Topological susceptibility from the Dirac spectrum and the Witten-Veneziano formula

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with
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Introduction

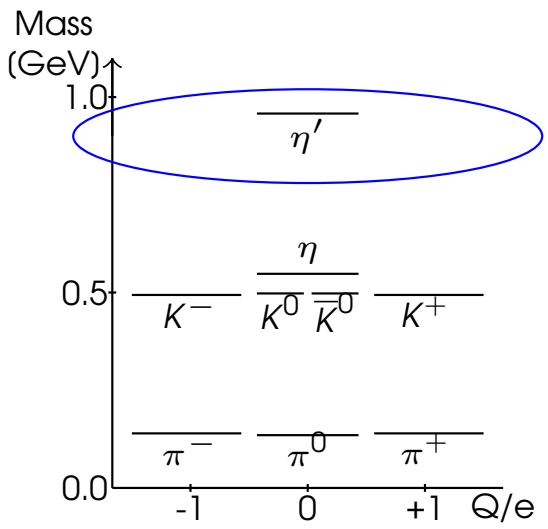
Spectral Projectors

Tests

Results

Conclusions

Witten-Veneziano Formula



related to χ_∞ through W-V formula

- ⋆ Relation between $m_{\eta'}$ and χ_∞

$$\frac{f_\pi^2}{4N_f}\left(m_\eta^2 + m_{\eta'}^2 - 2m_K^2\right) = \chi_\infty$$

- ⋆ Formula obtained in the large N_c limit. (Veneziano,1979) (Witten, 1979)
- ⋆ In the chiral limit $\frac{f_\pi^2}{4N_f} m_{\eta'}^2 = \chi_\infty$

Setup

- ⋆ $N_f = 2 + 1 + 1$ dynamical fermions (Baron et al., 2010, 2011)
- ⋆ **Wilson Twisted Mass Action at maximal twist** (Frezzotti & Rossi, 2003, 2004)
- ⋆ Iwasaki Gauge Action (Iwasaki, 1985)

ensemble	β	L	$a\mu_l$	$a\mu_\sigma$	$a\mu_\delta$	r_0/a	a (fm)	L	κ_c
A30.32	1.90	32	0.003	0.015	0.190	5.231(38)	0.0863(4)	2.8	0.1632720
A40.24	1.90	32	0.004	0.015	0.190	5.231(38)	0.0863(4)	2.1	0.1632700
A40.32	1.90	32	0.004	0.015	0.190	5.231(38)	0.0863(4)	2.8	0.1632700
A60.24	1.90	24	0.006	0.015	0.190	5.231(38)	0.0863(4)	2.1	0.1632650
A80.24	1.90	24	0.008	0.015	0.190	5.231(38)	0.0863(4)	2.1	0.1632600
A80.24s	1.90	24	0.008	0.015	0.197	5.231(38)	0.0863(4)	2.1	0.1632040
A100.24	1.90	24	0.01	0.015	0.190	5.231(38)	0.0863(4)	2.1	0.1632550
A100.24s	1.90	24	0.01	0.015	0.197	5.231(38)	0.0863(4)	2.1	0.1631960
B25.32	1.95	32	0.0025	0.035	0.197	5.710(41)	0.0779(4)	2.5	0.1612400
B35.32	1.95	32	0.0035	0.035	0.197	5.710(41)	0.0779(4)	2.5	0.1612400
B55.32	1.95	32	0.0055	0.035	0.197	5.710(41)	0.0779(4)	2.5	0.1612360
B75.32	1.95	32	0.0075	0.035	0.197	5.710(41)	0.0779(4)	2.5	0.1612320
B85.24	1.95	24	0.0085	0.035	0.197	5.710(41)	0.0779(4)	1.9	0.1612312
D45.32	2.10	32	0.0045	0.120	0.1385	7.538(58)	0.0607(2)	1.9	0.156315
D30.48	2.10	48	0.0030	0.120	0.1385	7.538(58)	0.0607(2)	2.9	0.156355
D20.48	2.10	48	0.0020	0.120	0.1385	7.538(58)	0.0607(2)	2.9	0.156357
D15.48	2.10	48	0.0015	0.120	0.1385	7.538(58)	0.0607(2)	2.9	0.156361

Setup

- ⋆ Quenched ensembles generated matching the physical situation of the dynamical ensembles.
- ⋆ Iwasaki Action

β	Volume	r_0/a	$a\mu_l^{valence_l}$	$r_0\mu$	κ_c^χ
2.37	$20^3 \times 40$	3.59(2)(3)	0.0087	0.0312	0.158738
2.48	$24^3 \times 48$	4.28(1)(5)	0.0073	0.0309	0.154928
2.67	$32^3 \times 64$	5.69(2)(3)	0.0055	0.0314	0.150269
2.85	$40^3 \times 80$	7.29(7)(1)	0.0043	0.0313	0.147180

