The temperature dependence of the chiral condensate in the Schwinger model with Matrix Product State

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with M. C. Banuls, K. Cichy, I. Cirac and K. Jansen

Schwinger model for $N_f = 1$

J. Schwinger Phys.Rev. 128 (1962)

- 1+1 dimensional QED model N. L. Pak and P. Senjanovic, Phys.Let.B71, 2 (1977), K. Johnson Phys.Let. 5, 4(1963)
  - ✳ not QCD, but **similar to QCD** :
    confinement, chiral symmetry breaking (via anomaly for $N_f=1$)
  - ✳ exactly solvable in massless case ⇒ a good test case
- Hamiltonian form. To solve eigen equation with Hamiltonian H:
  $H\,|\psi\rangle = \lambda\,|\psi\rangle$ λ :eigenvalue, $|\psi\rangle$:eigenstate
- Hamiltonian of Schwinger model T. Banks, L. Susskind and J. Kogut, PRD13, 4 (1973)

$$H = x\sum_{n=0}^{N-2}\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right] + \frac{\mu}{2}\sum_{n=0}^{N-1}\left[1+(-1)^n\sigma_n^z\right] + \sum_{n=0}^{N-2}\left[l + \frac{1}{2}\sum_{k=0}^{n}\left((-1)^k + \sigma_k^z\right)\right]^2$$

hopping term mass term gauge part

Gauss law

$$= H_{\rm hop} + H_{\rm mass} + H_{\rm g}$$

Tensor network (TN)

- Efficient approximation of quantum many-body state from quantum information
- **Matrix product state (MPS)**: tensor network for 1d

$$|\psi\rangle \approx \sum_{i_1, i_2, \ldots i_N} \mathrm{Tr}\left[M^{i_1} M^{i_2} \ldots M^{i_N}\right] |i_1 i_2 \ldots i_{N-1}\rangle$$

i_k: physical indices at site k, $M^{i_k}_{mn}$: tensor,

m, n (=1,…, D) : indices from this approximation, D **: bond dimension**

Ex. 1/2-spin 2 particle system

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big) = \sum_{i_1, i_2 = \uparrow, \downarrow} \mathrm{Tr}\left[M^{i_1} M^{i_2}\right] |i_1 i_2\rangle$$

$$M^{i_1=\uparrow} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}, \quad M^{i_1=\downarrow} = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad M^{i_2=\uparrow} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M^{i_2=\downarrow} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

One choice
Note: This example is exact description, not approximation because of large value of bond dimension.

Advantages of TN and Variational method

Bond dimension D

- ✳ dominant in computational effort
- ✳ way to express sub-space of Hilbert space
- ✳ smaller than $D \sim d^{N/2}$ can be enough
  Ex.) In our studies, $D \sim 100$ is enough ($\ll d^{N/2} \sim 10^{15}$)
- ✳ Hilbert space growing exponentially as increasing system size,
  ⇔ With TN, one can investigate sub-space growing **polynomially**

Ex.) 1d spin system

size of the whole Hilbert space		By using MPS with D,
d^N	⇔	NdD^2

d : d.o.f of physical index at each site, N : chain length

⇒ If $D \sim d^{N/2}$, no advantage of TN

F. Verstraete et al. PRL 93, 227204

Variational method

- ✳ For computing ground state, some excited states
- ✳ Updating each element of one tensor with keeping the others fixed by searching for the minimum of $\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$ with linear eq. of the derivative, then sweeping through the chain.

Lattice gauge theory (LGT) with TN approach

T. Byrnes, et al. PRD.66.013002 (2002)

- Earlier Study: critical behavior of Schwinger model with Density Matrix Renormalization Group
- Nowadays: various branches
  - ✳ Strong coupling exp. K. Cichy, et al. Comput.Phys.Commun. 184 1666 (2013)
  - ✳ TN rep. of LGT with continuous group L. Tagliacozzo, et al. arXiv:1405.4811
  - ✳ LGT with TN on higher dimension
  - ✳ Real time evolution B. Buyens, et al. arXiv:1312.6654
  - ✳ (as Hamiltonian) Quantum link model D. Banerjee, et al. PRL 110 125303 (2013) D. Banerjee, et al. PRL 109 175302 (2012)
  - ✳ (as different app.) Tensor Renormalization Group Y. Shimizu, Y. Kuramashi arXiv:1403.0642 (With Lagrangian)
  - ✳ Our studies M. C. Banuls et al JHEP 1311, 158, LAT2013, 332 (2013)

