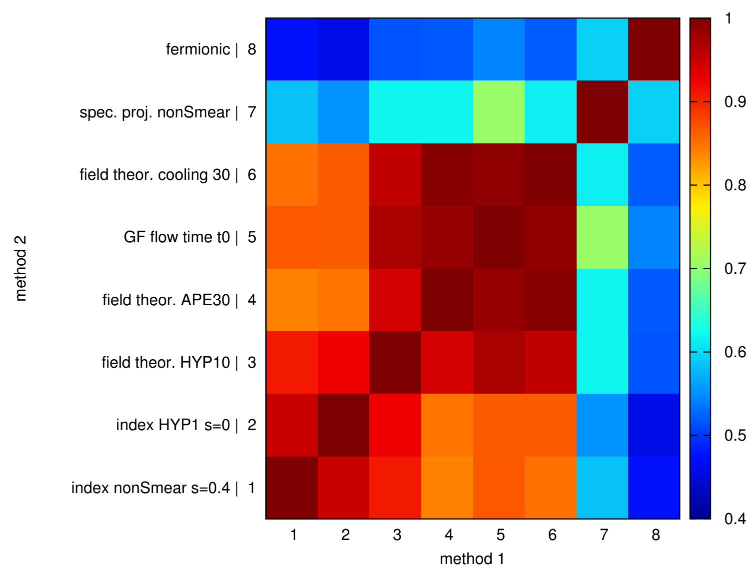
$$m_\pi = 300 \text{ MeV}, \ a = .081 \text{ fm}, \ L = 1.3 \text{ fm}$$

- deviations possible because of different lattice scale dependence.

Preliminary: χ_t values



Q_t correlation



$\Longrightarrow$ only stronger deviation found with fermionic (Smit-Vink) method.

$\Longrightarrow$ quenched case with projector method not enhanced (?)

4. Status of $\eta' - \eta$ mixing

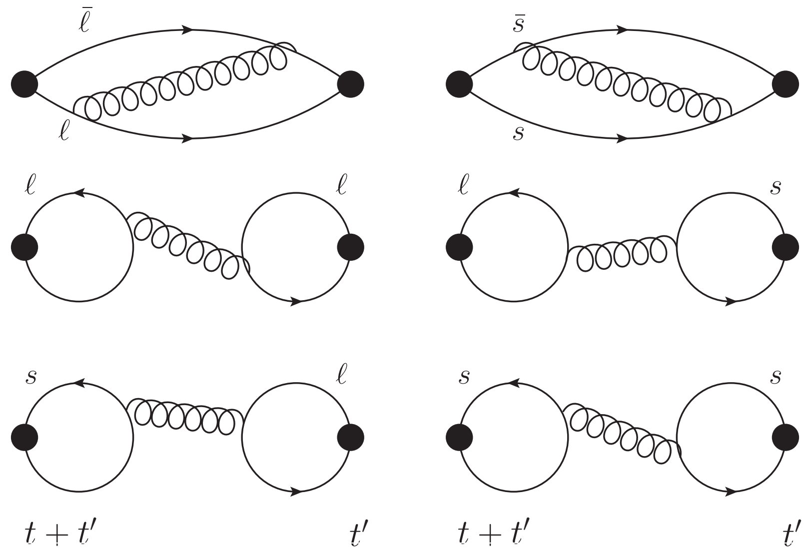
Earlier investigations:

[N. Christ et al, '10; Dudek et al, '11, '13; Gregory et al, '12]

Recent convincing $N_f = 2 + 1 + 1$ twisted mass fermion analysis

[C. Michael, K. Ottnad, C. Urbach (ETMC), PRL 111 (2013) 18, 181602]

Most important to estimate disconnected quark diagrams:



