Chiral condensate in $n_f = 2$ QCD from the Banks–Casher relation

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Collaborators:
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Based on: GPE, L. Giusti, S. Lottini, R. Sommer: “Chiral symmetry breaking in QCD Lite”, arXiv:1406.4987

Banks-Casher relate condensate Σ to spectral density ρ of Dirac operator

$$\Sigma \equiv -\frac{1}{2}\langle\bar{\psi}\psi\rangle = \pi \lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty}\rho(\lambda, m), \qquad \rho(\lambda, m) = \frac{1}{V}\sum_{k=1}^{\infty}\langle\delta(\lambda-\lambda_k)\rangle \quad (1)$$

Calculated on the lattice: mode number $\nu \equiv$ integrated density

The number of modes of the massive hermitian Dirac operator $D^\dagger D + m^2$, with eigenvalues $\alpha \leq \Lambda^2 + m^2$, is renormalization-group invariant[1]

$$\nu_{\mathrm{R}}(\Lambda_{\mathrm{R}}, m_{\mathrm{R}}) = \nu(\Lambda, m) = V\int_{-\Lambda}^{\Lambda} d\lambda\rho(\lambda, m) \quad (2)$$

To extract Σ: Define effective spectral density (removing threshold effects)

$$\tilde{\rho}_{\mathrm{R}} = \frac{\pi}{2V}\frac{\nu_{2,\mathrm{R}} - \nu_{1,\mathrm{R}}}{\Lambda_{2,\mathrm{R}} - \Lambda_{1,\mathrm{R}}} \xrightarrow{V\to\infty; a, m_{\mathrm{R}}, \Lambda_{\mathrm{R}}\to 0} \Sigma \quad (3)$$

[1] L. Giusti and M. Lüscher, JHEP903(2009)13

NLO (W)ChPT ($n_f = 2$) in the continuum[1,2] and GSM[3] regime[2]

$$\tilde{\rho}_{\mathrm{R}}^{\mathrm{NLO}} = \Sigma\left\{1 + \frac{m_{\mathrm{R}}\Sigma}{(4\pi)^2 F^4}\left[3\,\bar{l}_6 + 1 - \ln(2) - 3\ln\left(\frac{\Sigma m_{\mathrm{R}}}{F^2 M^2}\right) + \tilde{g}_\nu\left(\frac{\Lambda_{1,\mathrm{R}}}{m_{\mathrm{R}}}, \frac{\Lambda_{2,\mathrm{R}}}{m_{\mathrm{R}}}\right)\right]\right\}$$
$$-32(W_0 a)^2 \frac{W_8' m_{\mathrm{R}}}{\Lambda_{1,\mathrm{R}}\Lambda_{2,\mathrm{R}}} \tag{4}$$

$$\text{with } \tilde{g}_\nu(x_1, x_2) = \frac{f_\nu(x_1) + f_\nu(x_2)}{2} + \frac{1}{2}\frac{x_1 + x_2}{x_2 - x_1}\left[f_\nu(x_2) - f_\nu(x_1)\right]$$
$$f_\nu(x) = \left(x - \frac{1}{x}\right)\arctan(x) - \frac{\pi}{2}x - \ln(x + x^3)$$

- No chiral logs for fixed Λ_{R}; $\tilde{g}_\nu(.)$ mild function
- 1+2 NLO LECs; W_8' expected negative[4]

[1] L. Giusti and M. Lüscher, JHEP903(2009)13
[2] S. Necco and A. Shindler, JHEP1104(2010)31
[3] "Generally small quark mass"
[4] M.T. Hansen and S.R. Sharpe, PRD85(2012)14593; K. Splittorff and J. Verbaarschot, PRD85(2012)105008

Parameters of the simulation: $n_f = 2$ CLS-lattices[5]

id	L/a	m_π [MeV]	$m_\pi L$	a [fm]	$R\tau_{\rm exp}$ [MDU]	$R\tau_{\rm int}(m_\pi)$ [MDU]	$R\tau_{\rm int}(\nu)$ [MDU]	$Rn_{\rm it}(\nu)$ [MDU]	$N_{\rm cnfg}$
A3	32	490	6.0	0.075	40	7	3	48	55
A4		380	4.7			5		53	55
A5		330	4.0			5		36	55
B6	48	280	5.2			6		24	50
E5	32	440	4.7	0.065	56	9	6	36	92
F6	48	310	5.0			8		30	50
F7		270	4.3			7		27	50
G8	64	190	4.1			8		24-48	50
N5	48	440	5.2	0.048	200	30	23	281	60
N6		340	4.0		100	10		128	60
O7	64	270	4.2		100	15		76	50

- Autocorrelation under control, $\tau_{\rm int}(\nu) < n_{\rm it}(\nu)$
- Finite volume effects under control, found tiny for $\Lambda_{\rm R} \geq 20$ MeV
- 9 values of cutoff $\Lambda_{\rm R}$ for each ensemble, $20 \leq \Lambda_{\rm R} \leq 120$ MeV