This study

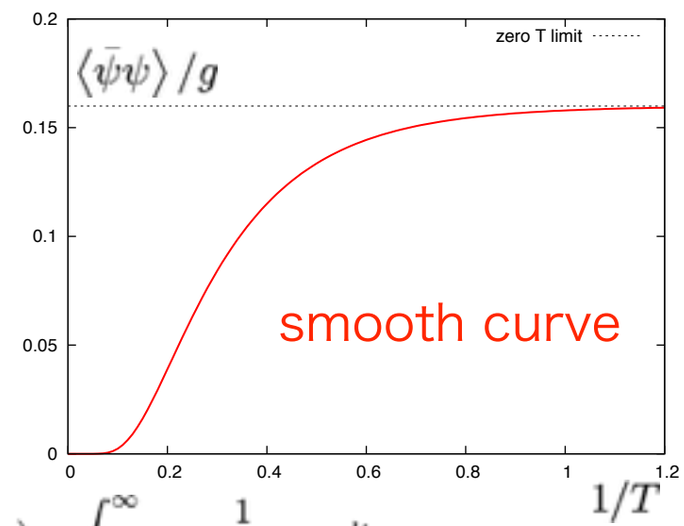
Chiral symmetry restoration of Schwinger model for $N_f = 1$

- Chiral symmetry breaking at $T = 0$ (anomaly)
  ⇔ At finite T, the symmetry restoration
- Order parameter : chiral condensate $\langle \bar{\psi}\psi \rangle$
- Analytic formula [I. Sachs and A. Wipf, arXiv:1005.1822]
  with fermionic zero mode & instanton for gauge

$$\langle \bar{\psi}\psi \rangle = \frac{m_\gamma}{2\pi} e^{\gamma} e^{2I(\beta m_\gamma)} = \begin{cases} \frac{m_\gamma}{2\pi} e^{\gamma} & \text{for} \quad T \to 0 \\ 2T e^{-\pi T/m_\gamma} & \text{for} \quad T \to \infty \end{cases}$$

where $I(x) = \int_0^\infty \frac{1}{1 - e^{x\cosh(t)}} dt$

$\gamma = 0.57721\ldots$ (Euler constant), $m_\gamma = g/\sqrt{\pi}$



- Operator of chiral condensate: $\frac{\sqrt{x}}{N} \sum_n (-1)^n \left[\frac{1 + \sigma_n^z}{2} \right]$ in spin language
- Expectation value at finite T : $\frac{\langle \bar{\psi}\psi \rangle_\beta}{g} = \frac{\mathrm{tr}\left[\bar{\psi}\psi\, \rho(\beta)\right]}{\mathrm{tr}\left[\rho(\beta)\right]}$ thermal density operator: $\rho(\beta) \equiv e^{-\beta H}$ where $\beta = 1/T$

technical details

- ✳ $\rho(\beta/2)$ to ensure positivity : $\rho(\beta) = \rho(\beta/2)\, \rho(\beta/2)^\dagger$ F. Verstraete *et al* PRL 93, 20 (2004)
- ✳ T-dep. by evolution of T with : $\rho(\beta/2) = \underbrace{e^{-\frac{\delta}{2}H} \cdots e^{-\frac{\delta}{2}H}}_{N_{\mathrm{step}} = \beta/\delta}$ Ex.) For fixed δ, larger N_{step} corresponds to lower T
- ✳ For each step, performing variational method with MPS approx. to $e^{-\frac{\delta}{2}H}$

Simulation setup

- Open Boundary condition
- Four simulation parameters
  1. From T evol., step size δ — $\delta \to 0$
  2. From MPS approx., bond dimension D — enough large D
  3. chain length N
  4. inverse coupling x

