Rapidity evolution of Wilson lines at the next-to-leading order

Giovanni Antonio Chirilli

The Ohio State University

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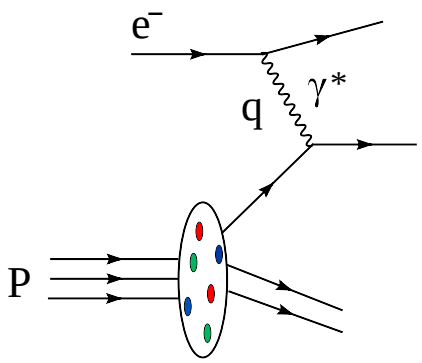
Outline

- High-energy/high-density QCD scattering processes.
- BFKL equation and violation of unitarity.
- High-energy Operator Product Expansion: high-energy QCD factorization.
- LO and NLO non-linear BK/JIMWLK evolution equation.
- Conclusions.

Incoherent-vs-Coherent

- Do DGLAP equations describe high parton-density dynamics?
- No. DGLAP is evolution equation towards dilute regime.

Incoherent Interactions



Bjorken Limit

$$-q^2 = Q^2 \to \infty, \ (P+q)^2 = s \to \infty$$

$$x_{\rm B} = \frac{Q^2}{s+Q^2} \ \text{ fixed}$$

resum $\alpha_s \ln \dfrac{Q^2}{\Lambda^2_{\rm QCD}}$

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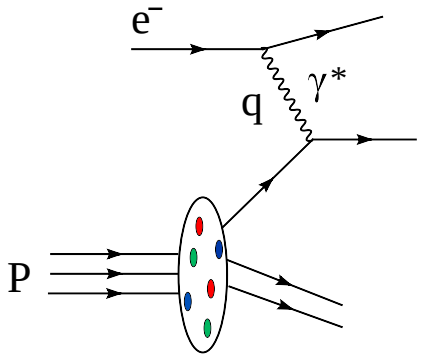

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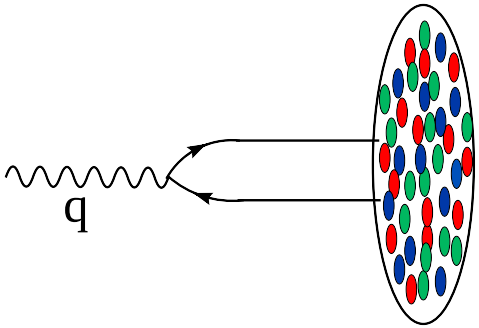

Regge Limit

$$Q^2 \ \text{fixed}, \ s \to \infty$$

$$x_B = \frac{Q^2}{s} \to 0$$

resum $\alpha_s \ln \frac{1}{x_B}$

DGLAP vs. BFKL

$x_B \sim \frac{Q^2}{s}$, $\Delta x_\perp \sim \frac{1}{Q}$ Resolution of γ^* in transverse direction (Breit frame)

Partons which participate in DIS sca. have transverse momenta $k_\perp^2 \lesssim Q^2$

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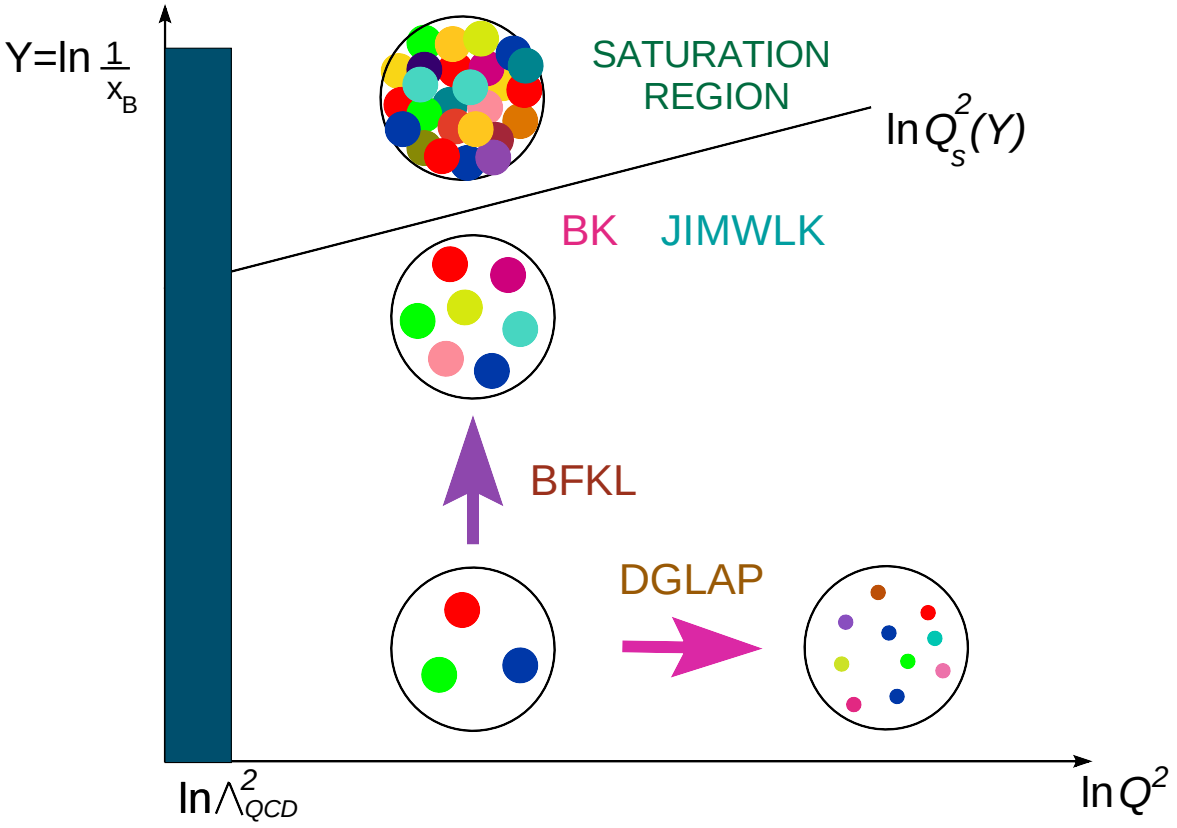

BFKL: Balitsky, Fadin, Kuraev, Lipatov

DGLAP: Dokshitzer, Gribov, Lipatov, Altarelli, Parisi

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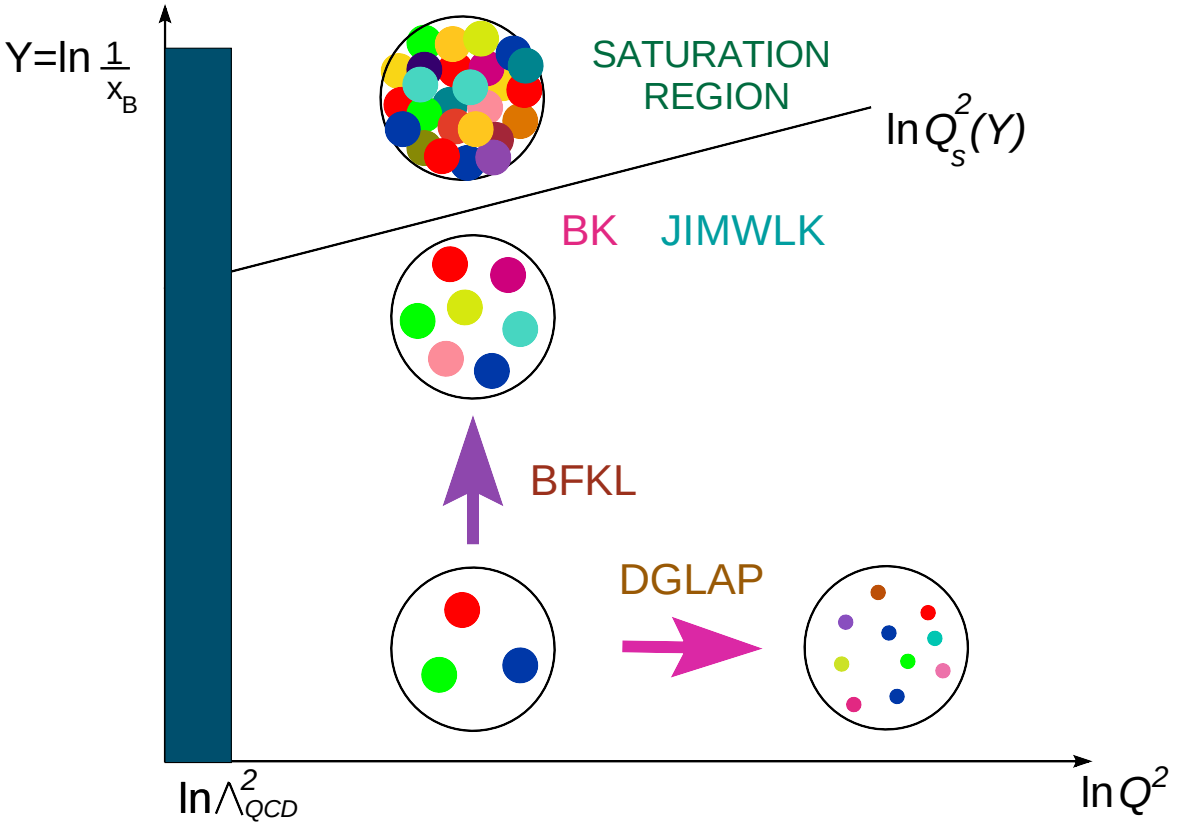