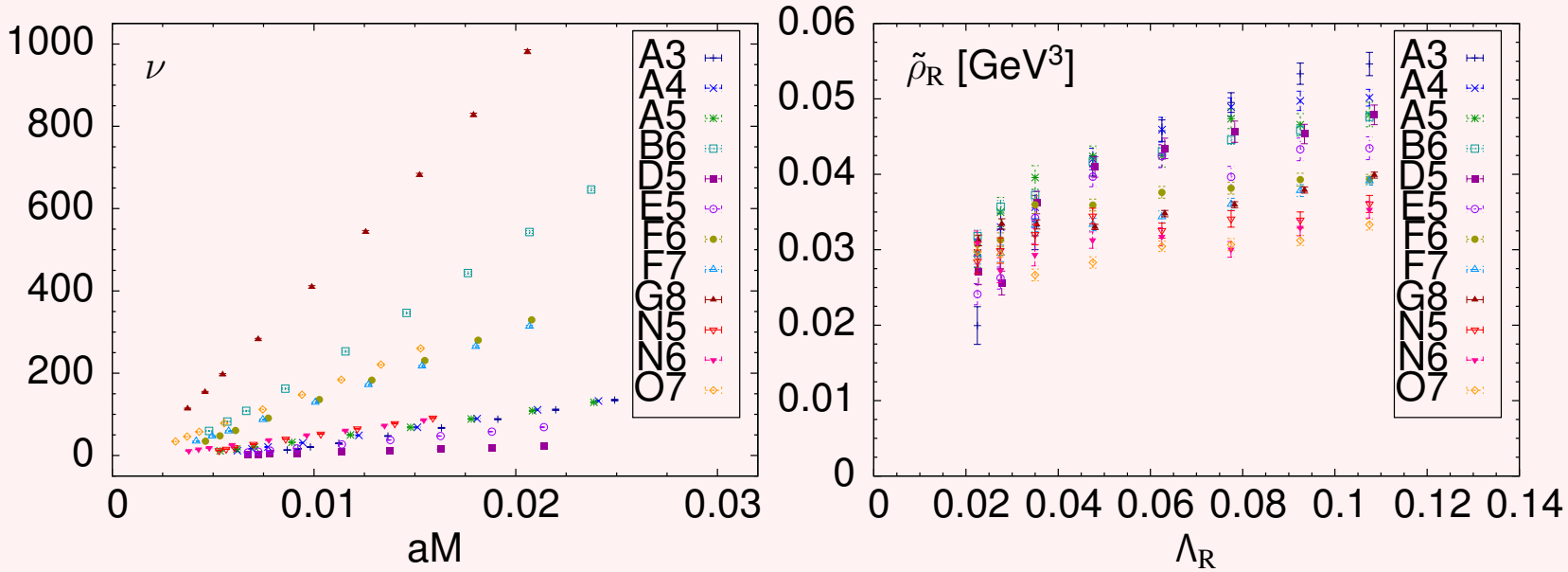
[5] P. Fritzsch *et al.*, NPB865(2012)397; M. Marinkovic *et al.*, PoS Lat(2011)232

Stochastic evaluation of ν through VEV of projector to low modes

$$\nu = \langle \mathrm{tr}[\mathbb{P}_M] \rangle \,, \qquad M = \sqrt{\Lambda^2 + m^2} \tag{5}$$

$$= \frac{1}{N} \sum_{k=1}^{N} (\eta_k, \mathbb{P}_M \eta_k) \,, \qquad \eta_k \ldots \text{ pseudo-fermion fields} \tag{6}$$

First look at numerical data



- ν roughly linear in all ensembles.

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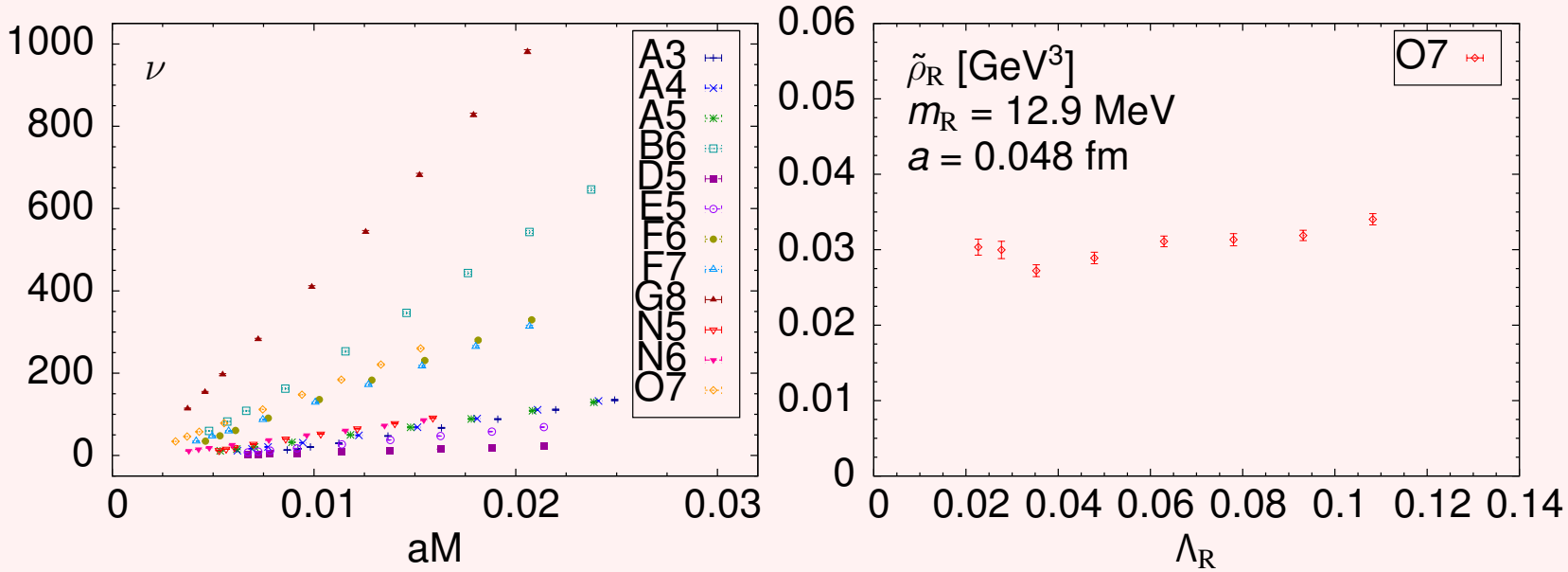


- ν roughly linear in all ensembles.
- $\tilde{\rho}$ non-zero and flat in Λ_R toward small m_R and a.