connected $l = (u, d)$ and s diagrams

disconnected l, s diagrams

from t' to $t' + t$

Possible only due to various powerful noise reduction techniques

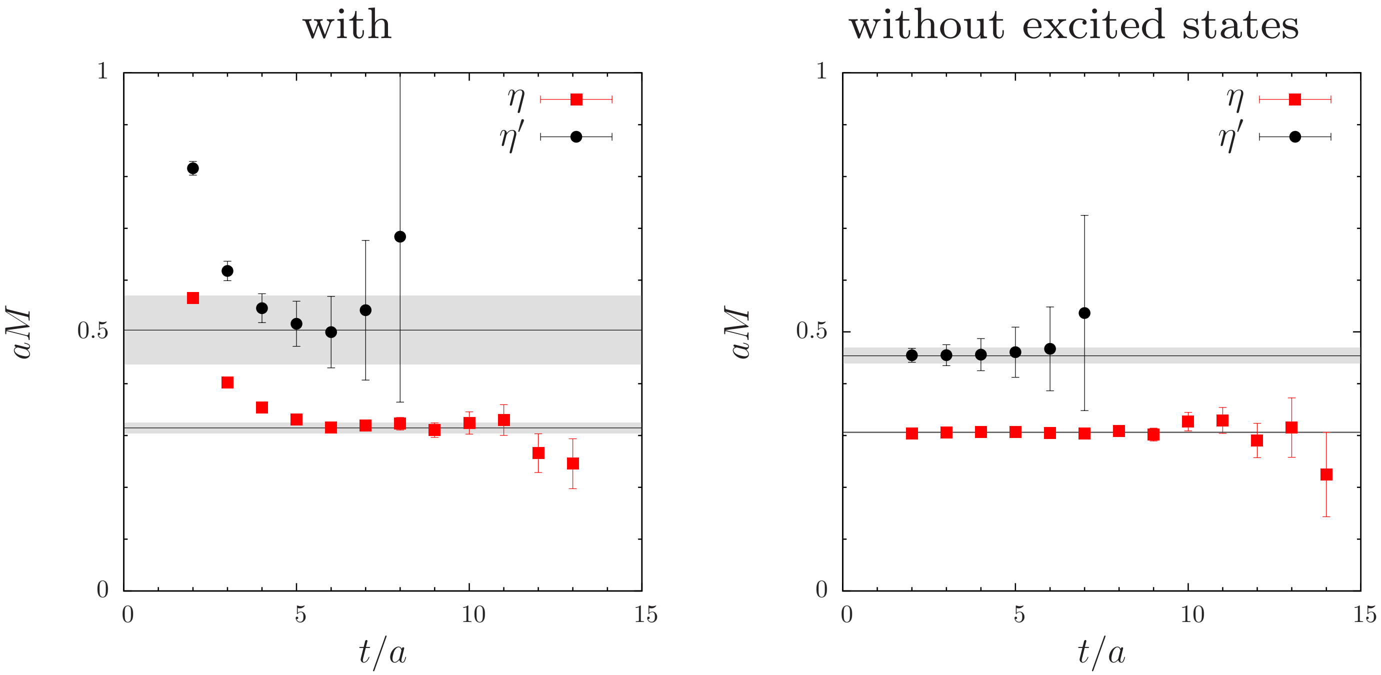
[Boucaud et al. (ETMC), '08; Jansen, Michael, Urbach (ETMC), '08]

Compute correlators

$$\mathcal{C}(t)_{qq'} = \langle \mathcal{O}_q(t'+t)\mathcal{O}_{q'}(t') \rangle \,, \quad q, q' \in l, s, c \,,$$

$\mathcal{O}_l = (\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d)/\sqrt{2}$, $\mathcal{O}_s = \bar{s}i\gamma_5 s$, $\mathcal{O}_c = \bar{c}i\gamma_5 c$ (including fuzzy op's.)

solve generalized eigenvalue problem and find effective η', η masses
(assume that excited states in connected contributions can be removed)