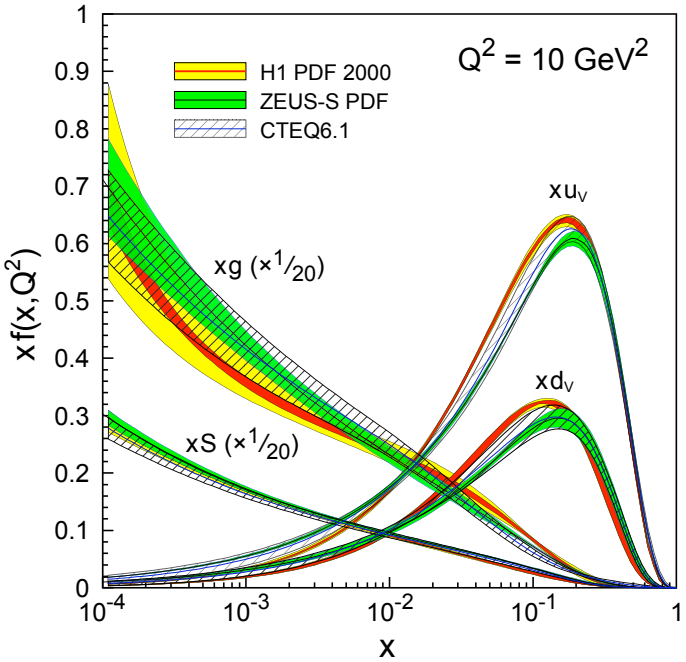

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DIS cross section at Leading Log Approximation

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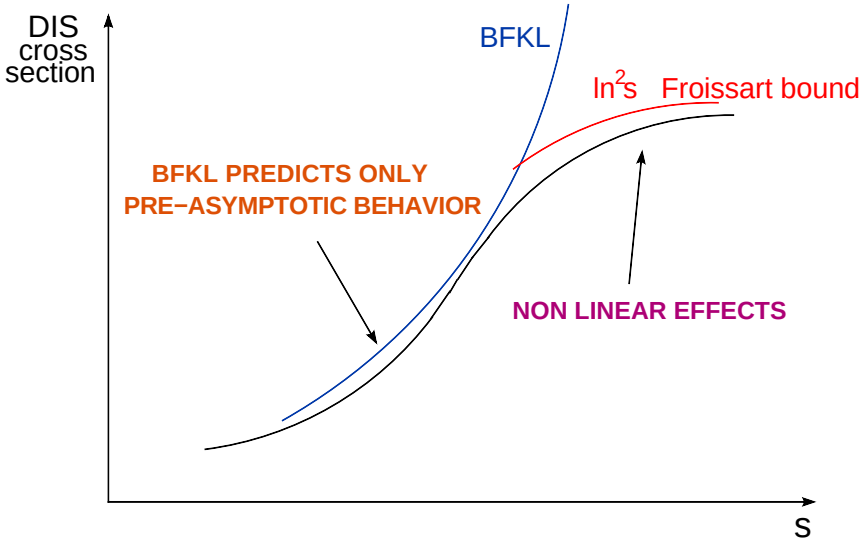

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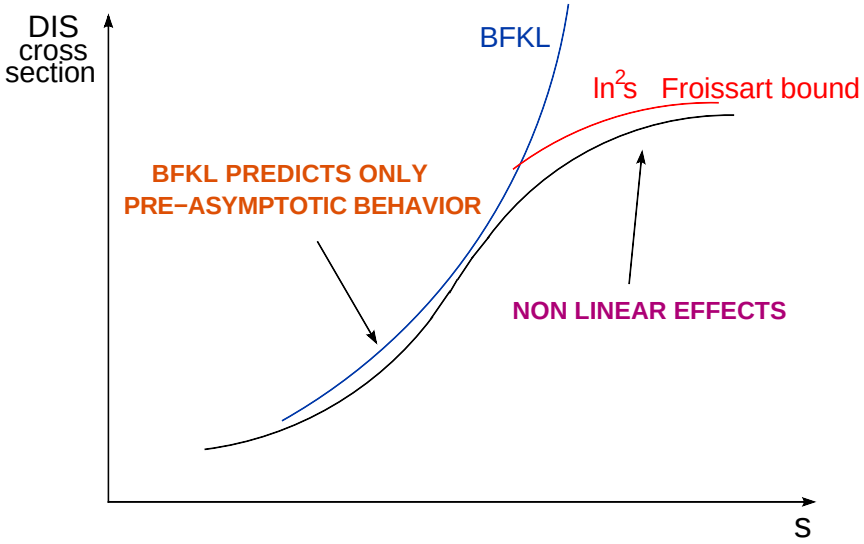

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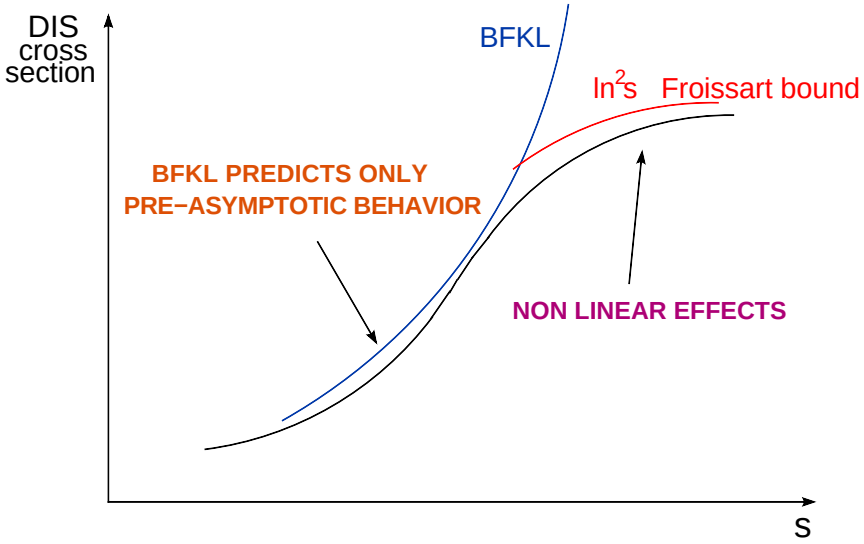

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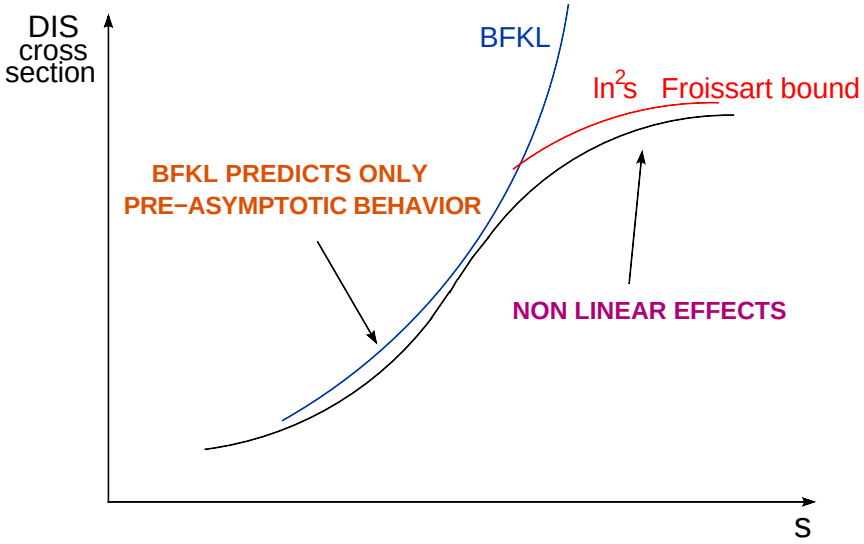

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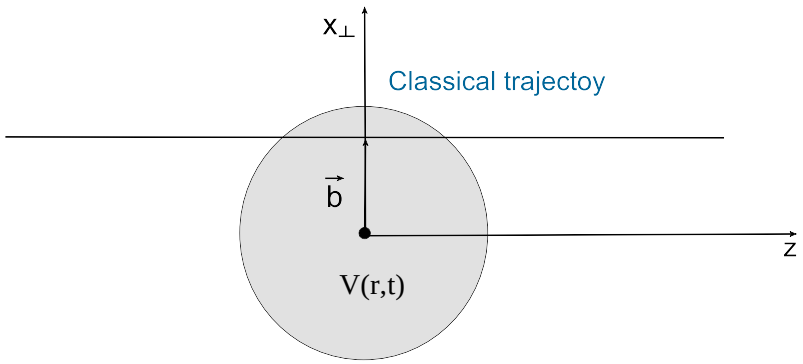
- At very high energy recombination begins to compensate gluon production. Gluon density reaches a limit and does not grow anymore. So does the total DIS cross sections. Unitarity is restored!
- In order to take in to account rgluons recombination the evolution equation for the structure function has to be non-linear.