First studies[6] indicated:

- Higher order effects in $\tilde{\rho}_{\mathrm{R}}$ observed.
- Functional form not well known at finite lattice spacing.

Suggest Strategy A:

- First perform continuum limit at each $(\Lambda_{\mathrm{R}}, m_{\mathrm{R}})$ following Symanzik.
- Finite spectral density near the origin will suggest chiral SSB.
- Use (continuum) ChPT to remove remaining corrections.

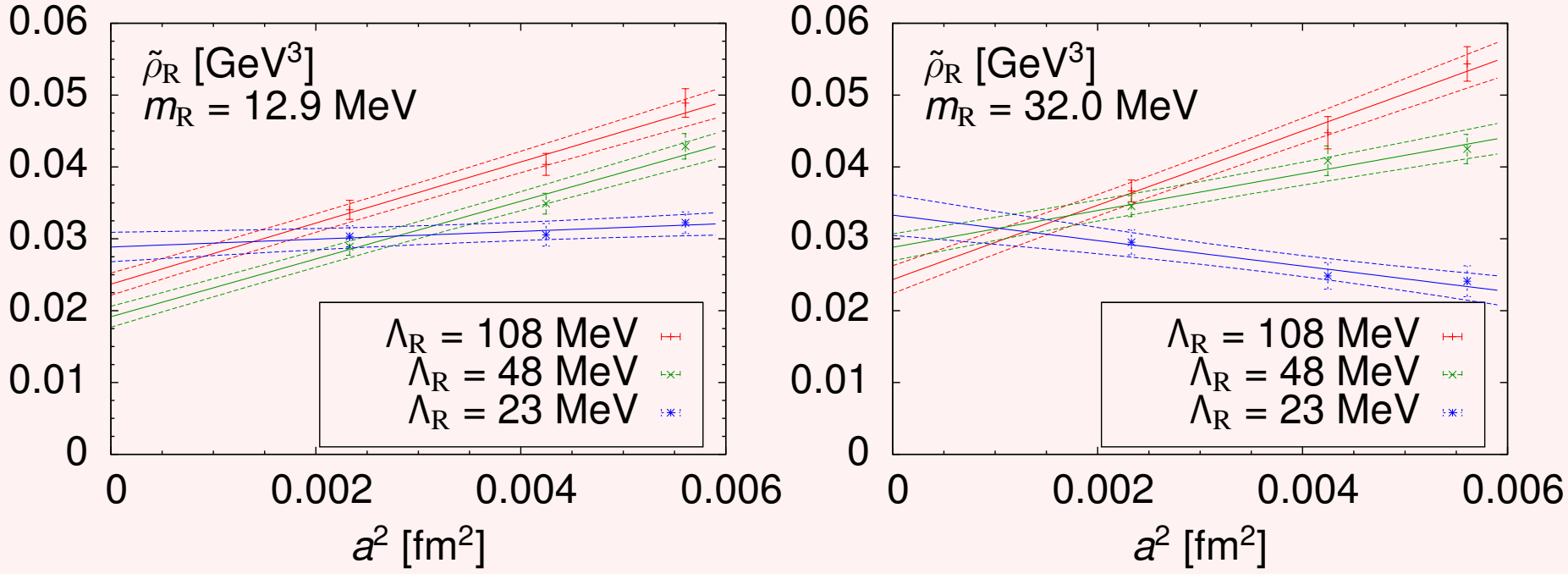
$$\Sigma = \lim_{\Lambda_{\mathrm{R}} \to 0} \lim_{m_{\mathrm{R}} \to 0} \lim_{a \to 0} \tilde{\rho}_{\mathrm{R}}(\Lambda_{\mathrm{R}}, m_{\mathrm{R}}, a) \tag{7}$$

- *A posteriori* self-consistency check:
  Agreement with $M_\pi^2 F_\pi^2/2$ vs. m (GMOR)?

[6] GPE *et al.*, PoS Lat(2013)119

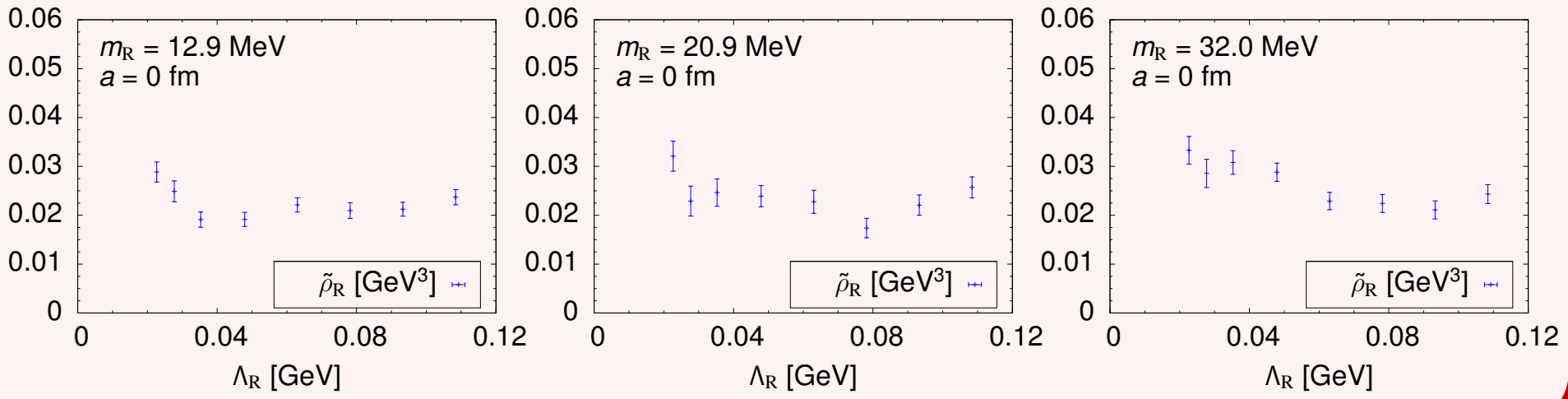
Fitting strategy A II: Extrapolate to the continuum