- ⋆ $N_f = 2 + 1 + 1$ ensemble matched:

ensemble	β	volume	r_0/a	$a\mu_l$	κ_c	a (fm)	L
B55.32	1.95	32×64	5.710(41)	0.0055	0.1612360	0.0779(4)	2.5

Topological Susceptibility I

(See Krzysztof Cichy's talk)

$\chi_{\rm top}$ is related to distribution of topological charge

$$\chi_{\rm top} = \int d^4x \langle q(x) q(0) \rangle$$

Topological Charge density $\rightsquigarrow$ property of the gauge fields:

$$Q = \int d^4x \; q(x); \qquad q(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \mathrm{tr}\, \{F_{\mu\nu} F_{\lambda\sigma}\}$$

- Direct computation $\rightsquigarrow$ non-integrable short-distance singularities.
- Index Theorem:

$$Q_{top} = n_- - n_+; \qquad \chi_{top} = \frac{\langle Q_{top}^2 \rangle}{V}$$

  - $\rightsquigarrow$ Relates topological charge and number of zero modes of the Dirac operator.
  - $\rightsquigarrow$ We have to simulate chiral fermions $\rightsquigarrow$ too expensive for large volumes.

Topological Susceptibility II

- Alternative definition using chiral fermions: (Giusti, Rossi & Testa, 2004)
- Topological susceptibility can be defined in a universal way: (Lüscher, 2004)

$$\chi_{top} = m_1 \ldots m_s \times a^{4s-4} \sum_{x_1 \ldots x_{s-1}} \langle P_{r1}(x_1) S_{12}(x_2) \ldots S_{r-1r}(x_r) \times P_{sr+1}(x_{r+1}) S_{r+1r+2}(x_{r+2}) \ldots S_{s-1s}(x_0) \rangle$$

where $P_{ab}(x) = \overline{\psi}_a(x)\gamma_5\psi_b(x)$ and $S_{ab}(x) = \overline{\psi}_a(x)\psi_b(x)$

- Topological susceptibility in the continuum:

$$\chi_{top} = m_1 \ldots m_5 \int d^4x_1 \ldots d^4x_4 \langle P_{31}(x_1) S_{12}(x_2) S_{23}(x_3) P_{54}(x_4) S_{45}(0) \rangle$$

this correlation function is free of short-distance singularities for number of densities ≥ 5

- Generalization for Wilson Fermions (Giusti & Lüscher 2008), (Lüscher & Palombi, 2010)
  $\rightsquigarrow$ Spectral Projectors

$$\chi_{top} = \frac{Z_S^2}{Z_P^2} \frac{\langle \mathrm{Tr}\,\{\gamma_5 \mathbb{P}_M\}\, \mathrm{Tr}\,\{\gamma_5 \mathbb{P}_M\} \rangle}{V}$$

- Application to Twisted Mass Fermions (K. Cichy, E.G.R, K.Jansen, 2014)

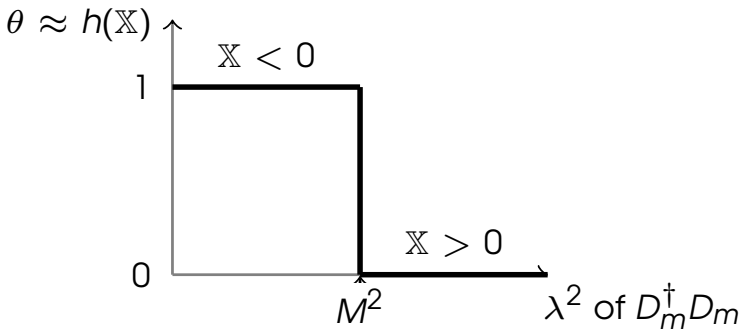
Easy way to understand Spectral Projectors

- mode number $\nu \rightsquigarrow$ number of evls below threshold M. (Giusti & Lüscher, 2008)
- Spectral Projector $\mathbb{P}_M$ to compute $\nu(M, m)$

$$\nu(M, m) = \langle \mathrm{Tr}\{\mathbb{P}_M\} \rangle$$

- Approximation of $\mathbb{P}_M$:

$$\mathbb{P}_M = \theta(\mathbb{X}) \approx h(\mathbb{X}),$$
$$h(x) = \frac{1}{2}\{1 - xP(x^2)\}$$
$$\mathbb{X} = 1 - \frac{2M_\star^2}{D_m^\dagger D_m + M_\star^2}$$



$\langle \mathrm{Tr}\{\mathbb{P}_M\} \rangle$ "simply" counts eigenvalues of $D^\dagger D$

$\rightsquigarrow$ $h(x)$ is an approximation to the step function $\theta(-x)$ in the interval $[-1, 1]$

- In the case of the topological susceptibility:

$$\chi_{top} = \frac{Z_S^2}{Z_P^2} \frac{\langle \mathrm{Tr}\,\{\gamma_5 \mathbb{P}_M\} \mathrm{Tr}\,\{\gamma_5 \mathbb{P}_M\} \rangle}{V} \tag{1}$$

$\mathcal{O}(a)$ improvement

even under $\mathcal{R}_5^{1,2}$: $\begin{cases} \chi(x_0,\mathbf{x}) & \to & i\tau^{1,2}\gamma_5\chi(x_0,\mathbf{x}), \\ \overline{\chi}(x_0,\mathbf{x}) & \to & i\overline{\chi}(x_0,\mathbf{x})\gamma_5\tau^{1,2}, \end{cases}$ automatic $\mathcal{O}(a)$ improvement

All odd terms vanish!!