High-energy scattering in quantum mechanics and QED



- High-energy: $E >> V(x)$ WKB approximation.
- Replace the exact wave function by the semi-classical wave function.
- $\Psi(\vec{r},t) = e^{-\frac{i}{\hbar}(Et-kx)} e^{-\frac{i}{\hbar}\int_{-\infty}^{z} dz' V(z')}$

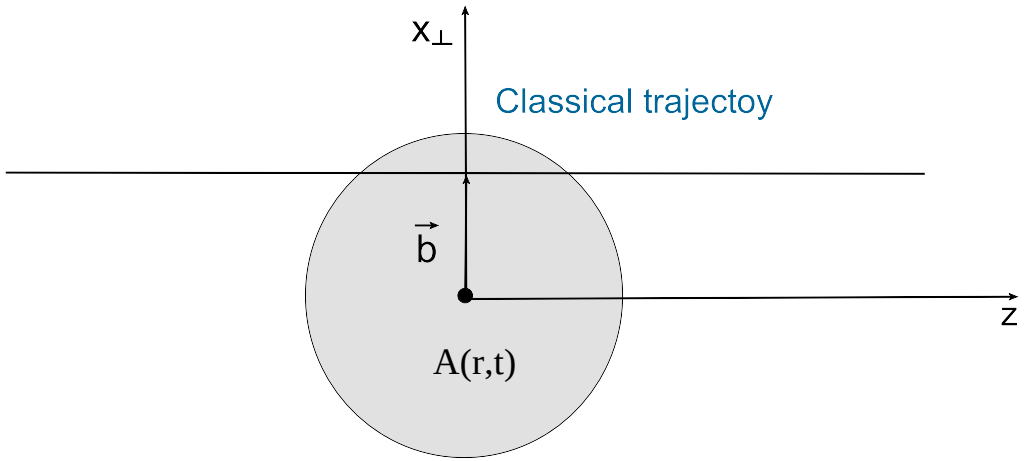
At high-energy $\Psi = \Psi_{\text{free}} \times$ phase factor ordered along the line parallel to $\vec{v}$. The scattering amplitude is proportional to $\Psi(t=-\infty)$

$$U(x_\perp) = e^{\frac{-i}{v\hbar}\int_{-\infty}^{+\infty} dz' V(z'+x_\perp)}$$

In QED

$$U(x_\perp) = e^{\frac{-ie}{c\hbar}\int_{-\infty}^{+\infty} dt \dot{x}_\mu A^\mu(x(t))}$$

High-energy scattering in QCD



phase factor for the high-energy scattering: Wilson-line operator

$$U(x_\perp, v) = \mathrm{P}e^{\frac{-ig}{c\hbar}\int_{-\infty}^{+\infty} dt\ \dot{x}_\mu A^\mu(x(t))}$$

$$\mathrm{P}e^{\int_{-\infty}^{+\infty} dt A(t)} = \int_{-\infty}^{+\infty} dt\ A(t) + \int_{-\infty}^{+\infty} dt\ A(t) \int_{-\infty}^{t} dt'\ A(t')$$

Propagation in the shock wave: Wilson line (Spectator frame)

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$[x,y] = Pe^{ig\int_0^1 du (x-y)^\mu A_\mu (ux+(1-u)y)} \qquad p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$

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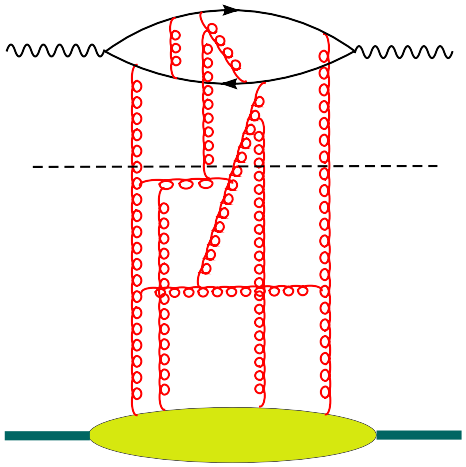

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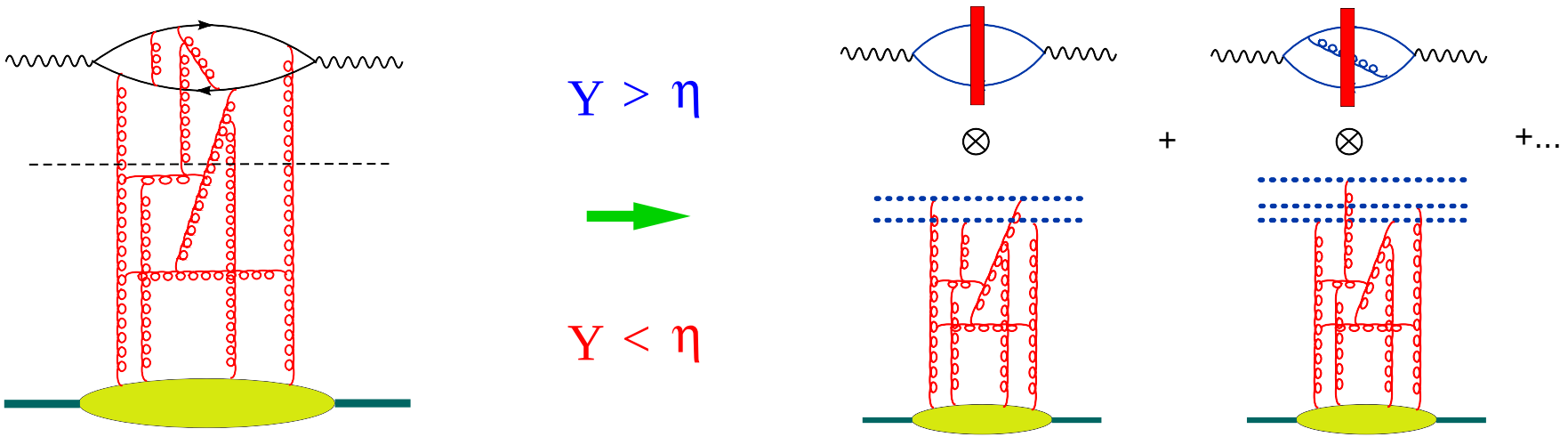
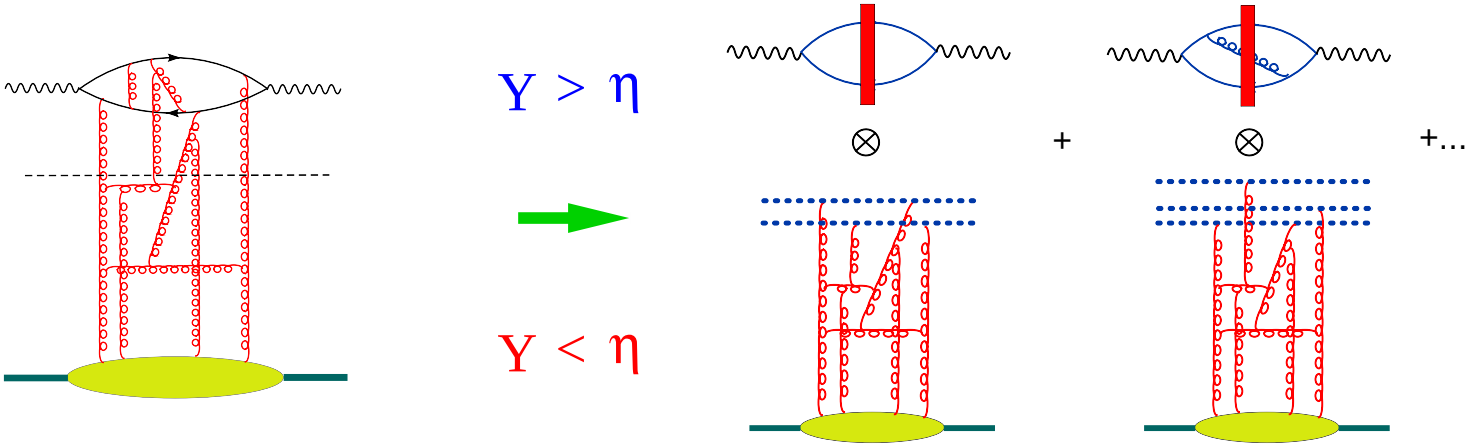