$\tilde{\rho}_R$ vs. a^2 for various (Λ_R, m_R)



- At each pair (Λ_R, m_R), extrapolate $a \to 0$.
- Data agree well with linear a^2-dependence ($\mathcal{O}(a)$-improved theory).
- Discretization effects show non-trivial (Λ_R, m_R)-dependence:
  $\tilde{\rho}_R^{1/3}$: mild (<5%) at lightest (Λ_R, m_R); i.g. up to $\mathcal{O}(20\%)$

$\tilde{\rho}_R$ vs. Λ_R in the continuum for various m_R



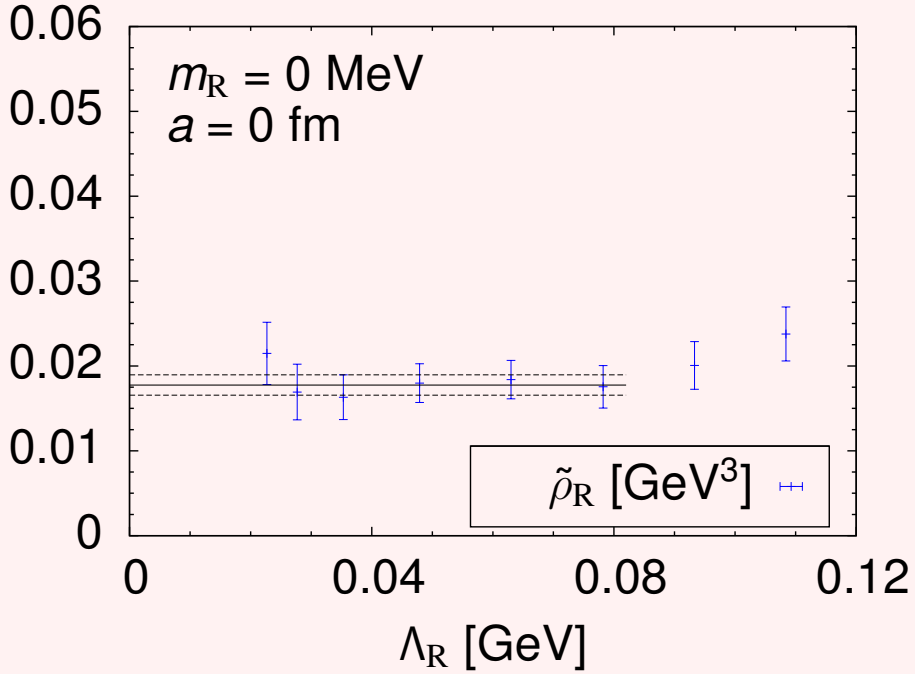
- Non-zero density at small (Λ_R, m_R) points to chiral SSB.
- Use generalized NLO ChPT to extrapolate to chiral limit[7]:

$$\tilde{\rho}_R = c_0(\Lambda_R) + c_1 m_R + c_2 g(\Lambda_R, m_R) \quad (8)$$

$$\Sigma = c_0(\Lambda_R) = \text{const.} \qquad \text{at NLO} \quad (9)$$

[7] Here and later we use the short-hand notation: $g(\Lambda, m) = m(\tilde{g}_\nu(\Lambda_1/m, \Lambda_2/m) - 3\ln(m/\mu))$

$\tilde{\rho}_R$ vs. Λ_R in the continuum and chiral limit



- $\tilde{\rho}_R|_{m_R=0} = c_0(\Lambda_R)$ shows plateau at NLO ChPT.
- Identify valid range of NLO: $\Lambda_R < 80$ MeV.
- $\Sigma^{1/3} = 261(6)$ MeV in $\overline{\text{MS}}$ at 2 GeV.

Combined 3-dim fit in (Λ_R, m_R, a)

- Include all data; no interpolation required.
- Fewer fit parameters compared to Strategy A.
- Model the discretization effects:
  - Linear in a^2 and m_R.
  - Still allow for arbitrary Λ_R-dependence.