- ⋆ Analogous representation of the topological susceptibility

$$\chi_{\rm top} = \int d^4x_1 \ldots d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle$$

$$S^\pm_{ab} = \bar{\chi}_a\tau^\pm\chi_b,\, P^\pm_{ab} = \bar{\chi}_a\gamma_5\tau^\pm\chi_b$$

- ⋆ Symanzik expansion at maximal twist:

$$\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle =$$

$$= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 + \text{contact terms} + \mathcal{O}(a^2)$$

⋆ Contacts terms → OPE

↓

Only terms which lead to $\mathcal{O}(a)$ contributions

↓

$$P^+_{ab}(x)S^-_{ac}(0) \overset{x\to 0}{\sim} C_1(x)P^\uparrow_{ac}(0) \qquad P^\uparrow_{ac} = \overline{\psi}_a\gamma_5\frac{1}{2}(\mathbb{1}+\tau^3)\psi_c$$

$$P^+_{ab}(x)P^-_{bc}(0) \overset{x\to 0}{\sim} C_2(x)S^\uparrow_{ac}(0) \qquad S^\uparrow_{ac} = \overline{\psi}_a\frac{1}{2}(\mathbb{1}+\tau^3)\psi_c$$

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$$S^\pm_{ab} = \bar{\chi}_a\tau^\pm\chi_b,\, P^\pm_{ab} = \bar{\chi}_a\gamma_5\tau^\pm\chi_b$$

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$\mathcal{O}(a)$ improvement

- ⋆ Symanzik expansion at maximal twist (K.Cichy, E.G.R, K.Jansen, A.Shinlder, in preparation)

$$
\begin{aligned}
&\int d^4x_1 \dots d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_I \\
&= \int d^4x_1 \dots d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \\
&+ ac_1 \int d^4x_2 d^4x_3 d^4x_4 d^4x_5 \left\langle P^\uparrow_{42}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \\
&- iac_3 \int d^4x_1 d^4x_3 d^4x_4 d^4x_5 \left\langle S^+_{41}(x_1)S^\downarrow_{13}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \\
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&+ ac_2 \int d^4x_1 d^4x_2 d^4x_3 d^4x_5 \left\langle P^\downarrow_{31}(x_1)P^-_{12}(x_2)P^+_{23}(x_3) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \\
&+ ac_5 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad P^\uparrow_{55}(0)\right\rangle_0 \\
&+ ac_5 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad P^\downarrow_{66}(0)\right\rangle_0 + \mathcal{O}(a^2)
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\begin{aligned}
&\int d^4x_1 \dots d^4x_5 \left\langle S^+_{41}(x_1) P^-_{12}(x_2) P^+_{23}(x_3) P^-_{34}(x_4) \quad S^+_{56}(x_5) P^-_{65}(0) \right\rangle_I \\
&= \int d^4x_1 \dots d^4x_5 \left\langle S^+_{41}(x_1) P^-_{12}(x_2) P^+_{23}(x_3) P^-_{34}(x_4) \quad S^+_{56}(x_5) P^-_{65}(0) \right\rangle_0 \\
&+ ac_1 \int d^4x_2 d^4x_3 d^4x_4 d^4x_5 \left\langle P^\uparrow_{42}(x_2) P^+_{23}(x_3) P^-_{34}(x_4) \quad S^+_{56}(x_5) P^-_{65}(0) \right\rangle_0 \\
&- iac_3 \int d^4x_1 d^4x_3 d^4x_4 d^4x_5 \left\langle S^+_{41}(x_1) S^\downarrow_{13}(x_3) P^-_{34}(x_4) \quad S^+_{56}(x_5) P^-_{65}(0) \right\rangle_0 \\
&- iac_4 \int d^4x_1 d^4x_2 d^4x_4 d^4x_5 \left\langle S^+_{41}(x_1) P^-_{12}(x_2) S^\uparrow_{24}(x_4) \quad S^+_{56}(x_5) P^-_{65}(0) \right\rangle_0 \\
&+ ac_2 \int d^4x_1 d^4x_2 d^4x_3 d^4x_5 \left\langle P^\downarrow_{31}(x_1) P^-_{12}(x_2) P^+_{23}(x_3) \quad S^+_{56}(x_5) P^-_{65}(0) \right\rangle_0 \\
&+ ac_5 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left\langle S^+_{41}(x_1) P^-_{12}(x_2) P^+_{23}(x_3) P^-_{34}(x_4) \quad P^\uparrow_{55}(0) \right\rangle_0 \\
&+ ac_5 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left\langle S^+_{41}(x_1) P^-_{12}(x_2) P^+_{23}(x_3) P^-_{34}(x_4) \quad P^\downarrow_{66}(0) \right\rangle_0 + \mathcal{O}(a^2)
\end{aligned}
$$