High-energy Operator Product Expansion



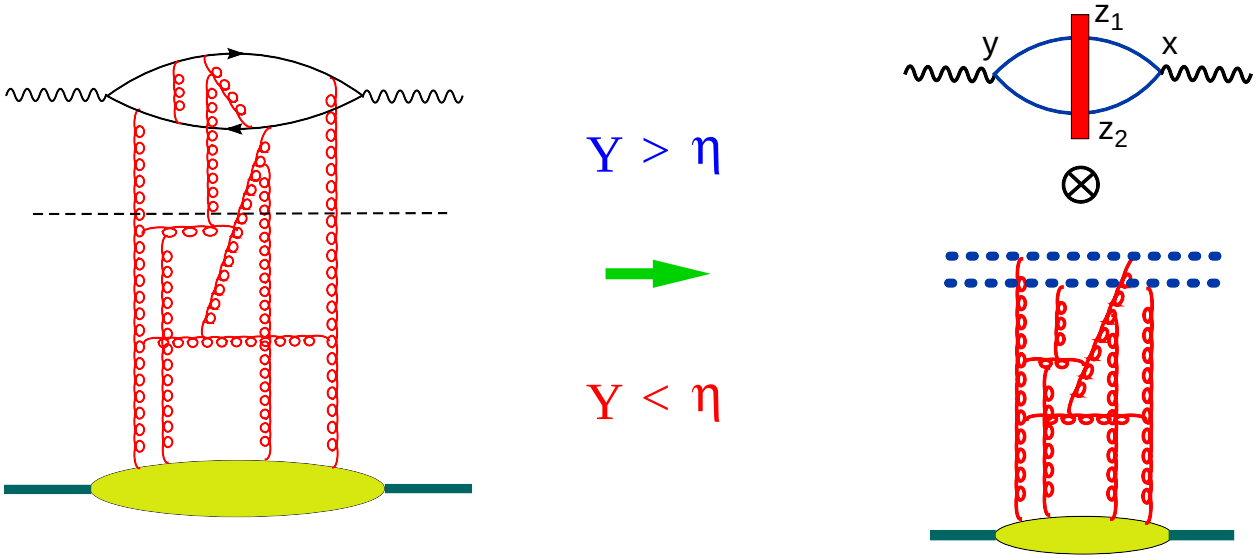
$$\langle B|\mathrm{T}\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle \simeq \int d^2z_1 d^2z_2\; I^{LO}_{\mu\nu}(z_1,z_2;x,y)\langle B|\mathrm{tr}\{U^\eta_{z_1}U^{\dagger\,\eta}_{z_2}\}|B\rangle$$

$$+\frac{\alpha_s}{\pi}\int d^2z_1 d^2z_2 d^2z_3\; I^{NLO}_{\mu\nu}(z_1,z_2,z_3;x,y)\langle B|\mathrm{tr}\{U^\eta_{z_1}U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{U^\eta_{z_3}U^{\dagger\eta}_{z_2}\}|B\rangle$$

- $\eta = \ln\frac{1}{x_B}$
- $|B\rangle$ Target state

Leading Order

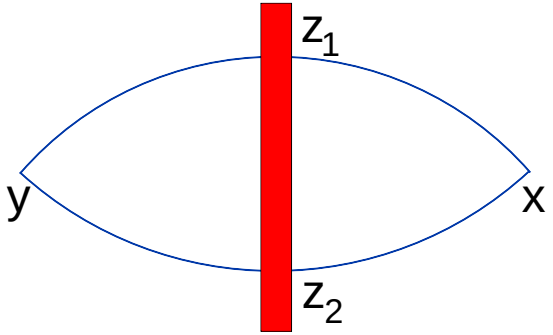
$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm LO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\,{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$



$$\langle B|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle B|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle + \ldots$$

LO Impact Factor

Conformal invariance: $(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2$ $\qquad x^+ = \frac{x^0+x^3}{sqrt2}$



Conformal vectors:

$$\kappa = \frac{1}{\sqrt{s}x^+}\left(\frac{p_1}{s} - x^2 p_2 + x_\perp\right) - \frac{1}{\sqrt{s}y^+}\left(\frac{p_1}{s} - y^2 p_2 + y_\perp\right)$$

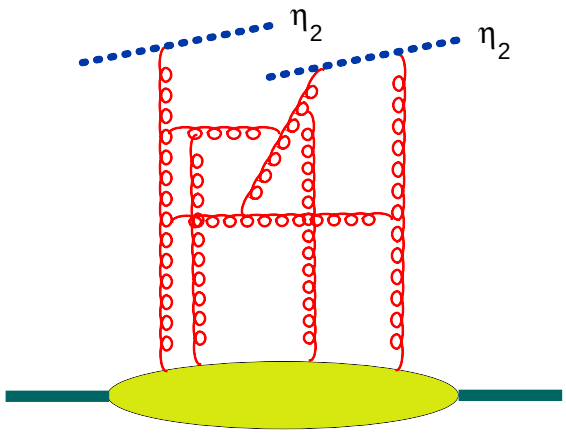
$$\zeta_1 = \left(\frac{p_1}{s} + z_{1\perp}^2 p_2 + z_{1\perp}\right), \qquad \zeta_2 = \left(\frac{p_1}{s} + z_{2\perp}^2 p_2 + z_{2\perp}\right)$$

Here $\quad x^2 = -x_\perp^2; \qquad \mathcal{R} = \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$

$$I_{\mu\nu}^{\rm LO}(z_1, z_2) = \frac{\mathcal{R}^2}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)} \frac{\partial^2}{\partial x^\mu \partial y^\nu}\left[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2) - \frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\right]$$

Regularization of the rapidity divergence

Matrix elements of Wilson lines: $\langle \mathrm{tr}\{U(x)U^\dagger(y)\}\rangle_A$ are divergent



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty$$

Regularization by: slope

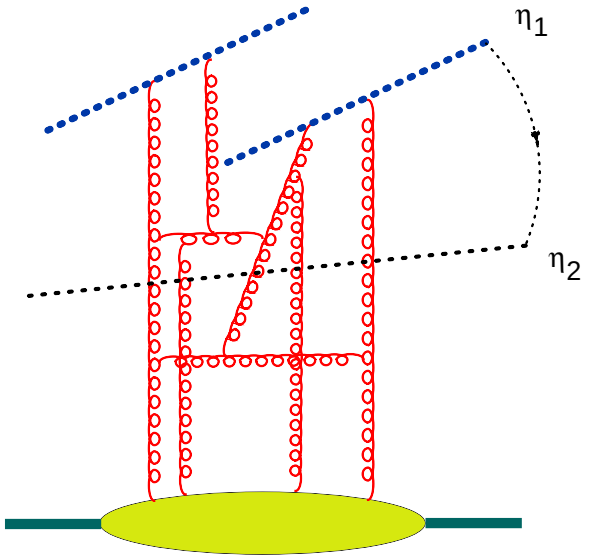
$$U^\eta(x_\perp) = \mathrm{Pexp}\left\{ig \int_{-\infty}^{\infty} du\, n_\mu\, A^\mu(un + x_\perp)\right\} \qquad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

Evolution Equation

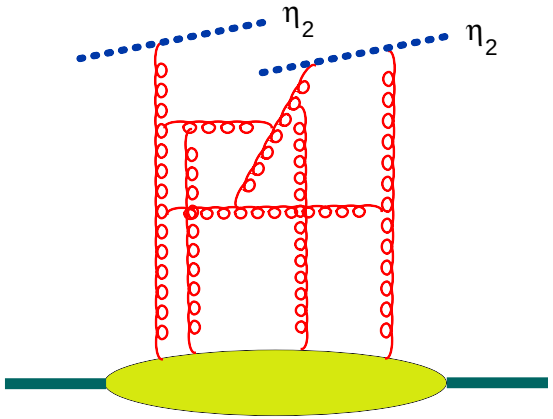
$$\frac{d}{d\eta}\mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\} \quad \Rightarrow \quad \frac{d}{d\eta}\langle \mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle$$

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidity $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to η_2).

In the frame $||$ to η_1 the gluons with $\eta < \eta_1$ are seen as pancake.



$$\alpha_s(\eta_1 - \eta_2)K_{\mathrm{evol}} \quad \otimes$$



Particles with different rapidity perceive each other as Wilson lines.

Non-linear evolution equation

- Linear case $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta \, K_{\rm evol} \otimes \mathcal{O}^{\eta_2}$

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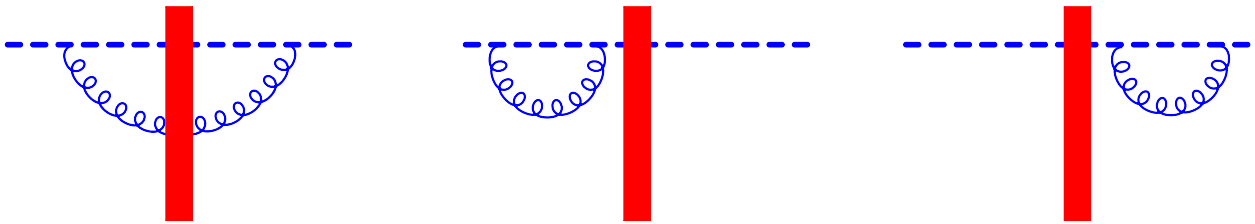

$$\langle \{U_x^{\eta_1}\}_{ij} \rangle_A = \frac{\alpha_s}{2\pi^2} \Delta\eta \int \frac{d^2 z_\perp}{(x-z)_\perp^2} \left[\langle {\rm tr}\{U_x^{\eta_2} U_z^{\eta_2 \dagger}\} \{U_z^{\eta_2}\}_{ij} \rangle_A - \langle \frac{1}{N_c} \{U_x^{\eta_2}\}_{ij} \rangle_A \right]$$