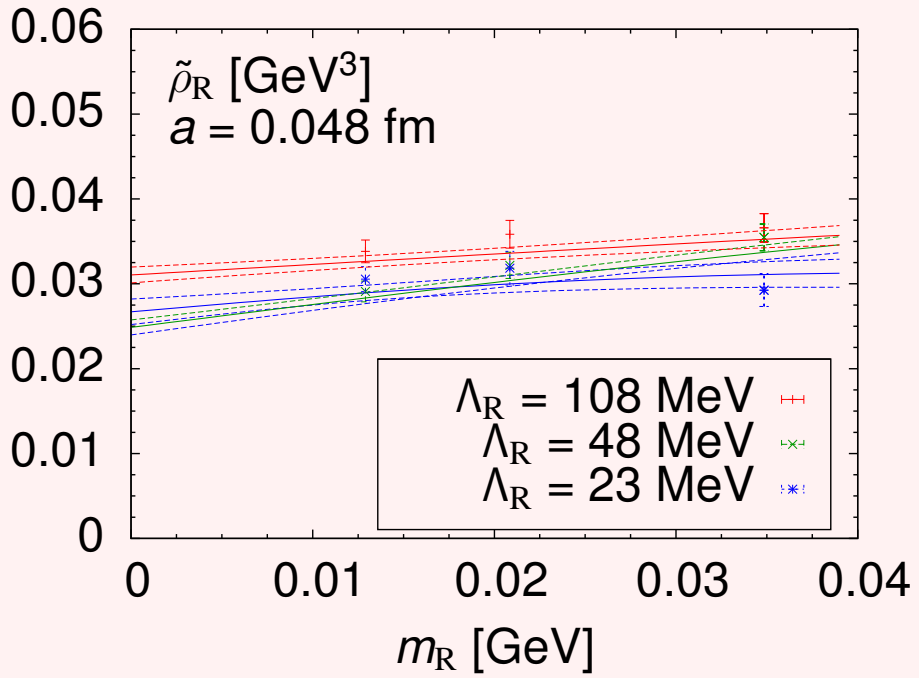
$$\begin{aligned} \tilde{\rho}_R &= c_{0,0}(\Lambda_R) + c_{0,1}(\Lambda_R)a^2 + c_{1,0}m_R + c_{1,1}(\Lambda_R)m_R a^2 + c_2 g(\Lambda_R, m_R) \\ \Sigma &= c_{0,0}(\Lambda_R) = \text{const.} \qquad \text{at NLO} \end{aligned} \tag{10}$$

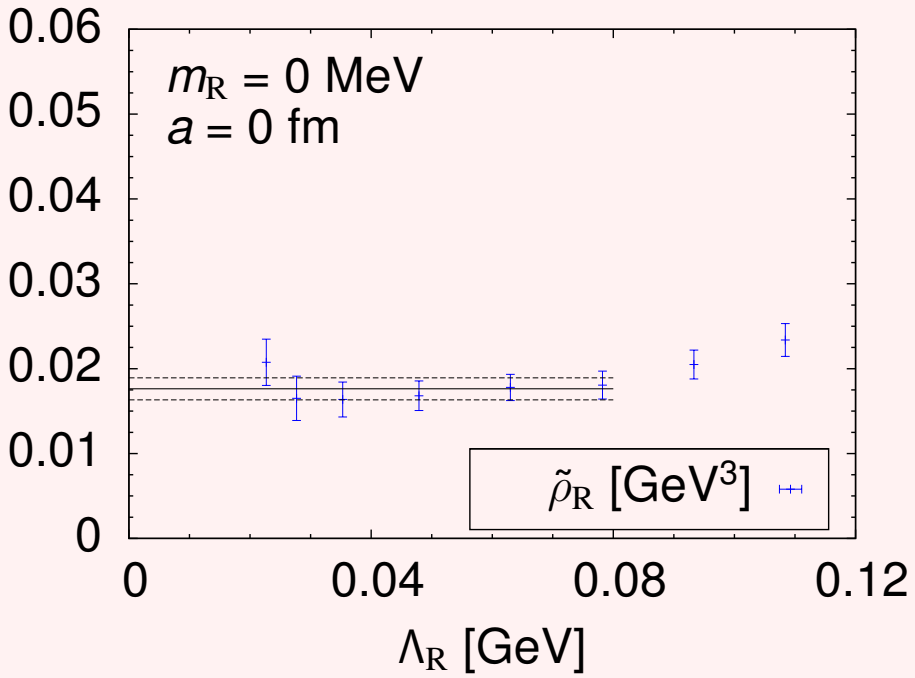
- Inspired by Symanzik and chiral power expansion.
- Model complies with results of Strategy A.
- Includes NLO WChPT GSM[3] as special case.

[3] "Generally small quark mass"

Fitting strategy B II: Continuum and chiral limit

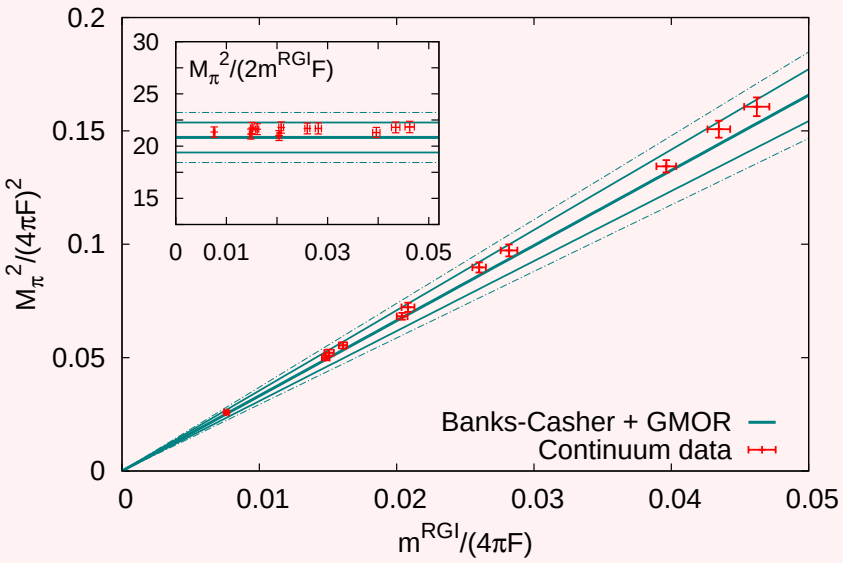
$\tilde{\rho}_R$ vs. m_R and vs. Λ_R





- NLO plateau in $c_{0,0}(\Lambda_R)$ for $\Lambda_R < 80$ MeV.
- $\Sigma^{1/3} = 260(6)$ MeV in $\overline{\text{MS}}$ at 2 GeV.
- Systematic error:
  - Neglect data at coarse lattices (a=0.075 fm): +8 MeV.
  - Include $\mathcal{O}(\Lambda_R^2, m_R^2)$-terms: -7 MeV.