$\mathcal{O}(a)$ improvement

- Symanzik expansion at maximal twist (K.Cichy, E.G.R, K.Jansen, A.Shinlder, in preparation)

$$
\begin{aligned}
&\int d^4x_1 \dots d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_I \\
&= \int d^4x_1 \dots d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \\
&+ ac_1 \int d^4x_2 d^4x_3 d^4x_4 d^4x_5 \left\langle P^\uparrow_{42}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \\
&- iac_3 \int d^4x_1 d^4x_3 d^4x_4 d^4x_5 \left\langle S^+_{41}(x_1)S^\downarrow_{13}(x_3)P^-_{34}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \\
&- iac_4 \int d^4x_1 d^4x_2 d^4x_4 d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)S^\uparrow_{24}(x_4) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \\
&+ ac_2 \int d^4x_1 d^4x_2 d^4x_3 d^4x_5 \left\langle P^\downarrow_{31}(x_1)P^-_{12}(x_2)P^+_{23}(x_3) \quad S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \\
&+ ac_5 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad P^\uparrow_{55}(0)\right\rangle_0 \\
&+ ac_5 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4) \quad P^\downarrow_{66}(0)\right\rangle_0 + \mathcal{O}(a^2)
\end{aligned}
$$

$\mathcal{O}(a)$ improvement

(K.Cichy, E.G.R, K.Jansen, A.Shinlder, in preparation)

- ⋆ All $\mathcal{O}(a)$ terms are $\mathcal{R}_5^1$ odd in pairs $\rightsquigarrow$ We recover $\mathcal{O}(a)$ improvement.

$$\int d^4x_1 \dots d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4)S^+_{56}(x_5)P^-_{65}(0)\right\rangle_I$$

$$= \int d^4x_1 \dots d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4)S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0$$

$$+ac_1 \int d^4x_2 d^4x_3 d^4x_4 d^4x_5 \left\langle P^\uparrow_{42}(x_2)P^+_{23}(x_3)P^-_{34}(x_4)S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \quad \mathcal{R}_5^1\text{odd}$$

$$-iac_3 \int d^4x_1 d^4x_3 d^4x_4 d^4x_5 \left\langle S^+_{41}(x_1)S^\downarrow_{13}(x_3)P^-_{34}(x_4)S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0 \quad \mathcal{R}_5^1\text{odd}$$

$$-iac_4 \int d^4x_1 d^4x_2 d^4x_4 d^4x_5 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)S^\uparrow_{24}(x_4)S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0$$

$$+ac_2 \int d^4x_1 d^4x_2 d^4x_3 d^4x_5 \left\langle P^\downarrow_{31}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)S^+_{56}(x_5)P^-_{65}(0)\right\rangle_0$$

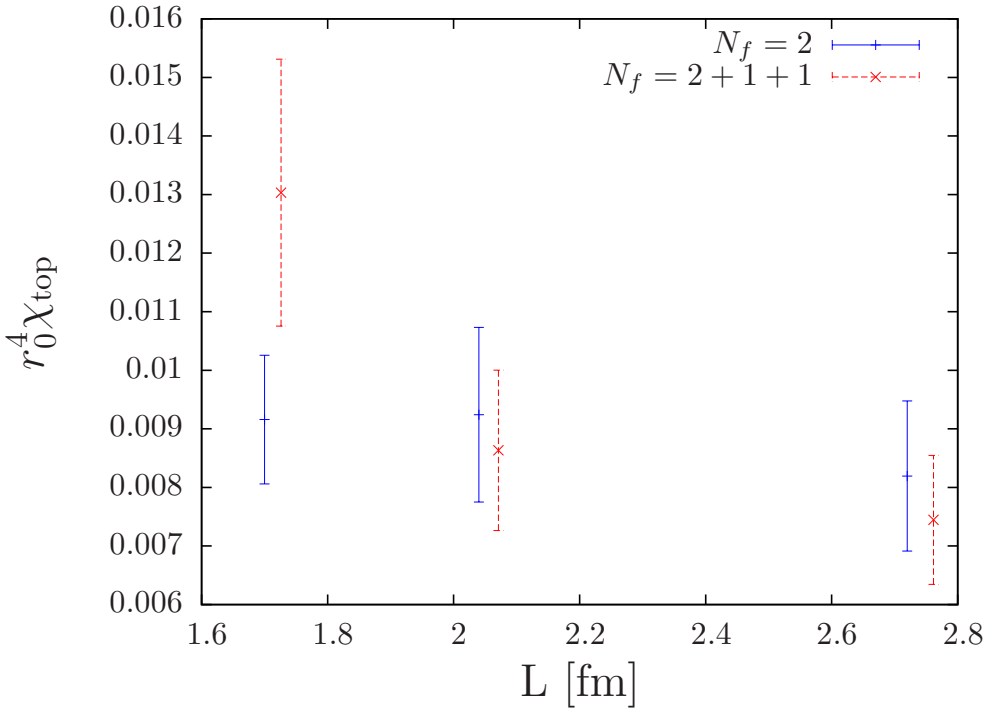
$$+ac_5 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4)P^\uparrow_{55}(0)\right\rangle_0 \quad \mathcal{R}_5^1\text{odd}$$