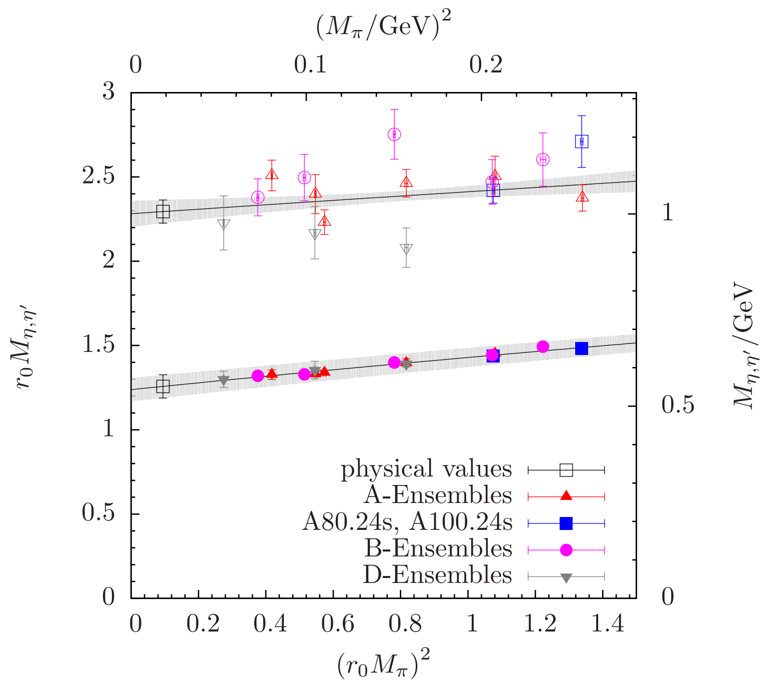


Simulations:

$a = (0.086,\ 0.078,\ 0.061)$ fm, $\quad L > 3$ fm, $\quad m_\pi L > 3.5$,

230 MeV $\leq m_\pi \leq$ 510 MeV, $\quad s$-quark mass tuned to phys. K mass.

Chiral extrapolation for η', η masses



$$M_\eta = 551(8)_{\mathrm{stat}}(6)_{\mathrm{sys}}\ \mathrm{MeV} \qquad (\mathrm{PDG\ \ exp.\ \ 547.85(2)\ MeV})$$

$$M_{\eta'} = 1006(54)_{\mathrm{stat}}(38)_{\mathrm{sys}}(+61)_{\mathrm{ex}}\ \mathrm{MeV} \qquad (\mathrm{PDG\ \ exp.\ \ 957.78(6)\ MeV})$$

Nice confirmation of the topological mechanism related to the axial anomaly.

Comment: η' physics studied with staggered fermions: what about *rooting* ?

massive Schwinger model investigated [S. Dürr, '12]

$\Longrightarrow$ anomaly correctly treated, *rooting* effectively works.

5. $T > 0$: $U_A(1)$ symmetry restoration puzzle

Question:

How $U_A(1)$ symmetry gets restored at or above T_c for $N_f = 2$ light flavors?

Common view:

- $U_A(1)$ monotonously restored for $T > T_c$ $\Longrightarrow$ 2nd order, $O(4)$ universality
- $U_A(1)$ restored at $T = T_c$ $\Longrightarrow$ 1st order

Recent theoretical work:

high-order pert. study of RG flow within 3D Φ^4 theory [Pelisetto, Vicari, '13].
Claim to find a stable FP, such that
$U_A(1)$ restored at $T = T_c$ can be accompanied by continuous transition,
but with critical behavior slightly differing from $O(4)$.

[see also talk by T. Sato]

Lattice studies:

LLNL/RBC (Buchoff et al., '13), Bielefeld (Sharma et al., '13),

JLQCD (Cossu, S. Aoki et al., '13), Regensbg.-Mainz-Frankfurt (Brandt et al.)

LLNL/RBC $N_f = 2 + 1$ domain wall fermion study:

- combined Iwasaki and dislocation suppressing gauge action
- compute Dirac eigenvalue spectrum, chiral condensates, susceptibities
- large volumes (up to 4 fm ... 5.6 fm), pion mass $\simeq$ 200 MeV
- pseudo-critical $T_c \simeq 165$ MeV.

- correlation functions of operators

$$\sigma = \overline{\psi}_l \psi_l, \qquad \delta^i = \overline{\psi}_l \tau^i \psi_l, \qquad \eta = i\overline{\psi}_l \gamma^5 \psi_l, \qquad \pi^i = i\overline{\psi}_l \tau^i \gamma^5 \psi_l.$$