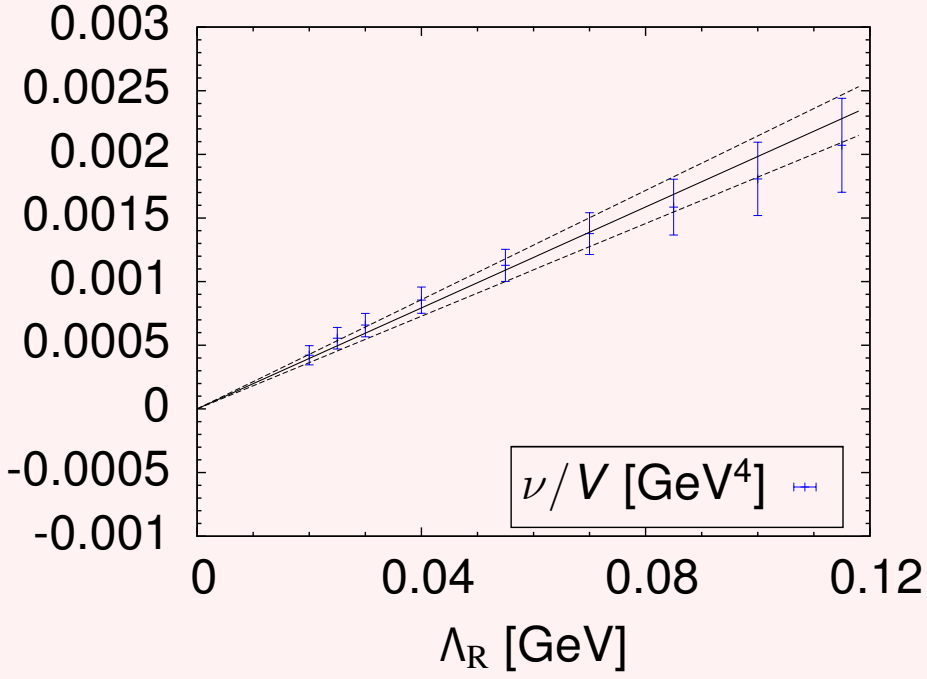
Conclusion and check with GMOR



- *Ab-initio* determination of the chiral condensate from Banks-Casher.
- Extensive study of the spectral density:
  - 3 lattice spacings: $0.048 \leq a \leq 0.076$ fm
  - 4 pion masses: $190 \leq m_\pi \leq 490$ MeV
  - 9 cutoffs: $20 \leq \Lambda_R \leq 120$ MeV
- Separate treatment of various effects, all systematics discussed.
- $\Sigma^{1/3}$ =261(6)(8) MeV in $\overline{\text{MS}}$ at 2 GeV.
- Final results agrees with GMOR-relation[8].

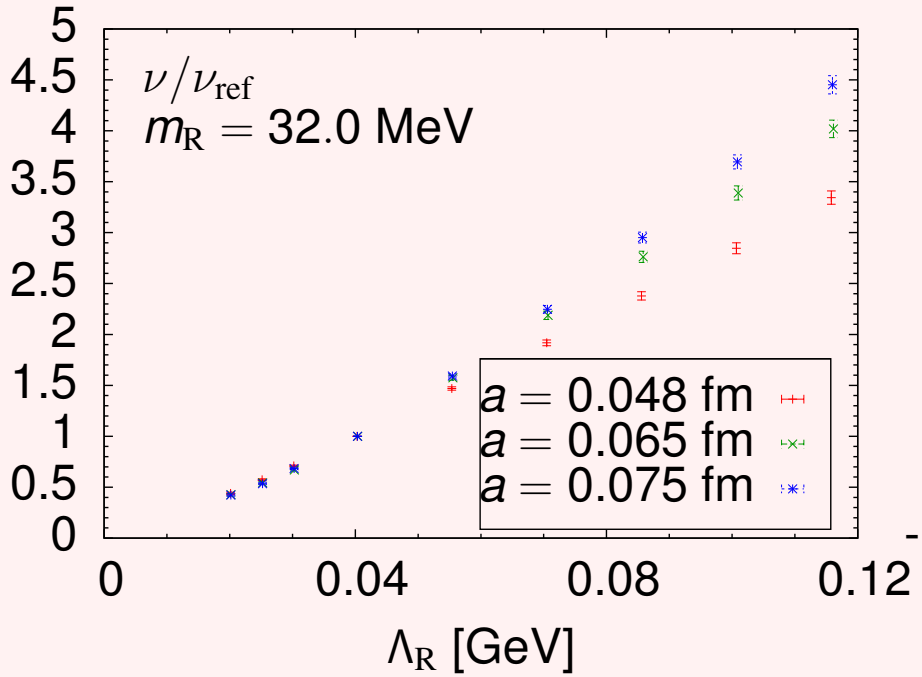
[8] GPE, L. Giusti, S. Lottini, R. Sommer: "Chiral symmetry breaking in QCD Lite", arXiv:1406.4987