$$+ac_5 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left\langle S^+_{41}(x_1)P^-_{12}(x_2)P^+_{23}(x_3)P^-_{34}(x_4)P^\downarrow_{66}(0)\right\rangle_0$$

$$+ \mathcal{O}(a^2)$$

Finite-Volume Effects

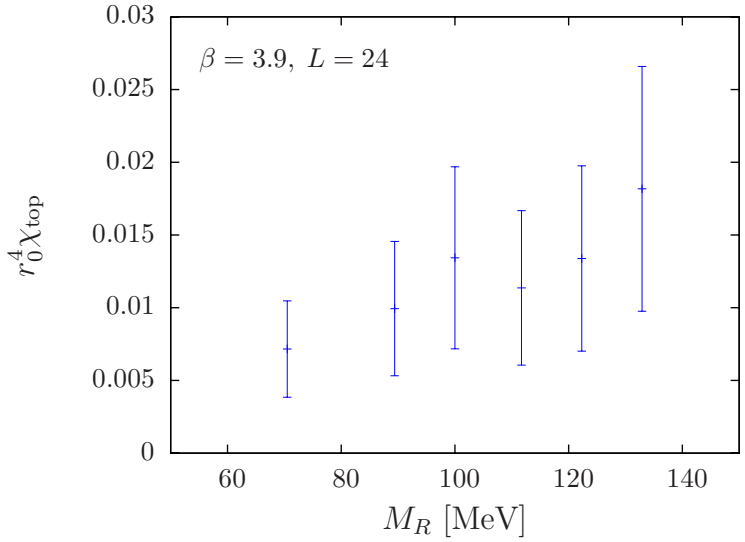
Study of the finite-size effects for the Topological Susceptibility



Choosing value of M_R for $\chi_{\rm top}$

β	volume	$M_\star^2$	ν
2.37	$20^3 \times 40$	0.000102	79.4(2)
2.48	$24^3 \times 48$	0.000068	78.7(2)
2.67	$32^3 \times 64$	0.000036	78.5(3)
2.85	$40^3 \times 80$	0.000025	78.1(4)

$$\frac{a_1}{a_2} = \left(\frac{\nu_1 n_2}{\nu_2 n_1}\right),$$



- For small volume choose M such that $\nu >> Q_{\rm top}$.

Z_P/Z_S with spectral projectors

- From the introduction we know:

$$\chi_{top} = \frac{Z_S^2}{Z_P^2} \frac{\langle \mathrm{Tr}\,\{\gamma 5 \mathbb{P}_M\}\,\mathrm{Tr}\,\{\gamma 5 \mathbb{P}_M\}\rangle}{V}$$

  - ⋆ We can compute the topological susceptibility
    - ⇝ quenched case Z_P/Z_S not known a priori
  - ⋆ We can also compute the ratio Z_P/Z_S (Giusti & Lüscher 2008)

$$\frac{Z_S^2}{Z_P^2} = \frac{\langle \mathrm{Tr}\,\{\mathbb{P}_M\}\rangle}{\langle \mathrm{Tr}\,\{\gamma_5 \mathbb{P}_M \gamma_5 \mathbb{P}_M\}\rangle}$$