$\Delta = \eta_1 - \eta_2$

$$\{U_x^{\dagger \eta_1}\}_{ij}, \quad \{U_x^{\eta_1} U_y^{\eta_1}\}_{ij}, \quad \{U_x^{\eta_1} U_y^{\dagger \eta_1}\}_{ij}, \quad \{U_x^{\dagger \eta_1} U_y^{\dagger \eta_1}\}_{ij}$$

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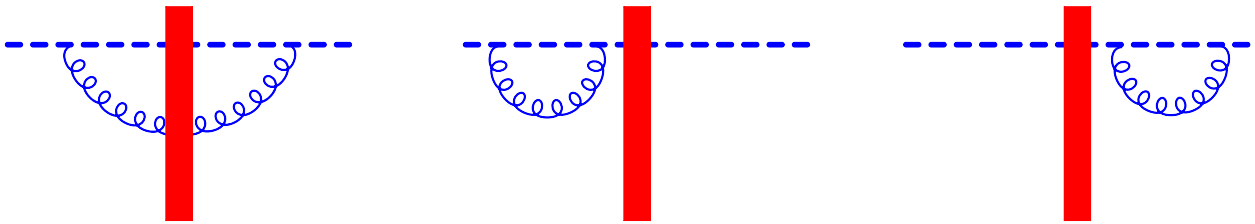

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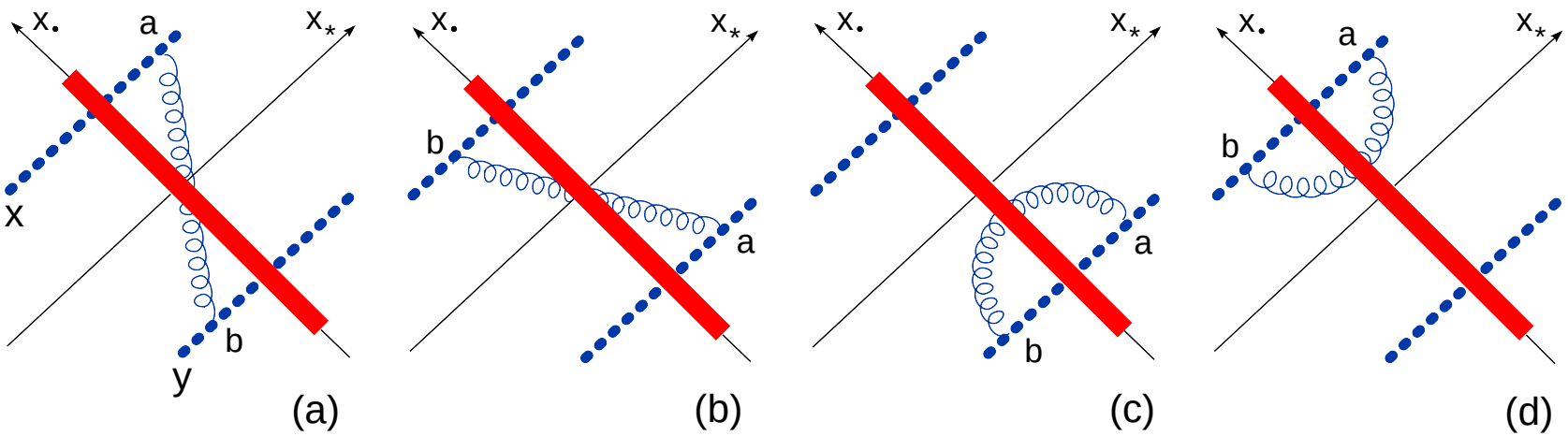
Obtain a set of rules that allow one to get the LO evolution of any trace or product of traces of Wilson lines

Leading order: BK equation

$$\frac{d}{d\eta}\mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle \mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$



$x_\bullet = \sqrt{\frac{s}{2}}x^-$ $\qquad$ $x_* = \sqrt{\frac{s}{2}}x^+$

Non-linear evolution equation: BK equation

$$U_z^{ab} = 2\mathrm{tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

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$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\left\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\right\}$$

Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

Non linear evolution equation: BK equation

$$U_z^{ab} = 2\mathrm{tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\left\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\right\}$$

Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1$)

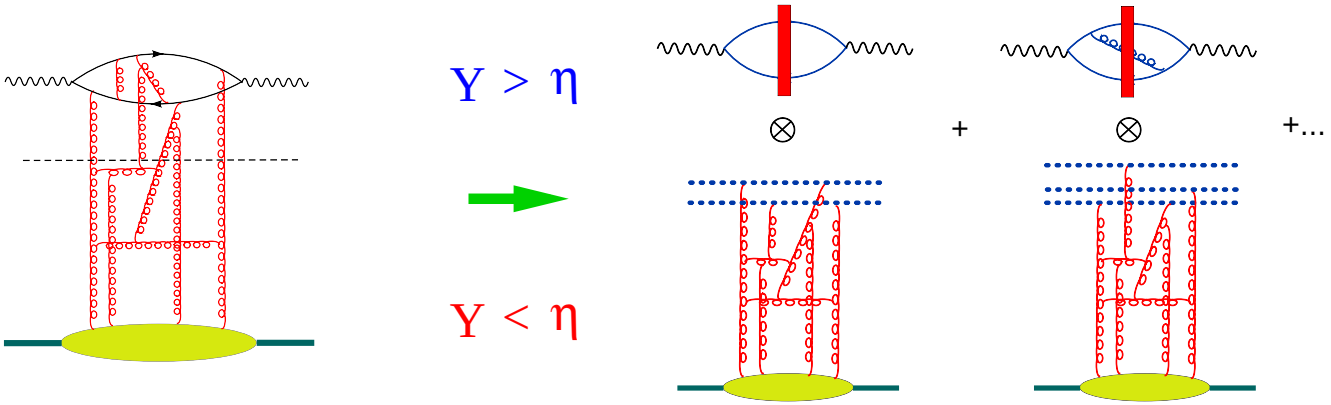
LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1,\ \alpha_s^2 A^{1/3} \sim 1$)

(s for semi-classical)

Motivation: Why NLO correction?

- How to take higher-order corrections into account (either for BFKL or non-linear evolution equation).
- Higher-order corrections are needed to improve phenomenology:
  - Determine the argument of the coupling constant.
  - Gives precision of LO.
- Get the region of application of the leading order evolution equation.
- Check conformal invariance (in $\mathcal{N}$=4 SYM)

High-energy Operator Product Expansion



$$\langle B|\hat{j}_\mu(x)\hat{j}_\nu(y)|B\rangle \simeq \int d^2z_1 d^2z_2\; I^{LO}_{\mu\nu}(z_1,z_2;x,y)\langle B|\mathrm{tr}\{U^\eta_{z_1}U^{\dagger\,\eta}_{z_2}\}|B\rangle$$

$$+\frac{\alpha_s}{\pi}\int d^2z_1 d^2z_2 d^2z_3\; I^{NLO}_{\mu\nu}(z_1,z_2,z_3;x,y)\langle B|\mathrm{tr}\{U^\eta_{z_1}U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{U^\eta_{z_3}U^{\dagger\eta}_{z_2}\}|B\rangle$$

- $\eta = \ln\frac{1}{x_B}$
- $|B\rangle$ Target state
- NLO Photon Impact Factor has been calculated I. Balitsly and G. A. C.
- NLO BK and Balitsky-JIMWLK eqution has been calculated I. Balitsly and G. A. C.

LO+NLO Impact Factor for BFKL pomeron in momentum space

G.A.C. and I. Balitsky (2013)

$$
\begin{aligned}
&I^{\mu\nu}(q,k_\perp)\\
&=\frac{N_c}{32}\int\frac{d\nu}{\pi\nu}\frac{\sinh\pi\nu}{(1+\nu^2)\cosh^2\pi\nu}\left(\frac{k_\perp^2}{Q^2}\right)^{\frac{1}{2}-i\nu}\Big\{\Big[\left(\frac{9}{4}+\nu^2\right)\left(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_1(\nu)\right)P_1^{\mu\nu}\\
&+\left(\frac{11}{4}+3\nu^2\right)\left(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_2(\nu)\right)P_2^{\mu\nu}\Big]\\
&+\frac{\frac{1}{4}+\nu^2}{2k_\perp^2}\left(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_3(\nu)\right)\left[\tilde{P}^{\mu\nu}\bar{k}^2+\bar{P}^{\mu\nu}\tilde{k}^2\right]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
P_1^{\mu\nu} &= g^{\mu\nu}-\frac{q_\mu q_\nu}{q^2}; \qquad P_2^{\mu\nu} = \frac{1}{q^2}\left(q^\mu-\frac{p_2^\mu q^2}{q\cdot p_2}\right)\left(q^\nu-\frac{p_2^\nu q^2}{q\cdot p_2}\right);\\
\bar{P}^{\mu\nu} &= \left(g^{\mu 1}-ig^{\mu 2}-p_2^\mu\frac{\bar{q}}{q\cdot p_2}\right)\left(g^{\nu 1}-ig^{\nu 2}-p_2^\nu\frac{\bar{q}}{q\cdot p_2}\right);\\
\tilde{P}^{\mu\nu} &= \left(g^{\mu 1}+ig^{\mu 2}-p_2^\mu\frac{\tilde{q}}{q\cdot p_2}\right)\left(g^{\nu 1}+ig^{\nu 2}-p_2^\nu\frac{\tilde{q}}{q\cdot p_2}\right);
\end{aligned}
$$

- Functions $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are defined in the paper.