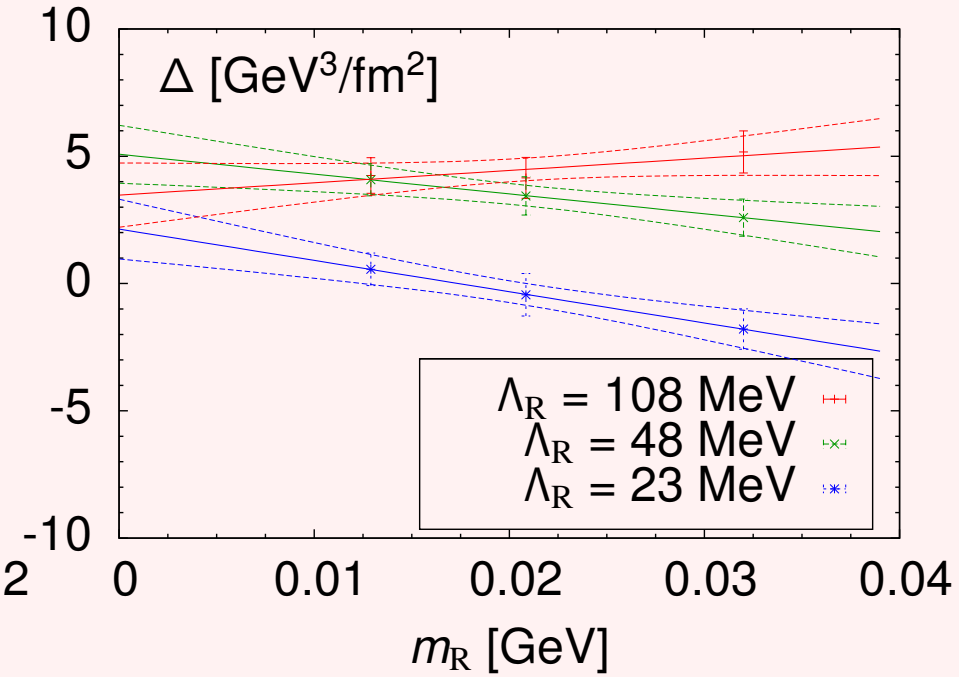
Backup: Strategy-A-analogously for the mode number



- Data well described by linear fit passing through origin.
- Strongly suggests to be actually within chiral regime.
- $\Sigma^{1/3} = 271(8)$ MeV.