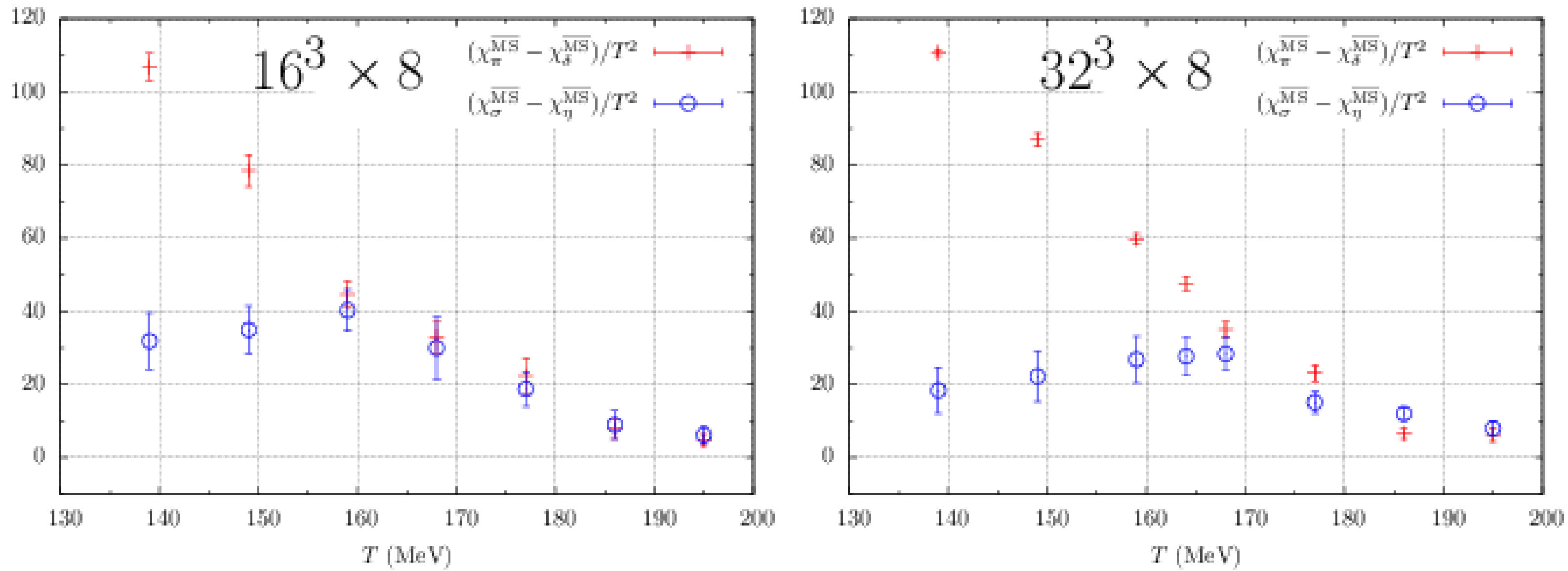
- for their susceptibilities $\chi_I, \quad I = \sigma, \delta^i, \eta, \pi^i \quad$ (correlators at $q^2 = 0$) hold symmetry relations

$$\left.\begin{array}{ccc} \chi_\sigma & = & \chi_\pi \\ \chi_\eta & = & \chi_\delta \end{array}\right\} \qquad SU(2)_L \times SU(2)_R,$$

$$\left.\begin{array}{ccc} \chi_\sigma & = & \chi_\eta \\ \chi_\pi & = & \chi_\delta \end{array}\right\} \qquad U(1)_A$$

Note: $\chi_\sigma = \chi_\delta + 2\chi_{disc}, \quad \chi_\eta = \chi_\pi - 2\chi_{5,disc}$ with disconnected parts.

Main result: $U_A(1)$-violating renorm. susceptibilities in $\overline{MS}$ scheme.



Not vanishing around $T_c \simeq 165$ MeV ! $\implies$ $U_A(1)$ breaking for $T > T_c$.

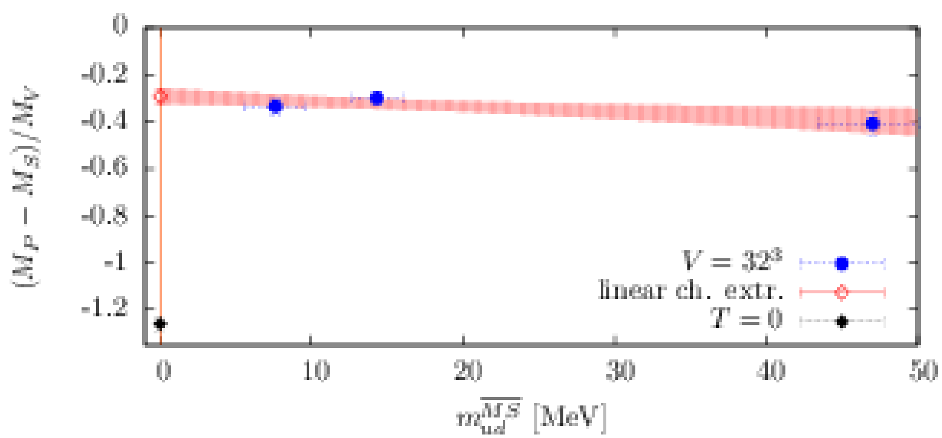
What about the chiral limit, where top. susceptibilities are expected vanish ?