For cont. limit extrapolation $N \to \infty$, $1/\sqrt{x} \to 0$

- Two setups

(i) Small N for check dep. on δ, D

x	N
16	20

D = 20-160,
δ = 0.0001-0.01

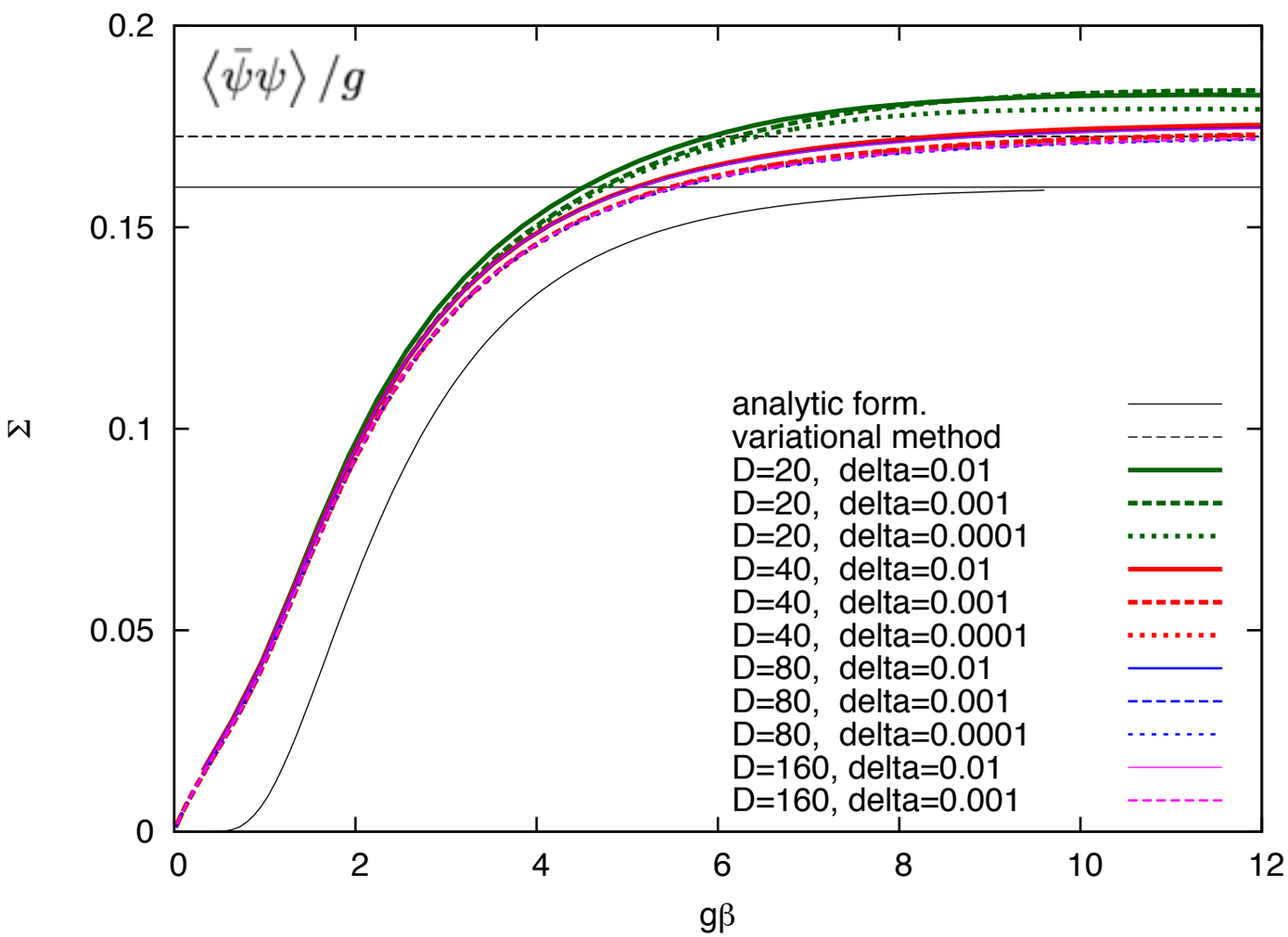
(ii) For continuum limit: several N, x

moderate values
D = 80,
δ = 0.00001
-0.00005

x	N
25	80-140
36	80-200
49	80-240
64	80-240
100	140-240
121	100-240

Chiral condensate at finite T with small system

- Result of (i) test case : small N
- dependence of bond dimension/step size



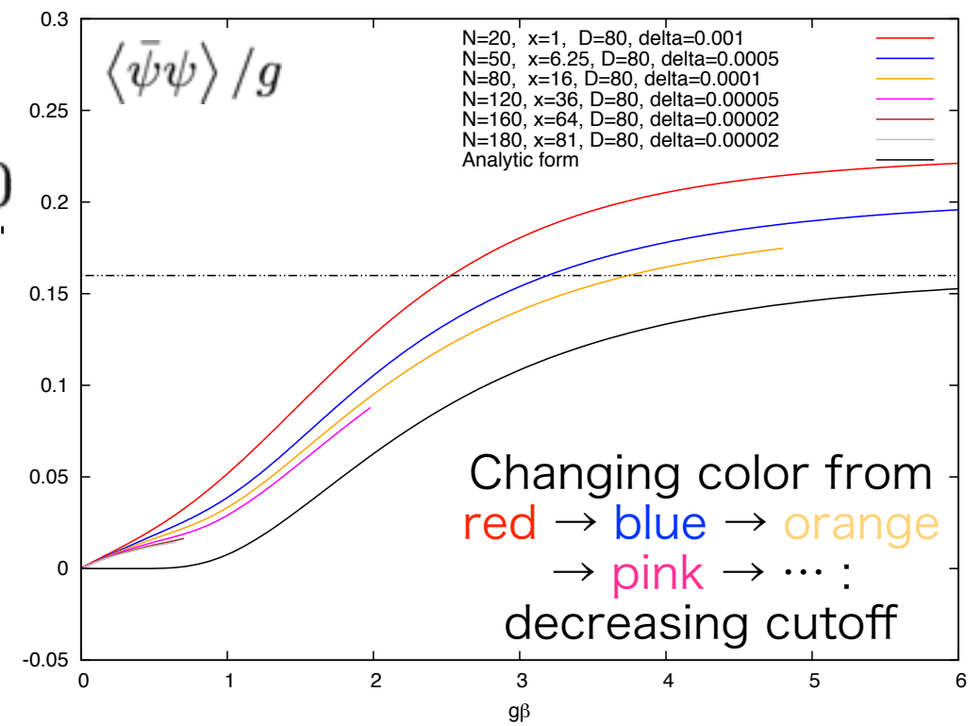
- ✳ different color ⇔ different D
  different line types
  ⇔ different δ
- ✳ smooth curve and similar to the analytic curve
- ✳ some differences between various colors and line types at low T
- ✳ less dependence at high T

Continuum extrapolation

Before extrapolations

- Naive estimate
  - ✳ Data of $N = 20\text{-}180$ fixed $N/\sqrt{x} = 20$

    $N/\sqrt{x} \gtrsim 20$ needed to see linear behavior of infinite volume extrapolation, from our results at $T = 0$
  - ✳ When taking cont. limit, result getting closer to analytic curve