Backup: Discretization effects from Strategy A



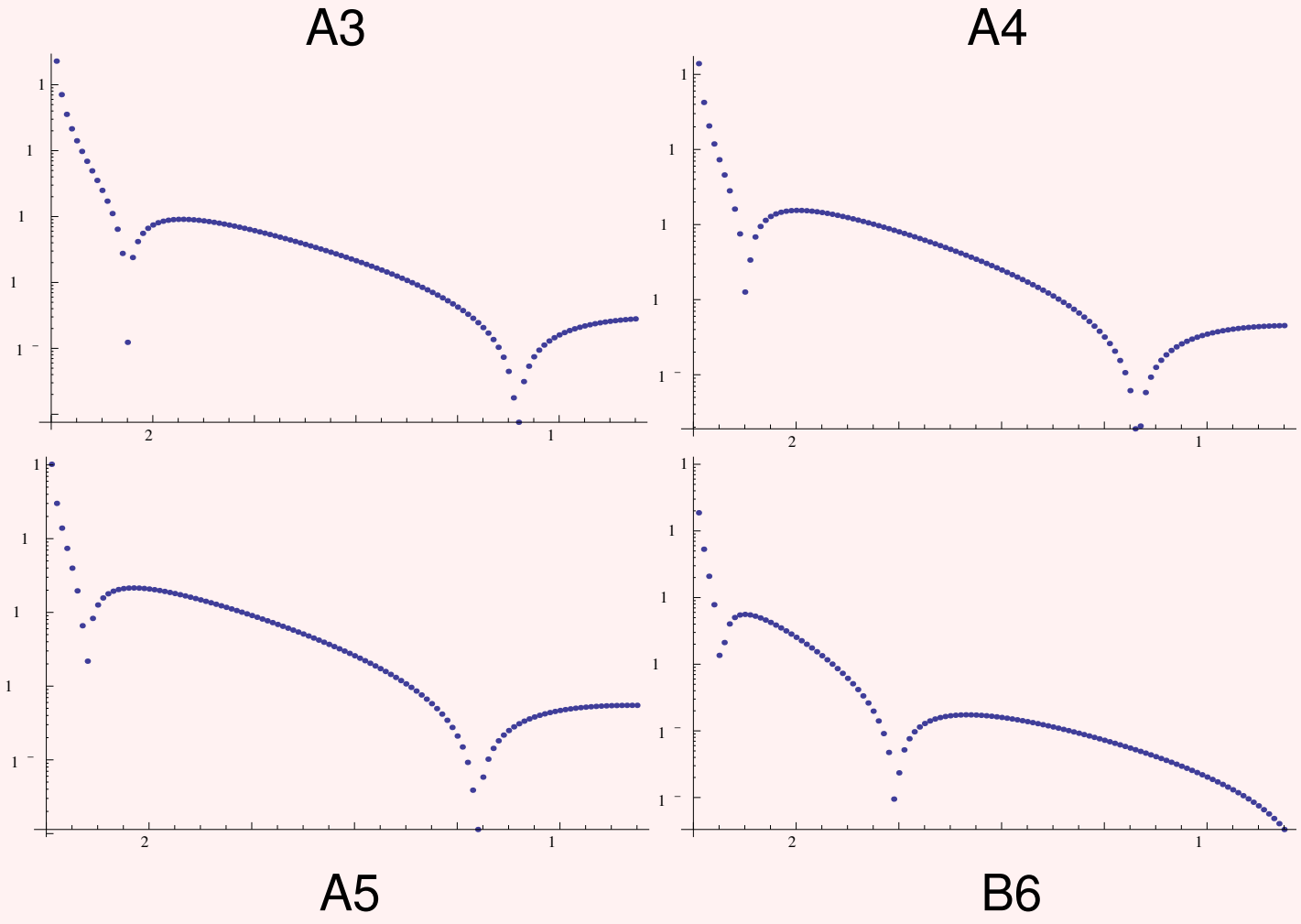
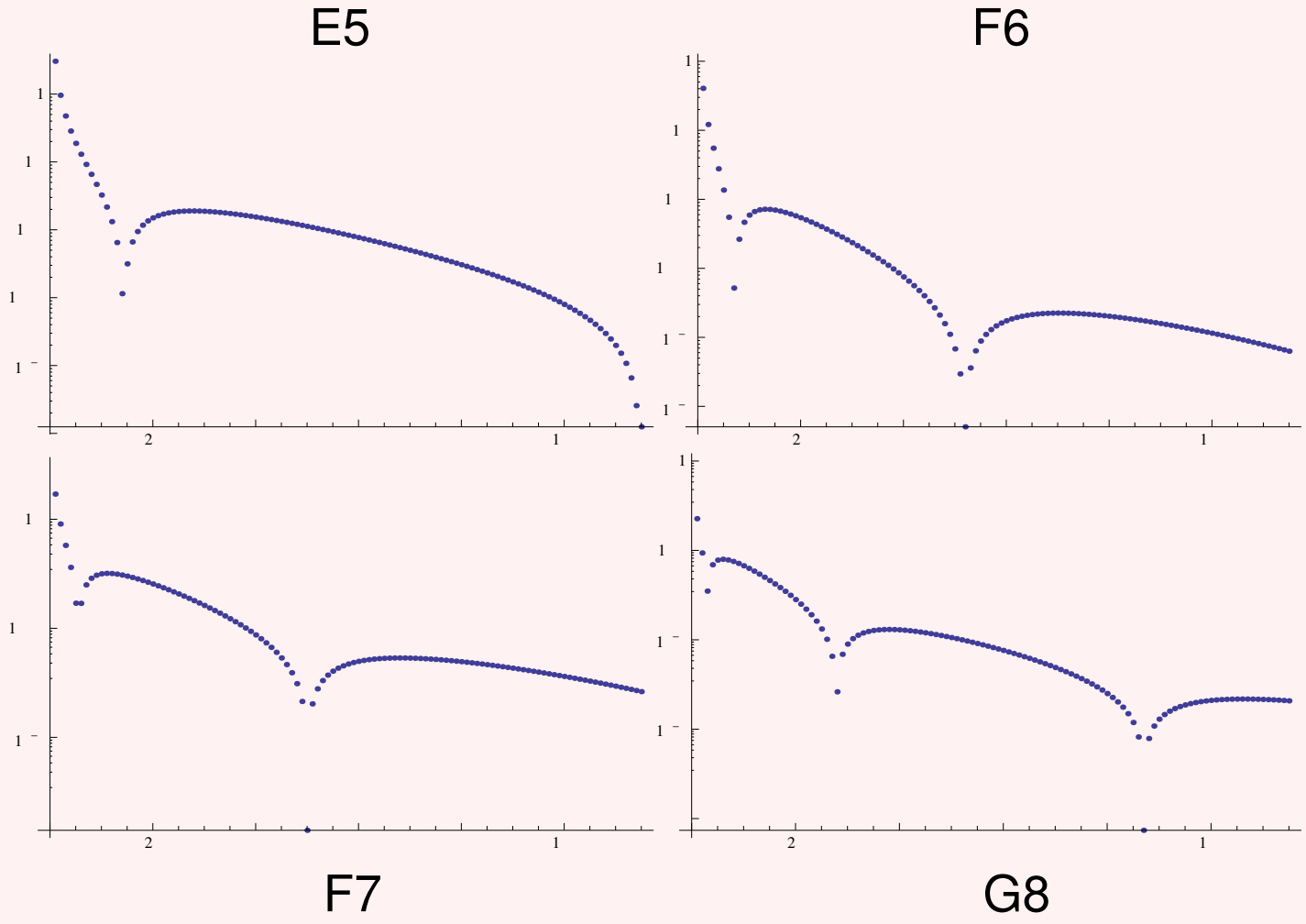


- $\nu_{\rm ref} = \nu|_{\Lambda_{\rm R}=40{\rm MeV}}$ shows flat a-dependence in dimensionful analysis.

$$\tilde{\rho}_{\rm R}(\Lambda_{\rm R}, m_{\rm R}, a) = \tilde{\rho}_{\rm R}(\Lambda_{\rm R}, m_{\rm R}, 0) + a^2\Delta(\Lambda_{\rm R}, m_{\rm R}) \quad (11)$$

$$\Delta(\Lambda_{\rm R}, m_{\rm R}) = \bar{c}_{0,1}(\Lambda_{\rm R}) + \bar{c}_{1,1}(\Lambda_{\rm R})m_{\rm R} \quad (12)$$

$|\Delta\widetilde{\Sigma}^{\mathrm{FV}}|/\Sigma$ vs. Λ_{R} for $\beta = 5.2$



$|\Delta\widetilde{\Sigma}^{\mathrm{FV}}|/\Sigma$ vs. Λ_{R} for $\beta = 5.3$