Result is supported by

- Bielefeld study with overlap valence quarks [Sharma et al., '13], and by

- Regensburg-Mainz-Frankfurt study: comparing screening masses in pseudoscalar and scalar channel.
  $N_f = 2$ LQCD with clover-improved fermions at $m_\pi = 540, 290, 200$ MeV.
  [Brandt et al., '12, '13 and priv. communication]

Show ratio $(M_P - M_S)/M_V$ vs. quark mass at T_c.
Does not seem to vanish in the chiral limit !



JLQCD's explorative study:

Dynamical overlap fermions ($N_f = 2$) studied in a fixed topological sector.
Topological susceptibility from finite-volume corrections

$$\lim_{|x|\to\infty} \langle mP(x)mP(0)\rangle_Q = \frac{1}{V}\left(\frac{Q^2}{V} - \chi_t - \frac{c_4}{2\chi_t V}\right) + O(e^{-m_\eta |x|}).$$

[for a systematic expansion see: Dromard, Wagner, '14 and talk by A. Dromard]

- small lattice size $16^3 \times 8$, but various quark masses,
- eigenvalue spectrum: close to T_c gap for decreasing m_q
- represent disconnected iso-singlet scalar and pseudo-scalar meson correlators through low-lying modes.
- compare (pseudo-) scalar singlet and triplet correlators degenerate close to T_c for small enough m_q.

$\Longrightarrow$ systematic finite volume error analysis required

$\Longrightarrow$ JLQCD switched to domain wall fermions, so far preliminary results.

[cf. talks by G. Cossu and A. Tomiya]

6. Properties of $SU(2)$ (single) calorons with non-trivial holonomy

[K. Lee, Lu, '98, Kraan, van Baal, '98 - '99, Garcia-Perez et al. '99]

$$P(\vec{x}) = \mathbf{P} \exp \left(i \int_0^{b=1/T} A_4(\vec{x}, t)\ dt \right) \stackrel{|\vec{x}| \to \infty}{\Longrightarrow} \quad \mathcal{P}_\infty = e^{2\pi i \omega \tau_3} \notin \mathbf{Z}(2)$$

Holonomy parameter: $0 \leq \omega \leq \frac{1}{2}$, $\omega = \frac{1}{4}$ – maximally non-trivial holonomy.

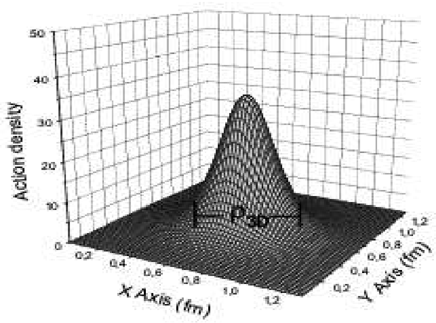
- (anti)selfdual with topological charge $Q_t = \pm 1$,
- at positions $\vec{x}_1, \vec{x}_2$, where local holonomy has identical eigenvalues, identify constituents $\Rightarrow$ "dyons" or "instanton quarks", carrying opposite magnetic charge (maximally Abelian gauge),
- limiting cases:
  - $\omega \to 0 \quad \Longrightarrow$ 'old' HS caloron,
  - $|\vec{x}_1 - \vec{x}_2|$ small $\Longrightarrow$ non-static single caloron (CAL),
  - $|\vec{x}_1 - \vec{x}_2|$ large $\Longrightarrow$ two static BPS monopoles or "dyon pair" (DD) with topological charges ($\sim$ masses) $|Q_t^{\rm dyon}| = 2\omega, \quad 1 - 2\omega$.
- $L(\vec{x}) = \frac{1}{2} {\rm tr} P(\vec{x}) \to \pm 1$ close to $\vec{x} \simeq \vec{x}_{1,2} \Longrightarrow$ "dipole" structure
- carries center vortex - percolating at maximally non-trivial holonomy