Z_P/Z_S with spectral projectors

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  - ⋆ We can compute the topological susceptibility
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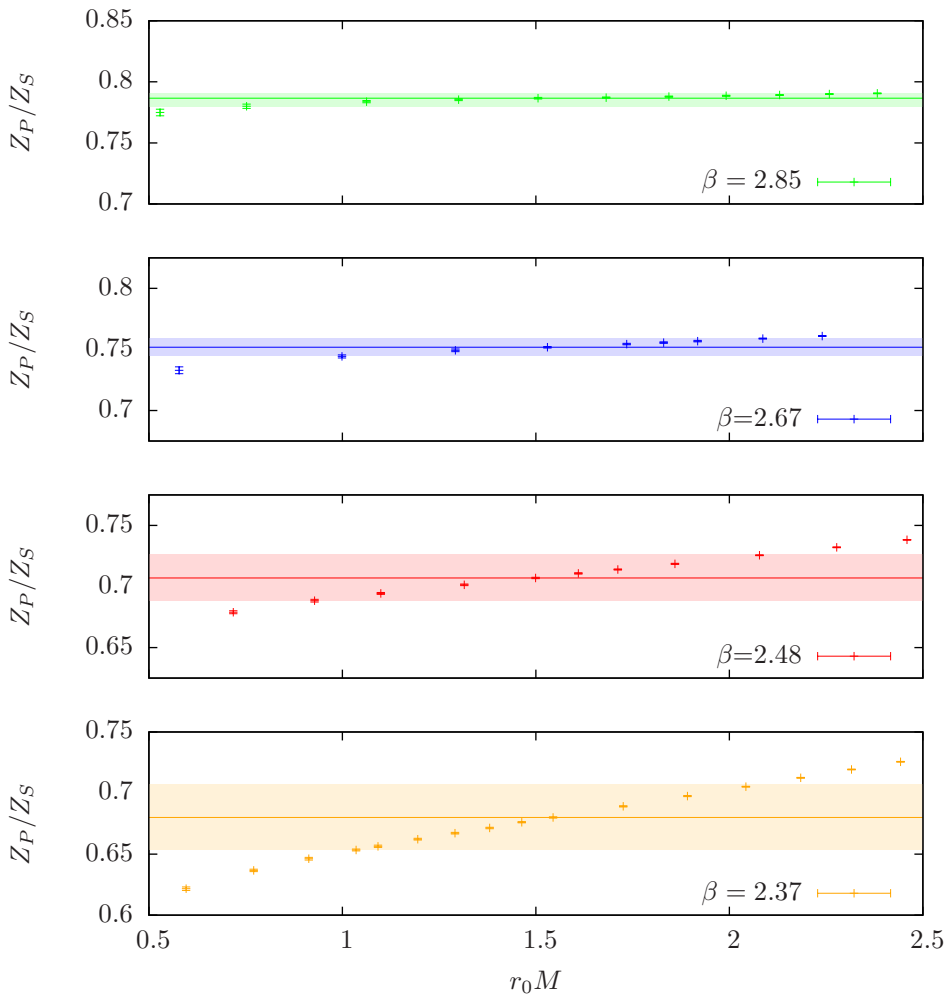
Z_P/Z_S with spectral projectors



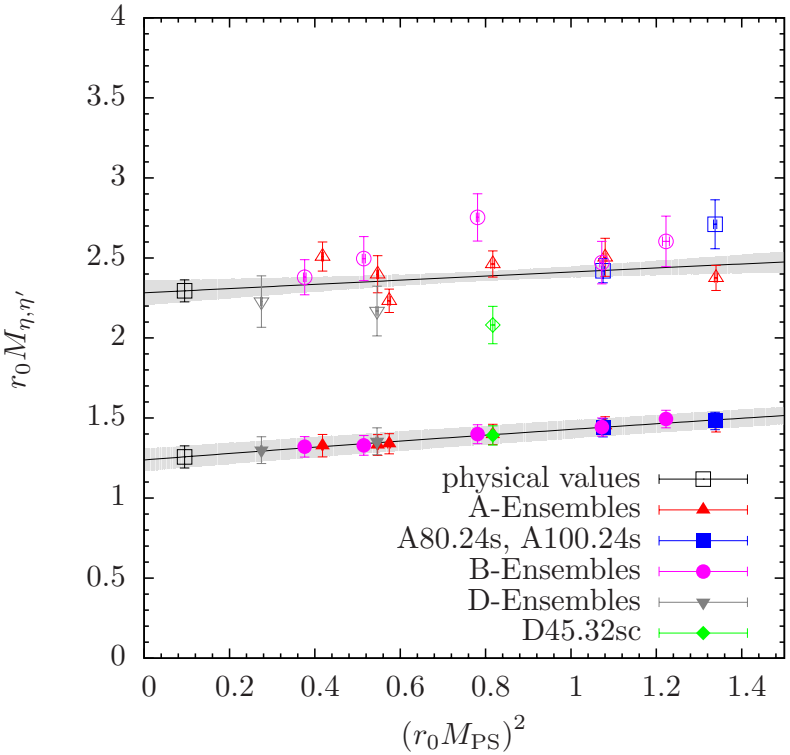
- Our results for quenched:

β	$L^3 \times T$	r_0/a	Z_P/Z_S
2.37	$20^3 \times 40$	3.60	0.680(1)(27)
2.48	$24^3 \times 48$	4.23	0.707(1)(19)
2.67	$32^3 \times 64$	5.69	0.752(1)(7)
2.85	$40^3 \times 80$	7.28	0.787(1)(3)

- For $N_f = 2$ compatible results with RI-MOM (K.Cichy, E.G.R, K.Jansen, 2014)

η and η' masses

(C. Michael, K. Ottnad, & C. Urbach, 2013)