As preliminary result, cont. limit at high T

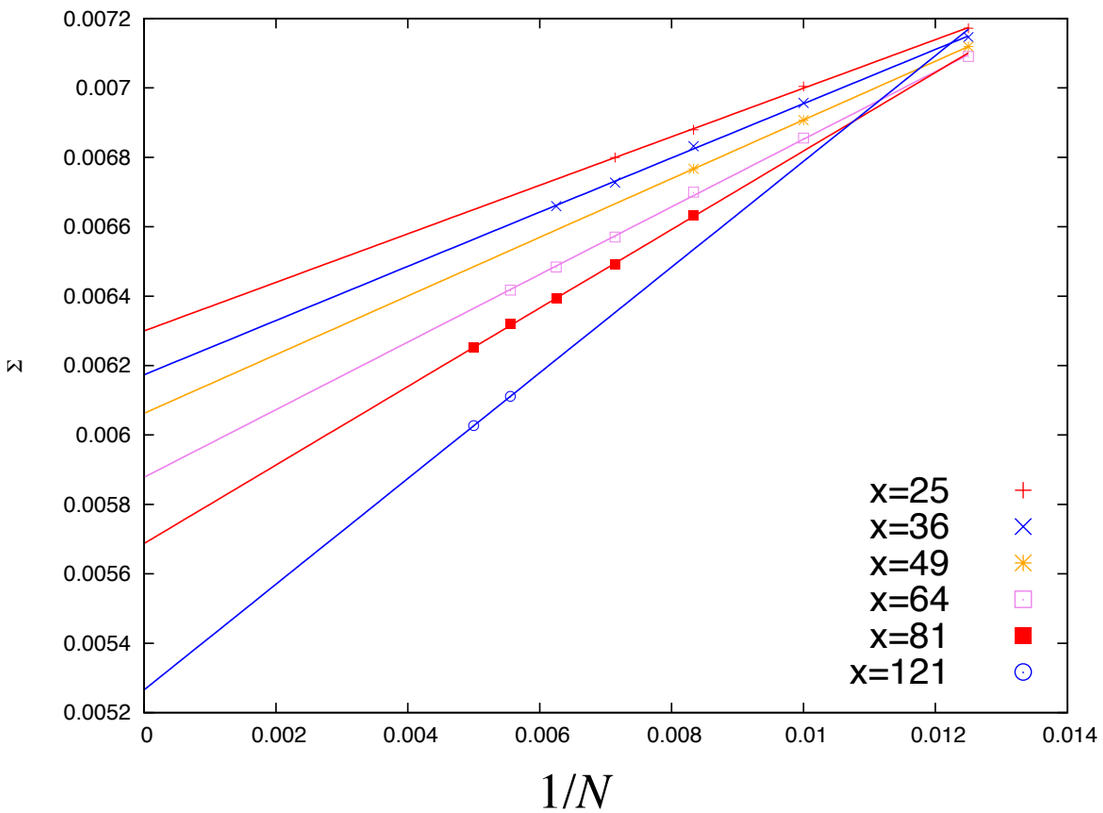
- Four extrapolations

  Less dependence on D, δ at high T, let me support
  - ✳ ~~zero step size δ limit~~ → sufficiently small δ
  - ✳ ~~large bond dimension D limit~~ → sufficiently large D
  - ✳ infinite volume limit
  - ✳ continuum limit