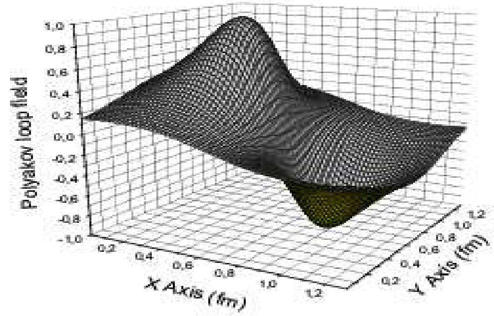
[Bruckmann; Ilgenfritz, Martemyanov, Bo Zhang, '09]

Portrait of an $SU(2)$ KvBLL caloron with max. non-trivial holonomy

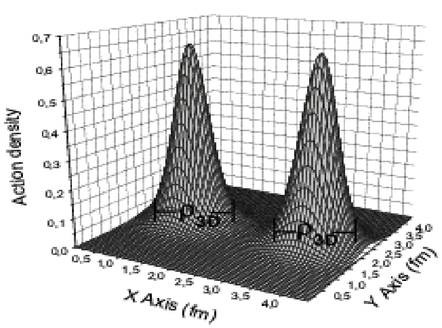
Action density | Polyakov loop

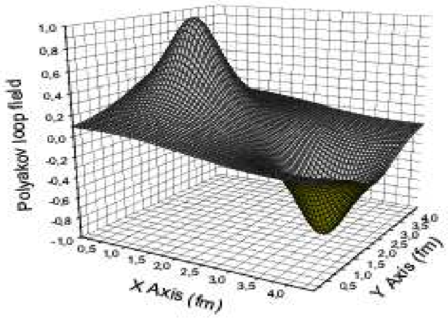
singly localized caloron (CAL)





caloron dissolved into dyon-dyon pair (DD)





Plotted with the help of Pierre van Baal's caloron codes available at:
http://www.lorentz.leidenuniv.nl/research/vanbaal/DECEASED/Caloron.html.

See also [Garcia Perez, Gonzalez-Arroyo, Montero, van Baal, '99;
Ilgenfritz, Martemyanov, Müller-Preussker, Shcheredin, Veselov, '02]

- Localization of the zero-mode of the Dirac operator:

- x_4-antiperiodic b.c.:
  around the center with $L(\vec{x}_1) = -1$,

$$|\psi^-(x)|^2 = -\frac{1}{4\pi}\partial_\mu^2 \left[\tanh(2\pi r\bar{\omega})/r\right] \quad \text{for large} \quad d,$$

- x_4-periodic b.c.:
  around the center with $L(\vec{x}_2) = +1$,

$$|\psi^+(x)|^2 = -\frac{1}{4\pi}\partial_\mu^2 \left[\tanh(2\pi s\omega)/s\right] \quad \text{for large} \quad d.$$

Search for signatures of KvBLL calorons / dyons in MC generated fields:

[Bornyakov, Ilgenfritz, Martemyanov, M.-P.,. . ., '02 - '13;

see also F. Bruckmann, P. van Baal et al., NPB (Proc.Suppl.) 140 (2005) 635]

- Apply smoothing and/or filtering with overlap Dirac operator eigenmodes.
- Find clusters of topological charge density.
- Study their local correlations with local holonomy and Abelian monopoles.
- Study hopping of localized modes while varying fermionic b.c.'s.
  [Gattringer, Pullirsch, '04]

Qualitative topological model emerging for YM theory at $T > 0$,
here for $SU(2)$ (analogously $SU(3)$):

Occurence of (anti) calorons and dyons at $T < T_c$ differs from $T > T_c$.

$T < T_c$: maximally non-trivial holonomy determined by $< L > \simeq 0$

$\longrightarrow$ dyons have same 'mass', i.e. identical statistical weight.

$\longrightarrow$ (dissociating) calorons dominate.

$\longrightarrow$ topological susceptibility $\chi_t \neq 0$.

$T \gg T_c$: trivial holonomy determined by $< L > \simeq \pm 1$

$\longrightarrow$ dyons have different 'mass', i.e. different statistical weight.

$\longrightarrow$ heavy dyons are missing, i.e. complete calorons are suppressed.