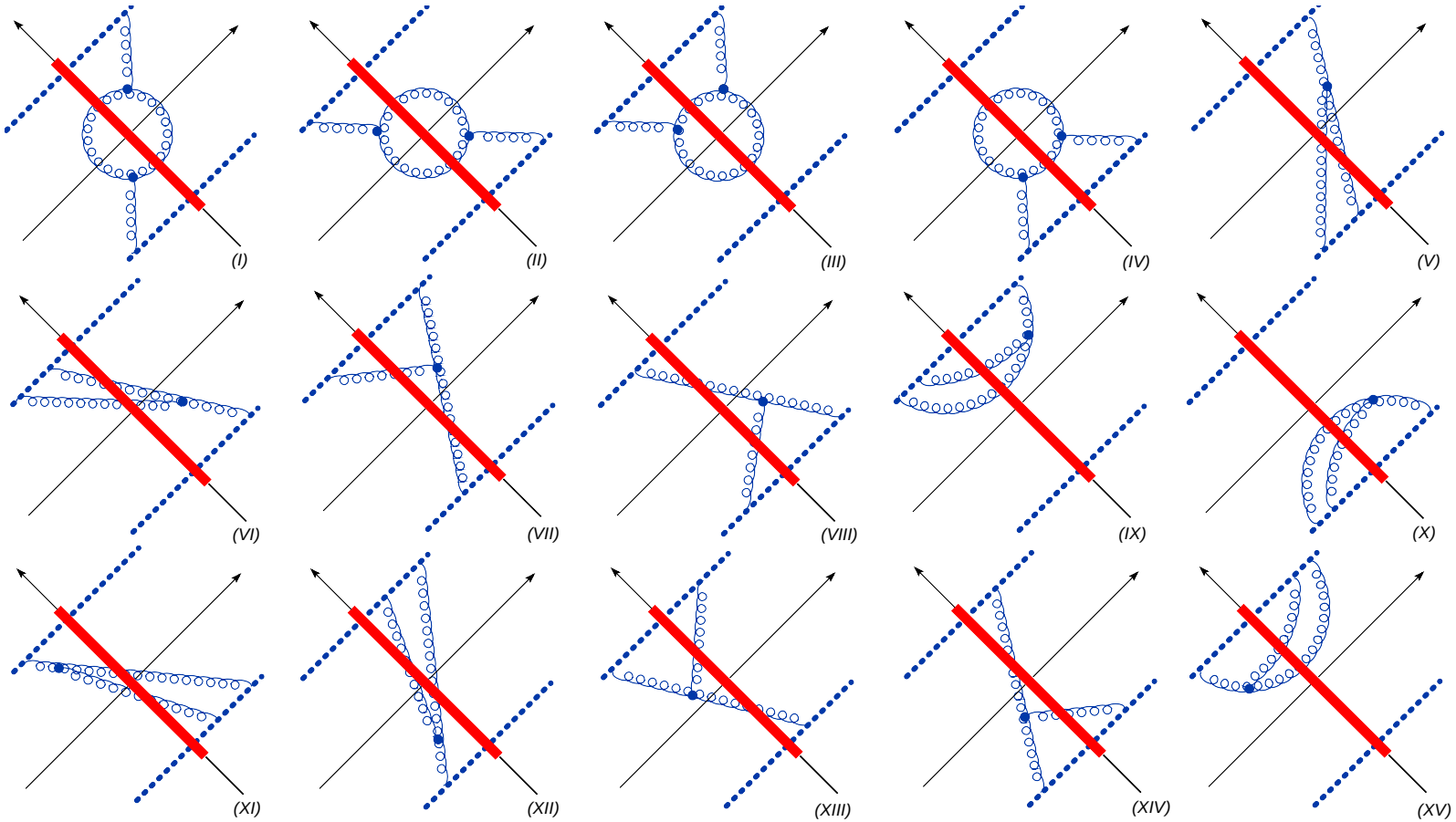
$$\frac{d}{d\eta}Tr\{U_x U_y^\dagger\} =$$
$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_x U_z^\dagger\}Tr\{U_z U_y^\dagger\} - N_c Tr\{U_x U_y^\dagger\}] +$$
$$\alpha_s^2 \int d^2z d^2z' \left(K_4(x,y,z,z')\{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z')\{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\}\right)$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K_4 and K_6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

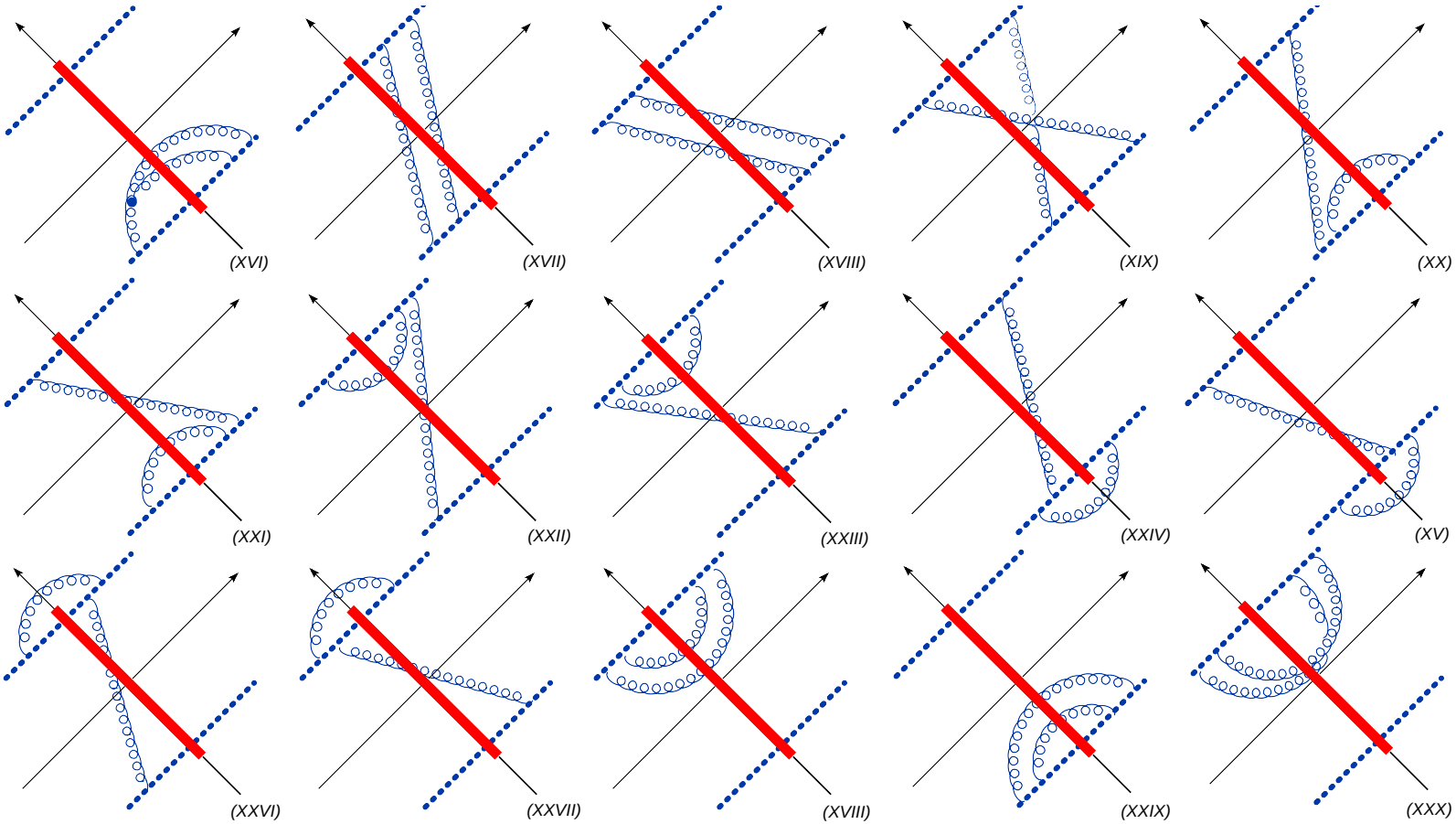
- We need to calculate some diagrams analytically (pen and paper).

Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction

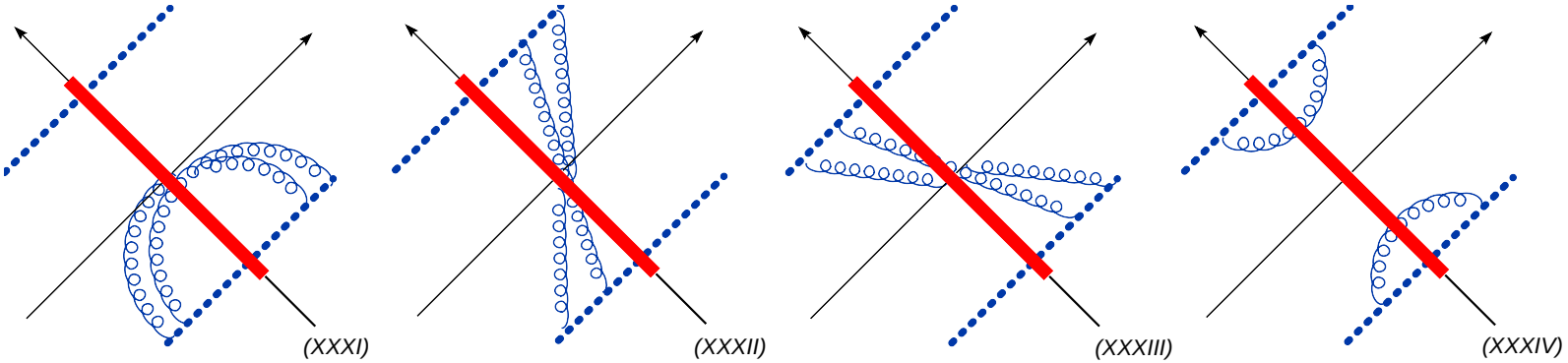


Diagrams with 2 gluons interaction



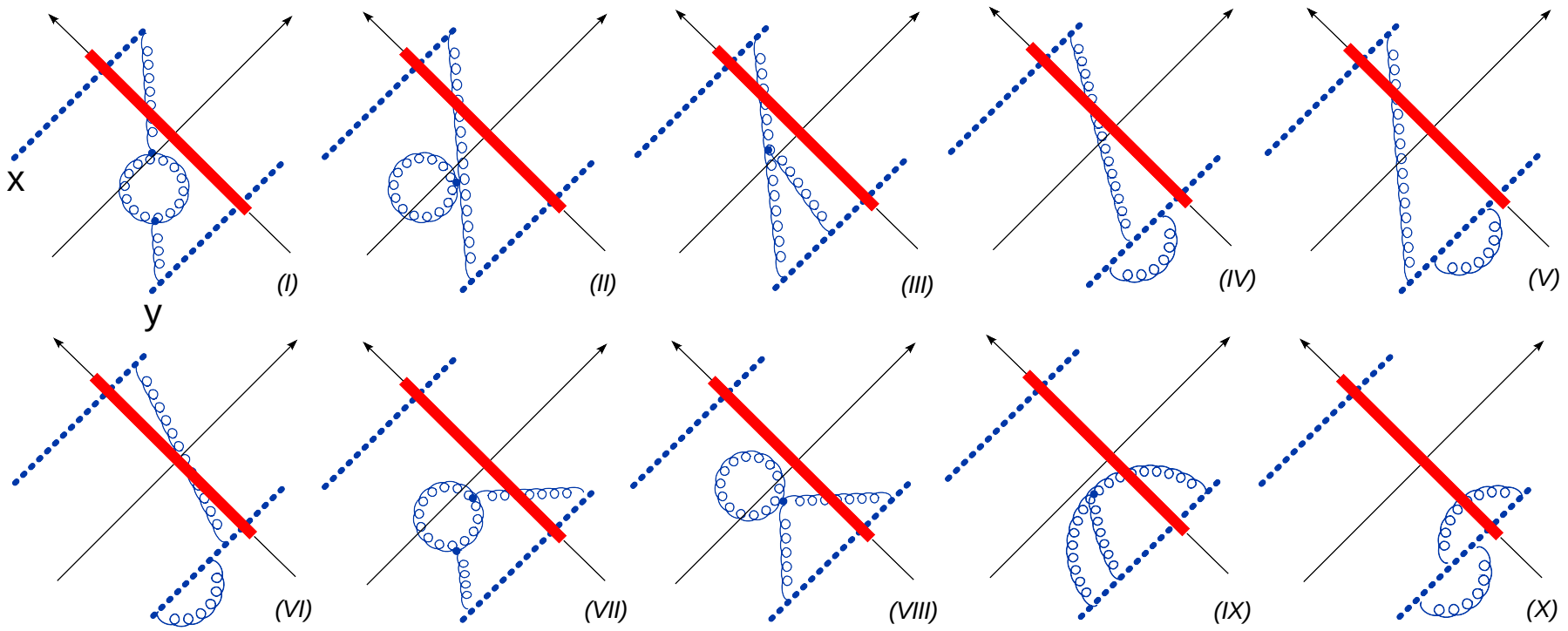
Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction



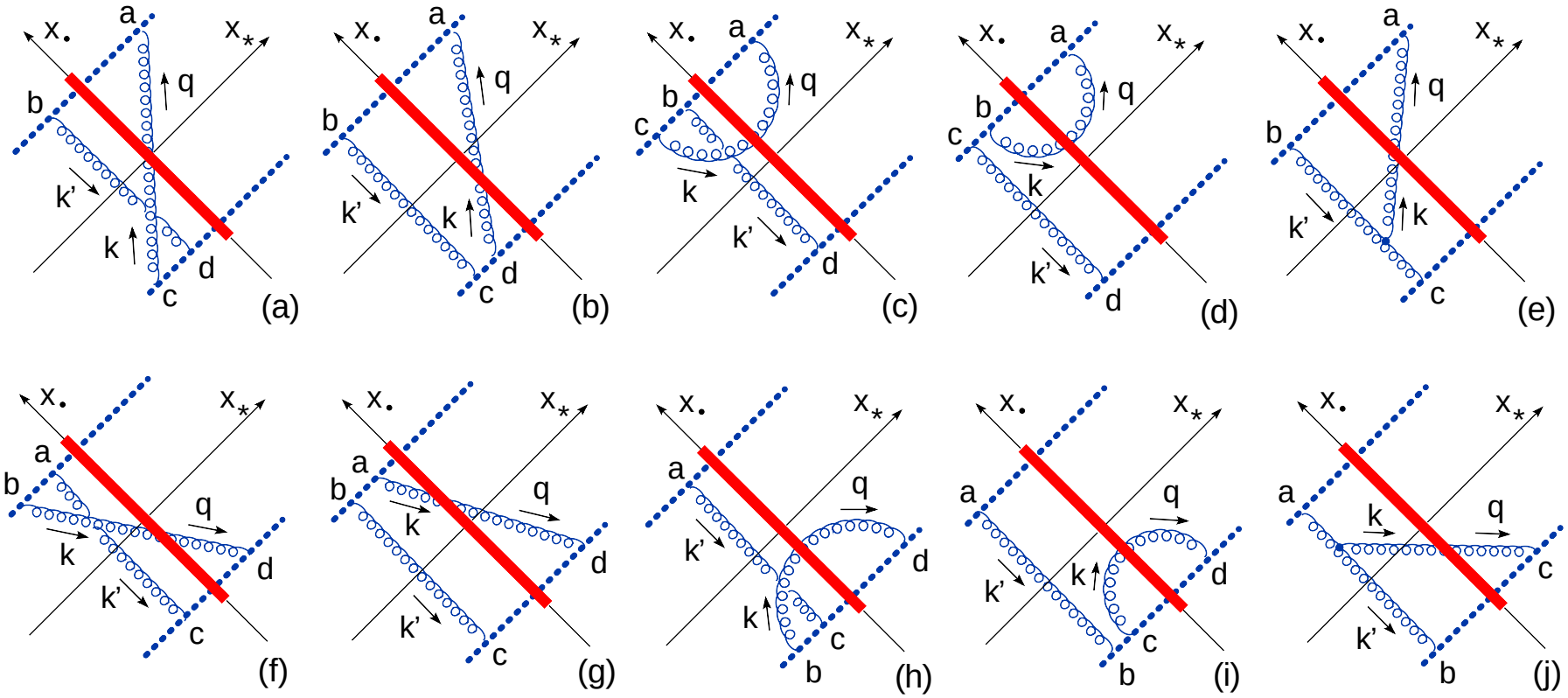
Diagrams of the NLO gluon contribution

"Running coupling" diagrams



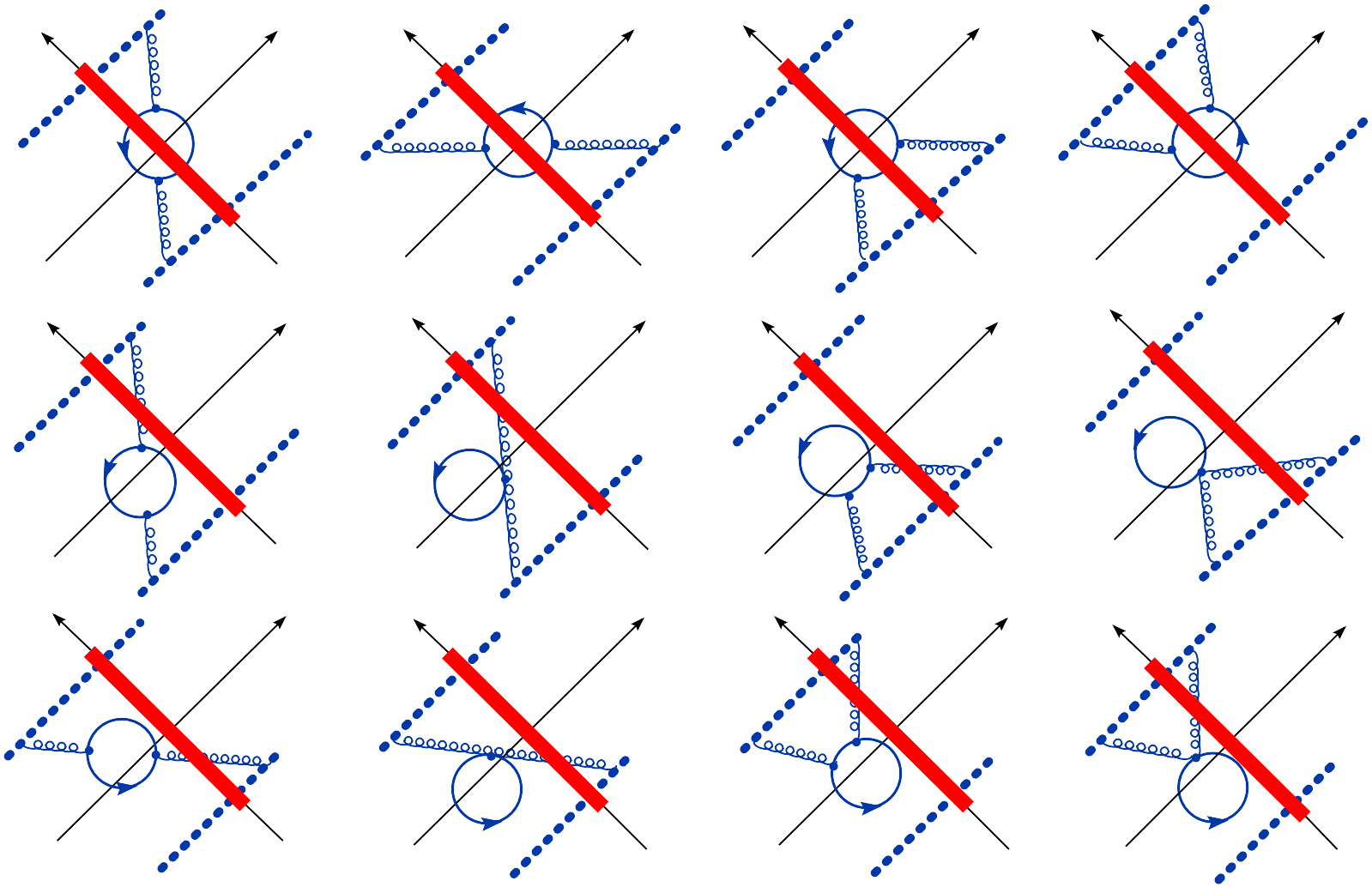
Diagrams of the NLO gluon contribution

$1 \to 2$ dipole transition diagrams



Diagrams of the NLO gluon contribution

$\mathcal{N} = 4$ SYM diagrams (scalar and gluino loops)



Evolution equation for composite conformal dipoles in $\mathcal{N}=4$ SYM

$$[\mathrm{tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\frac{\alpha_s}{4\pi}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[\frac{1}{N_c}\mathrm{tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-\mathrm{tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\Big]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2} + O(\alpha_s^2)$$

$$\frac{d}{d\eta}\Big[\mathrm{tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\Big]^{\mathrm{conf}}$$
$$=\frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\Big[\mathrm{tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\Big]^{\mathrm{conf}}$$
$$-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}$$
$$\times\,\mathrm{tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}[(\hat{U}^{\eta}_{z_3})^{aa'}(\hat{U}^{\eta}_{z_4})^{bb'}-(z_4\to z_3)]$$

Conformal (Möbius) invariant

NLO evolution of composite "conformal" dipoles in QCD

$$\frac{d}{d\eta}[\mathrm{tr}\{\hat{U}_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]^{\mathrm{conf}}$$

$$\times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{{z_{14}}^2z_{23}^2+{z_{24}}^2z_{13}^2-4z_{12}^2z_{34}^2}{2({z_{14}}^2z_{23}^2-{z_{24}}^2z_{13}^2)}\ln\frac{{z_{14}}^2z_{23}^2}{{z_{24}}^2z_{13}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\}\{\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\} - \mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\frac{z_{12}^2z_{34}^2}{z_{13}^2{z_{24}}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2{z_{24}}^2}{{z_{14}}^2z_{23}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\}\mathrm{tr}\{\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\} - \mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_4}\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\hat{U}_{z_4}\hat{U}^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$$b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

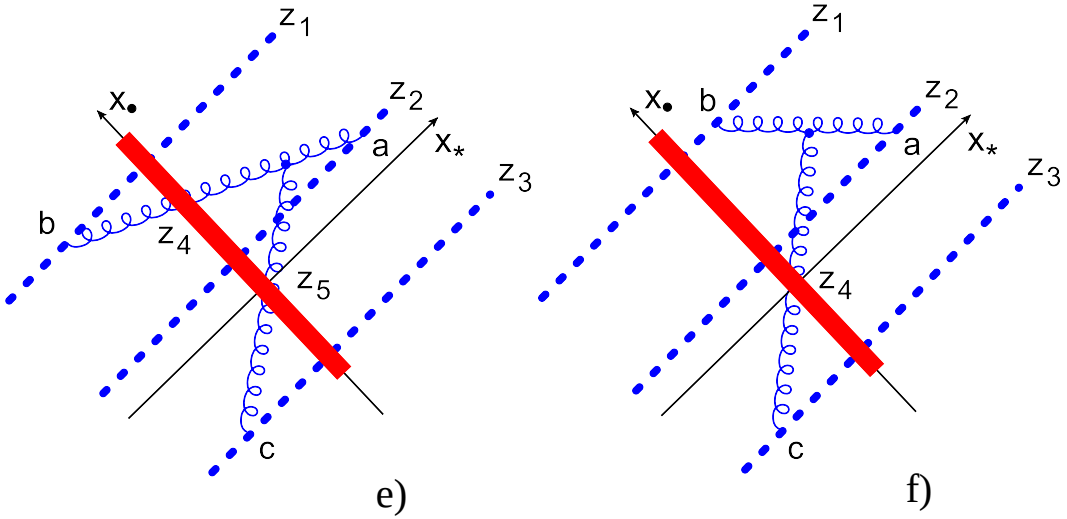
I. Balitsky and G.A.C

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel Fadin and Lipatov (1998).

NLO Balitsky-JIMWLK evolution equation

- In proton-Nucleus and Nucleus-Nucleus collisions there are also quadrupole Wilson line operators $\mathrm{tr}\{U_x U_y^\dagger U_w U_z^\dagger\}$.
- $\Rightarrow$ Need NLO Balitsky-JIMWLK evolution equation.



NLO Balitsky-JIMWLK evolution equation

G.A.C. and I. Balitsky (2013)

$$
\begin{aligned}
\frac{d}{d\eta}(U_1)_{ij}(U_2)_{kl}(U_3)_{mn} = \; & i\frac{\alpha_s^2}{2\pi^4}\int d^2z_4 d^2z_5 \left\{ \mathcal{J}_{12345} \ln \frac{z_{34}^2}{z_{35}^2} \right. \\
& \times f^{cde}\big[(t^a U_1)_{ij}(t^b U_2)_{kl}(U_3^\dagger t^c)_{mn}(U_4 - U_1)^{ad}(U_5 - U_2)^{be} \\
& - (U_1 t^a)_{ij}(U_2 t^b)_{kl}(t^c U_3)_{mn}(U_4 - U_1)^{da}(U_5 - U_2)^{eb}\big] \\
& + \mathcal{J}_{32145} \ln \frac{z_{14}^2}{z_{15}^2} f^{ade}\big[(U_1^\dagger t^a)_{ij}(t^b U_2)_{kl}(t^c U_3)_{mn}(U_4 - U_3)^{cd}(U_5 - U_2)^{be} \\
& - (t^a U_1)_{ij} \otimes (U_2 t^b)_{kl}(U_3 t^c)_{mn}(U_4^{dc} - U_3^{dc})(U_5^{eb} - U_2^{eb})\big] \\
& + \mathcal{J}_{13245} \ln \frac{z_{24}^2}{z_{25}^2} f^{bde}\big[(t^a U_1)_{ij}(U_2^\dagger t^b)_{kl}(t^c U_3)_{mn}(U_4 - U_1)^{ad}(U_5 - U_2)^{ce} \\
& - (U_1 t^a)_{ij}(t^b U_2)_{kl}(U_3 t^c)_{mn}(U_4 - U_1)^{da}(U_5 - U_3)^{ec}\big]
\end{aligned}
$$

Functions $\mathcal{J}_{12345}$, $\mathcal{J}_{32145}$, $\mathcal{J}_{13245}$ are defined in the paper *Phys.Rev. D88 (2013) 111501*

Conclusions

- Linear evolution equation cannot describe the dynamics of high-energy QCD.
- Dynamics of QCD at high-energy is non-linear.
- BK/JIMWLK evolution equations.
- NLO BK/JIMWLK evolution equation.
- No solution for NLO BK/JIMWLK equation is available at the moment.
- 15 years after the NLO BFKL equation was completed its solution has been found G.A.C and Y. Kovchegov.