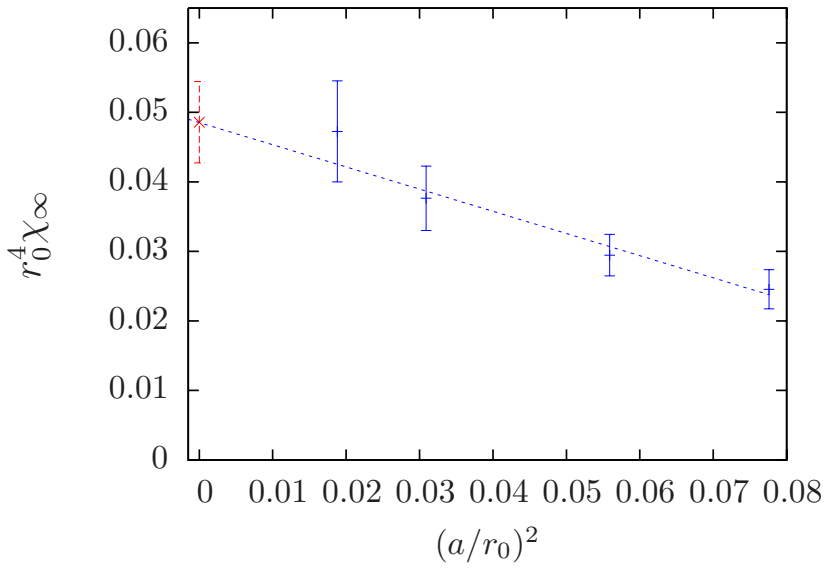
$r_0 m_\eta$	$r_0 m_{\eta'}$	$r_0 m_K$	$r_0 f_\pi$
1.256(22)	2.29(0.21)	1.13476(5)	0.312(11)
$r_0^4 \frac{f_\pi^2}{4N_f}\left(m_\eta^2 + m_{\eta'}^2 - 2m_K^2\right) = 0.043(4)$			

Continuum Limit of χ_∞

l.h.s.: PDG

m_η [MeV]	$m_{\eta'}$ [MeV]	m_K [MeV]	f_π [MeV]	χ_∞
547.85(2)	957.78(6)	497.61(2)	130(5)	
$\frac{f_\pi^2}{4N_f}\left(m_\eta^2 + m_{\eta'}^2 - 2m_K^2\right) = (180\ \mathrm{MeV})^4$				$(185(6)\ \mathrm{MeV})^4$

Compatible results with (Del Debbio, Giusti & Pica, 2005)



$r_0 m_\eta$	$r_0 m_{\eta'}$	$r_0 m_K$	$r_0 f_\pi$	$r_0^4\chi_\infty$
1.256(22)	2.29(0.21)	1.13476(5)	0.312(11)	
$r_0^4\frac{f_\pi^2}{4N_f}\left(m_\eta^2 + m_{\eta'}^2 - 2m_K^2\right) = 0.043(4)$				0.049(6)

(C. Michael, K. Ottnad, & C. Urbach, 2013)

Conclusions and Outlook

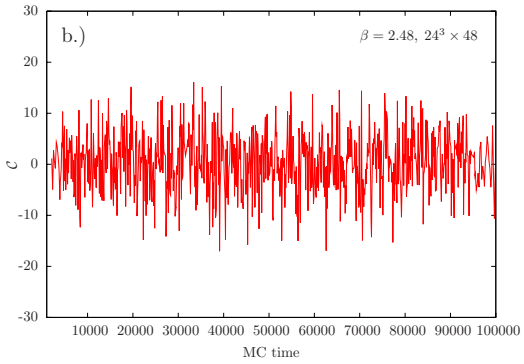
- Use of density chains leads to definition of $\chi_{\rm top}$ free of divergences.
- In practice we use the spectral projector method.
- Proof of $\mathcal{O}(a)$ improvement for twisted mass fermions.

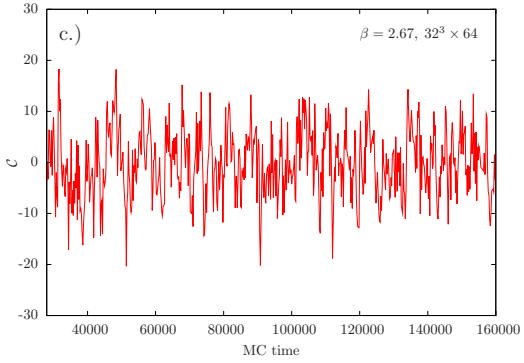
- Test on the Witten-Veneziano formula which confirms the relation between $m_{\eta'}$ and χ_∞.
  - ⋆ Continuum limit of χ_∞.
  - ⋆ Computation of Z_P/Z_S using spectral projectors.

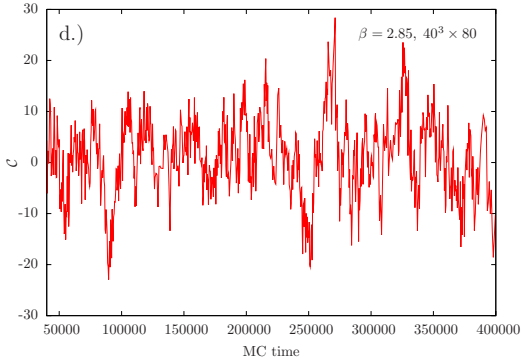
- Our plans include a dedicated analysis of the left-hand side which will presented in future discussions.