$\longrightarrow$ topological susceptibility gets suppressed $\chi_t \to 0$,
while (light) magnetic monopoles are surviving
(spatial Wilson loop area law).

Simulating caloron ensembles

[Gerhold, Ilgenfritz, M.-P., '07]

Model: random superpositions of KvBLL calorons.

Influence of the holonomy

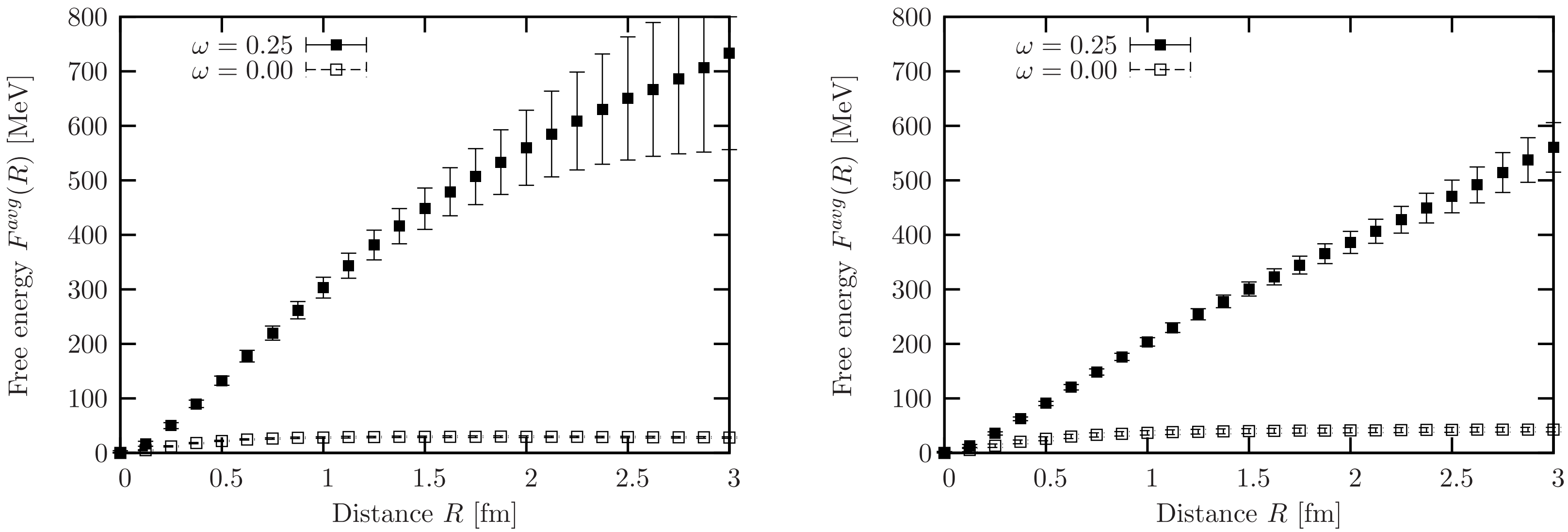
- put (anti-) calorons randomly in a 3d box with open b.c.'s, with same asymptotic holonomy for all (anti)calorons: $\mathcal{P}_\infty = \exp 2\pi i \omega \tau_3$, $\omega = 0$ – trivial versus $\omega = 1/4$ – maximally non-trivial,
- fix parameters as for IL model and lattice scale: temperature: $T = 1\ \mathrm{fm}^{-1} \simeq T_c$, density: $n = 1\ \mathrm{fm}^{-4}$, scale size: (a) fixed $\rho = 0.33$ fm (b) distribution $D(\rho) \propto \rho^{7/3} \exp(-c\rho^2)$, such that $\bar{\rho} = 0.33$ fm,
- for measurements use a $32^3 \times 8$ lattice grid and lattice observables.

Polyakov loop correlator $\rightarrow$ quark-antiquark free energy

$$F(R) = -T \log\langle L(\vec{x})L(\vec{y})\rangle, \quad R = |\vec{x} - \vec{y}|$$

with trivial ($\omega = 0$) and maximally non-trivial holonomy ($\omega = 0.25$).

(a) ρ fixed　　　　(b) ρ sampled with distribution



$\Rightarrow$ Non-trivial (trivial) holonomy creates long-distance coherence (incoherence) and (de)confines for standard instanton or caloron liquid model parameters.