Continuum Limit - High T

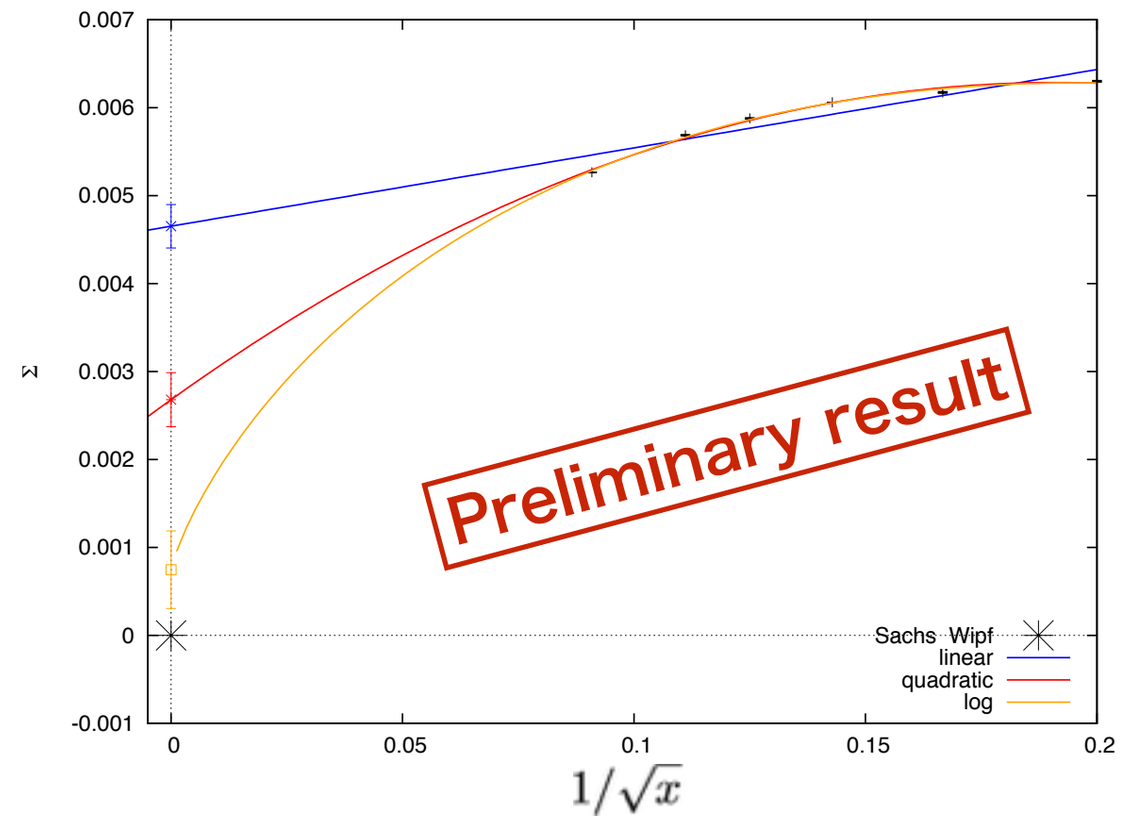
$g\beta = 0.2$ (dimensionless inverse of temperature)

(i) Infinite volume extrapolation



- Linear behavior
- cond. in infinite vol. limit for each x

(ii) Continuum limit extrapolation



- Three fit functions: linear, quadratic, logarithmic $ay + by\log(y) + c$ where $y = 1/\sqrt{x}$
- Logarithmic corr. from analytic calc. of free
- Consistent to the analytic curve

Summary

- Computing chiral condensate at finite T in Hamiltonian formalism with tensor network methods
- Evaluating dependence of bond dimension/step size
- As a preliminary result, by taking continuum limit at high temperature, we obtain results consistent with the analytic formula. I. Sachs and A. Wipf, arXiv:1005.1822
- Future plans