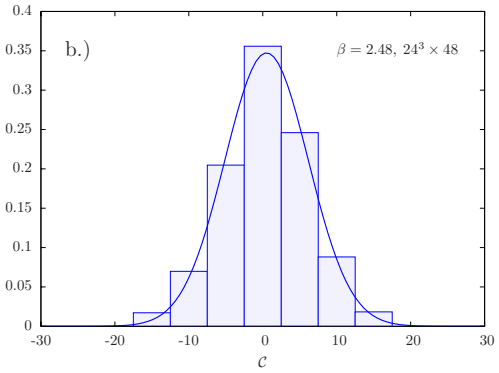
Thank you for your attention!

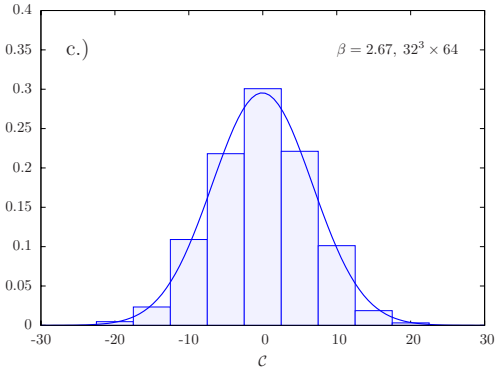
Autocorrelation

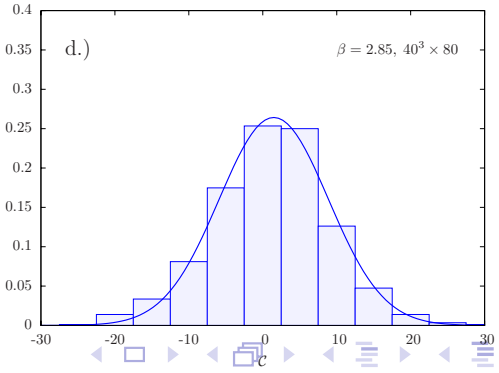






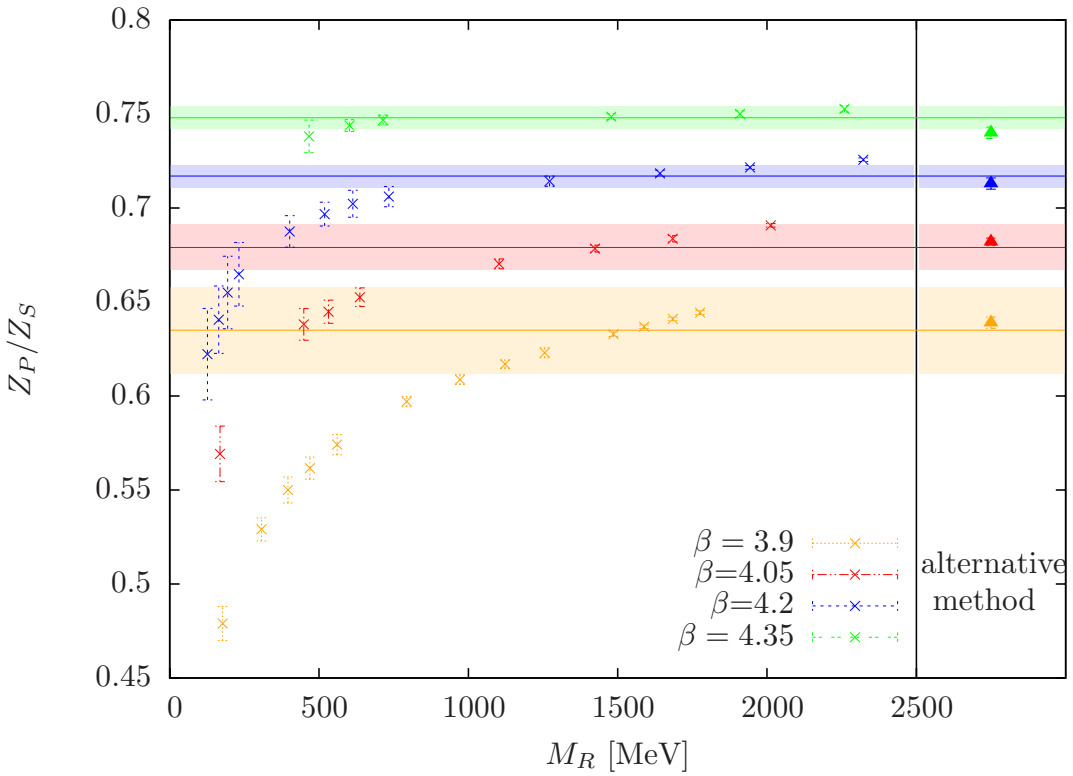






Z_P/Z_S with spectral projectors

(K. Cichy, E.G.R., K. Jansen, in preparation)



β	Z_P/Z_S spectral proj.	Z_P/Z_S RI-MOM/X-space
3.9	0.635(1)(23)	0.639(3)
4.05	0.679(2)(12)	0.682(2)
4.2	0.717(2)(5)	0.713(3)
4.35	0.749(2)(2)	0.740(3)