$\Rightarrow$ More realistic model describing the temperature dependence is possible

Dyon gas ensembles and confinement [cf. talk by E. Shuryak]

Polyakov, '77:

Confinement evolves from magnetic monopoles effectively in 3D.

Here: monopoles = dyons (KvBLL caloron constituents) for $0 < T < T_c$.

Conjecture for Yang-Mills theory at $0 < T < T_c$**:**

rewrite integration measure over KvBLL caloron moduli space

in terms of dyon degrees of freedom.

Diakonov, Petrov, '07 :

proposed integration measure (Abelian fields; no antidyons, i.e. CP is violated).

Dyon ensemble statistics analytically solved $\Longrightarrow$ confinement.

However, observation from numerical simulation:

Moduli space metric satisfies positivity only for a small fraction

of dyon configurations and only for low density.

[Bruckmann, Dinter, Ilgenfritz, M.-P., Wagner, '09].

Simplify the model:

- Far-field limit, i.e. purely Abelian monopole fields, non-trivial holonomy.
- Neglect moduli space metric, take random monopole gas.
- Compute free energy of a static quark-antiquark pair $F_{\bar{Q}Q}(d)$ from Polyakov loop correlators.

  [Bruckmann, Dinter, Ilgenfritz, Maier, M.-P., Wagner, '12]

- Exact solution: $F_{\bar{Q}Q}(|\mathbf{r}-\mathbf{r}'|) = -T \ln \left\langle P(\mathbf{r})P^\dagger(\mathbf{r}') \right\rangle \sim \frac{\pi}{2}\,\frac{\rho}{T}\,|\mathbf{r}-\mathbf{r}'| + \text{const.}$

- Simulation in a finite box requires to deal with long-range tails of the fields.

$\Longrightarrow$ Ewald's method used e.g. in plasma physics [P. Ewald, '21]

$\Longrightarrow$ find nice agreement with exact result.

$\Longrightarrow$ Further work required !

7. Miscellaneous

I apologize for not having discussed various topics in detail, which might have been also of interest for Pierre van Baal:

- open b.c.s suppressing HMC's autocorrelation for Q_t:
  [Chowdhury et al., '14; Bruno, S. Schäfer, Sommer, '14; cf. talk by G. Mc Glynn]
- simulation of θ-vacua with Langevin techniques or dual variables:
  [cf. talks by L. Bongiovanni; T. Kloiber]
- fixed topology considerations:
  [cf. talks by J. Verbaarschot; U. Gerber; A. Dromard; H. Fukaya]
- ongoing discussions about the vacuum structure and topological excitations:
  [cf. talks by M. Ünsal; M. Ogilvie; A. Shibata; M. Hasegawa; N. Cundy; H.B. Thacker; D. Trewartha; P. de Forcrand]
- phase structure at differing m_u, m_d masses:
  [Creutz, '13; cf. talk by S. Aoki]
- topology in related theories (G_2 YM theory; $N = 1$ SUSY on the lattice):
  [Ilgenfritz, Maas, '12; cf. talk by P. Giudice]
- chiral magnetic effect in QCD with constant magnetic background field:
  [Bruckmann, Buividovich, Sulejmanpasic, '13; Bali et al. '14]
- . . .

8. Summary

- Topological aspects in QCD occur naturally and have phenomenological impact. Standard instanton gas/liquid remains phenomenologically important: chiral symmetry breaking, solution of $U_A(1)$, ..., but fails to explain confinement.

- Computation of the topological susceptibility with new methods (gradient flow, spectral projector method) on a promising way. Keep track of lattice artifacts and study the continuum limit !!

- Solution of the $\eta' - \eta$ mixing problem now in a good shape.

- $U_A(1)$ restoration at $T > T_c$ seems to be close to be solved, but chiral limit ? Looks like slow restauration above T_c. Then for $N_f = 2$ more likely $O(4)$ scenario.

- $0 < T < T_c$: KvBLL caloron and dyon gas models with non-trivial holonomy very encouraging for description of confinement
  [→ talk by E. Shuryak]

- Calorons and dyon dissociation provide way to improve systematically semiclassical approach [→ talk by M. Ünsal].

Thanks to all those who provided material,
sorry to those, I could not mention,
thank you all for your attention.

Thank you, Pierre,
your vision and ideas are alive.