i) Continuum limit in low T region

ii) Many flavor Schwinger model

iii) Schwinger model at finite μ

iv) Non-Abelian gauge theory

v) Real time evolution

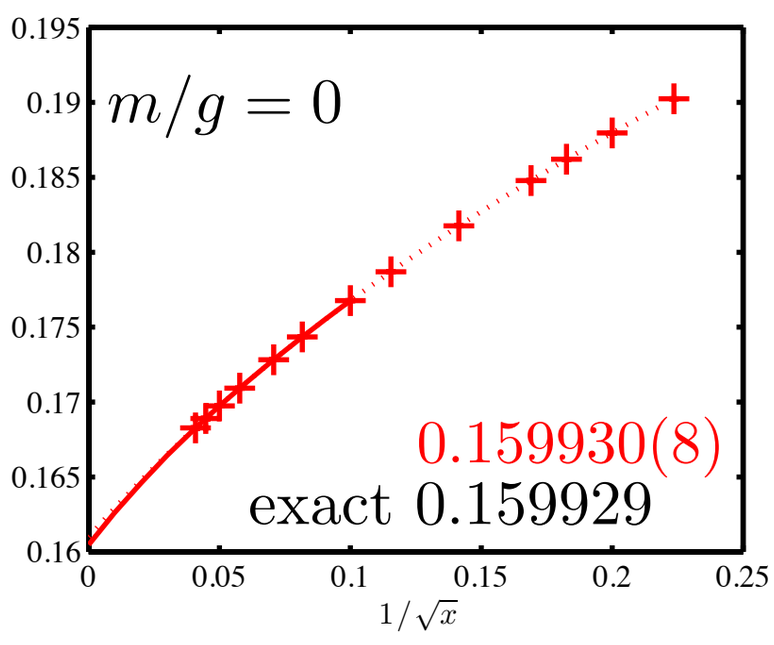
vi) Higher dimension of TN

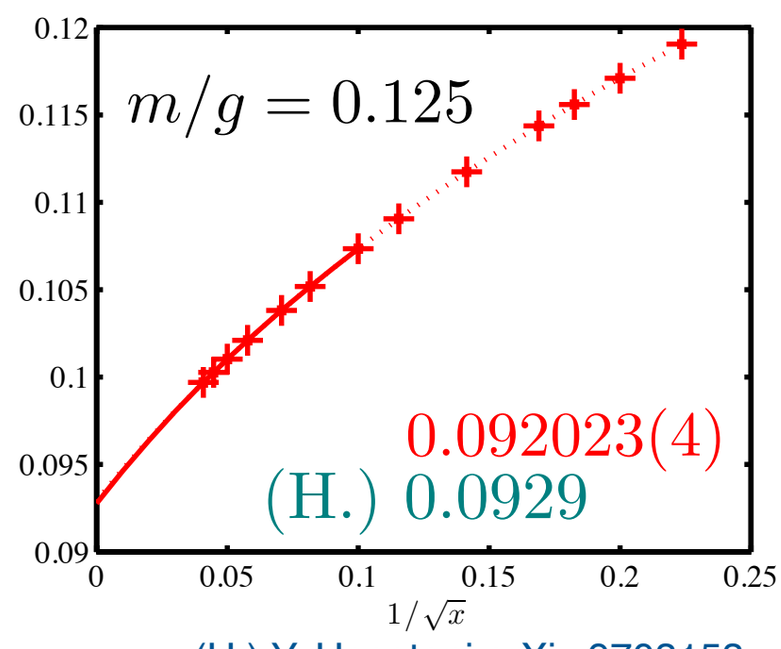
Backup slides

Our previous study

M. C. Banuls et al JHEP 1311, 158, LAT2013, 332

- Schwinger model with MPS method
- With variational method, computing:
  - spectrum
  - (subtracted) chiral condensate: $\bar{\psi}\psi = \frac{\sqrt{x}}{L}\sum_n (-1)^n \left[\frac{1+\sigma_n^z}{2}\right]$ in spin language
- Continuum limit: $1/\sqrt{x} \to 0$
  with inverse coupling $x = 1/g^2a^2$





Fit function:

$$f(x) = A + F\frac{\log(x)}{\sqrt{x}} + B\frac{1}{\sqrt{x}} + C\frac{1}{x}$$

Logarithmic correction from analytic form of free theory

(H.) Y. Hosotani arXiv:9703153

Thermal state calculation in detail

F. Verstraete *et al* PRL 93, 20 (2004)

- Expectation value at finite T: $\langle \mathcal{O} \rangle_\beta = \dfrac{\mathrm{tr}\left[\mathcal{O}\ \rho(\beta)\right]}{\mathrm{tr}\left[\rho(\beta)\right]}$
- How to calculate the $\rho(\beta)$

thermal density operator $\rho(\beta) \equiv e^{-\beta H}$
where $\beta = 1/T$

technical details

- ✳ $\rho(\beta/2)$ to ensure positivity: $\rho(\beta) = \rho\left(\beta/2\right)\rho\left(\beta/2\right)^\dagger$
- ✳ Evolution of temperature: $\rho(\beta/2) = \underbrace{e^{-\frac{\delta}{2}H} \cdots e^{-\frac{\delta}{2}H}}_{N=\beta/\delta}$

high $T \rightarrow$ low T

Ex.) For fixed δ, larger N corresponds to lower T

- ✳ Our thermal density operator

$$e^{-\frac{\delta}{2}H} \approx e^{-\frac{\delta}{4}H_g} \underbrace{e^{-\frac{\delta}{2}(H_{\mathrm{hop}}+H_{\mathrm{mass}})}}_{\approx e^{-\frac{\delta}{4}H_e} e^{-\frac{\delta}{2}H_o} e^{-\frac{\delta}{4}H_e}} e^{-\frac{\delta}{4}H_g}$$

H_g : including long range int.

additional Trotter exp. for fermionic terms into even site and odd site (2nd order Trotter exp.)

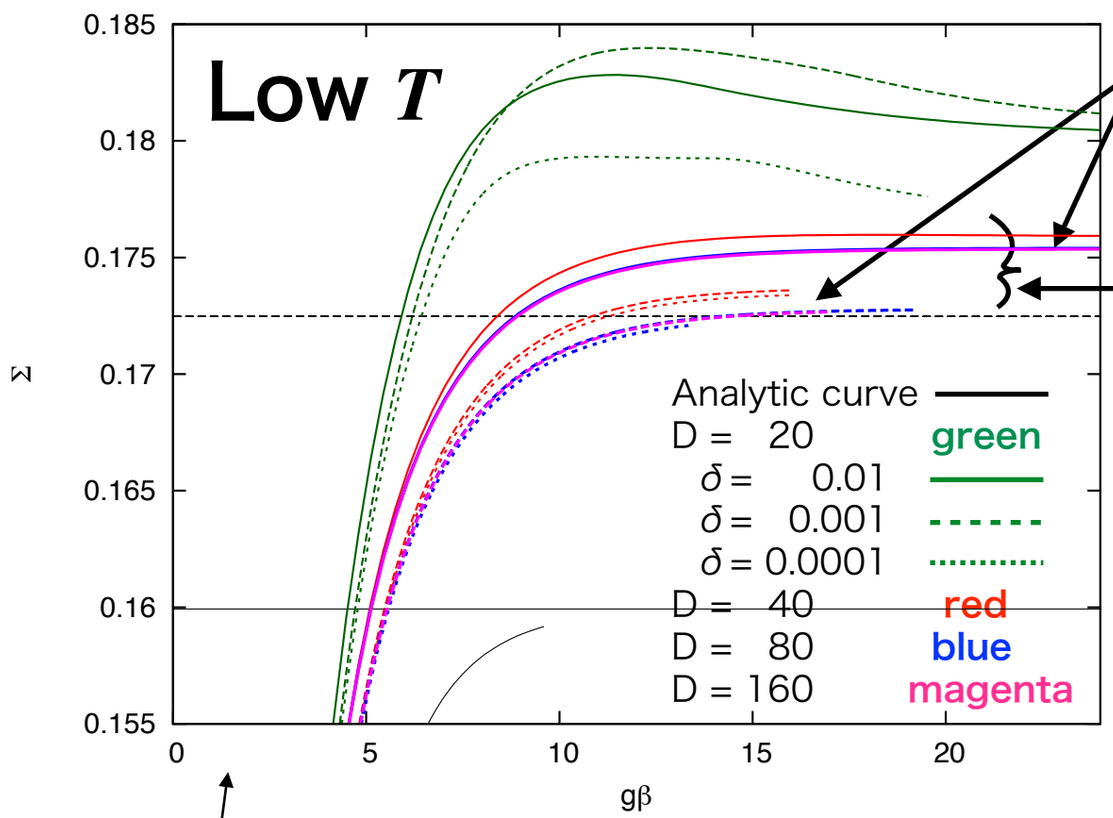
multiplication of five $e^{-\delta H/4}$'s for each step of $\rho(\beta/2)$

- ✳ In each one of the five, updating tensor by searching the minimum of $\mathrm{Tr}\left[\Pi_i\ e^{-\delta H_{\alpha(i)}/4} \Pi_j \left(e^{-\delta H_{\alpha(j)}/4}\right)^\dagger\right]$ and sweeping (variational method)

$H_{\alpha(i)} = H_g$ for i=5l, 5l+4, H_e for i=5l+1, 5l+3, H_o for i=5l+2

Chiral condensate at finite T with small system (fine)

- Focusing on high/low temperature region



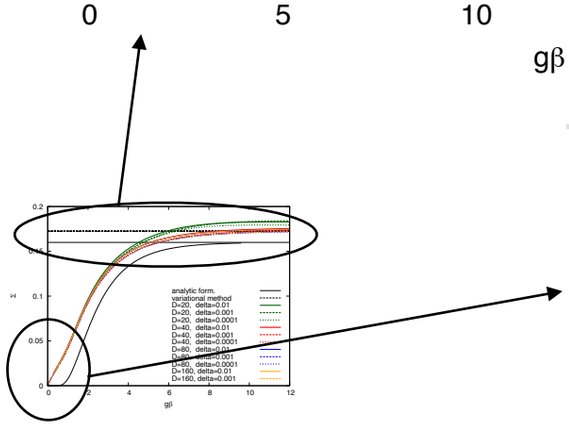
overlapping blue and magenta lines

from comparing three red lines, decreasing deviation as δ decreasing

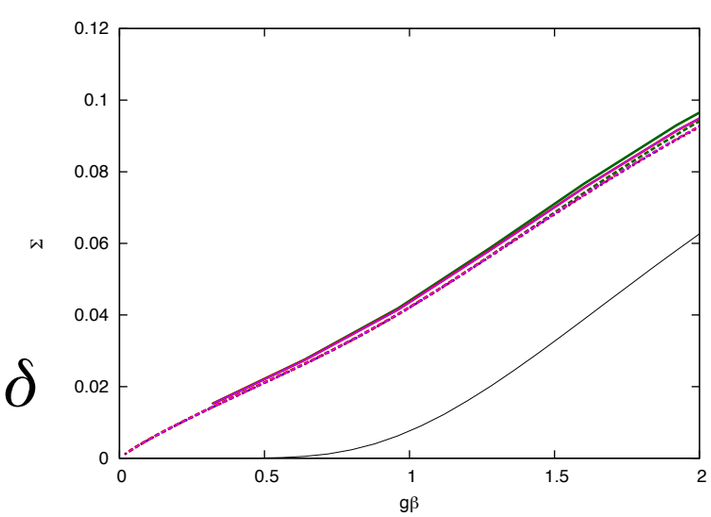
(i) convergence of D for $D \gtrsim 80$

(ii) approaching to convergence of δ for $\delta \lesssim 0.001$

(iii) less dependence on δ for larger D



High T

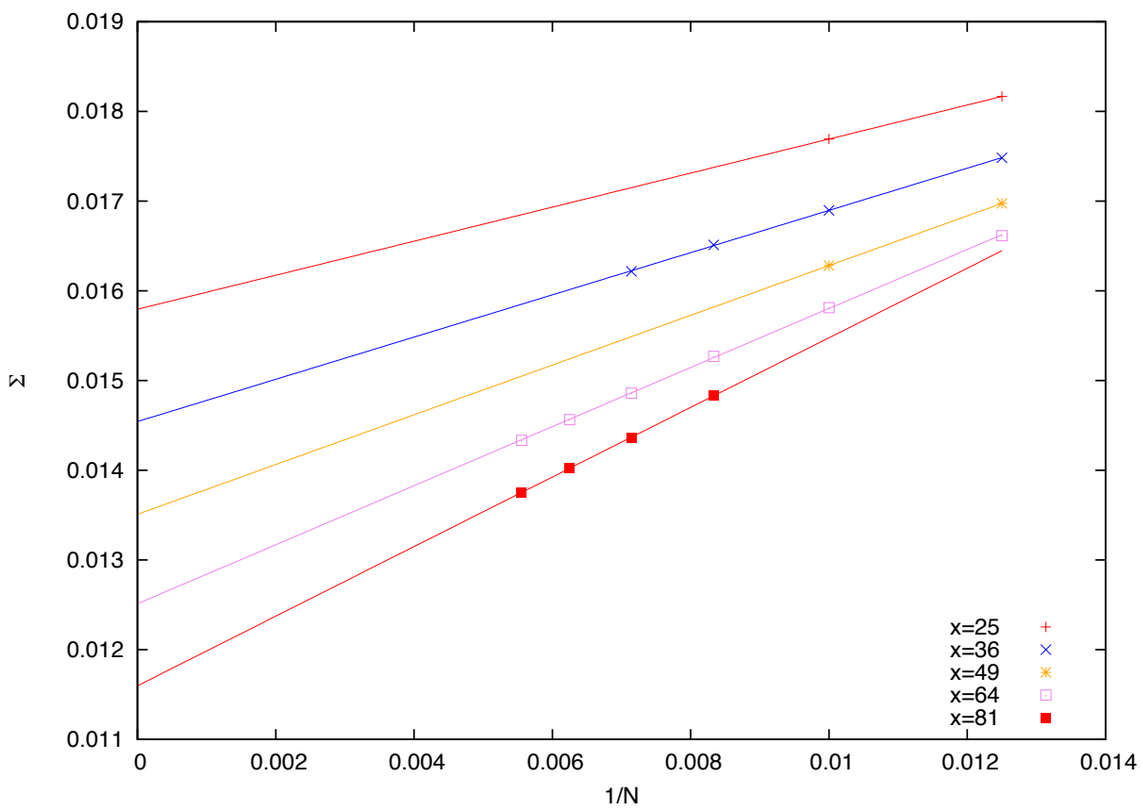


Little dependence on D, δ

Continuum Limit - High T (2)

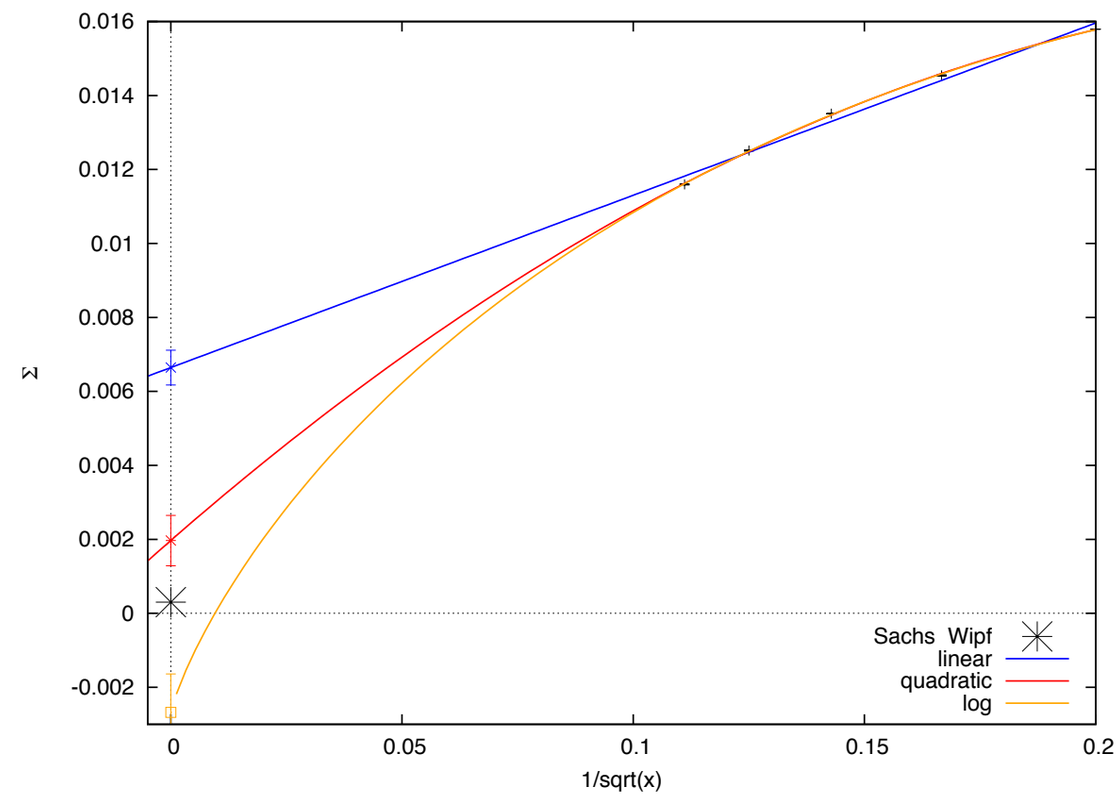
$g\beta = 0.6$

(i) Infinite volume limit



Linear behavior

(ii) Continuum limit



Three fit functions:
linear, quadratic, logarithmic
$ay + by\log(y) + c$ where $y = 1/\sqrt{x}$