Lattice simulations of G_2-QCD at finite density II

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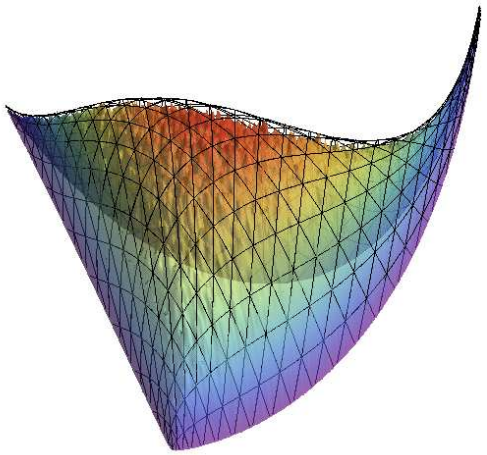


JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

- G_2 is the smallest simple and simply connected Lie group with trivial center
- Rank 2 $\Rightarrow$ two fundamental representations

$$(7) \quad \text{Quarks} \quad \text{and} \quad (14) \quad \text{Gluons}$$

- First order deconfinement transition (pure Gauge Theory) and many more similarities to $SU(3)$ Gauge Theory and QCD
- No fermion sign problem at finite density
  - $\Rightarrow$ We can investigate the full phase diagram of a Gauge Theory with fundamental fermions and fermionic baryons
  - $\Rightarrow$ We can use G_2-QCD to test methods for QCD at finite density
- Compared to QCD additional bound states like diquarks, hybrids ... present

K. Holland, P. Minkowski, M. Pepe and U. J. Wiese, *Exceptional confinement in G_2 gauge theory*, Nucl. Phys. **B668** (2003) 207

G. Cossu, M. Massimo D'Elia, A. Di Giacomo, B. Lucini, C. Pica, *G_2 gauge theory at finite temperature*, JHEP 0710 (2007)

L. Lptak and S. Olejnik, *Casimir scaling in G_2 lattice gauge theory*, Phys.Rev. D78 (2008)

A. Maas, L. von Smekal, B. H. Wellegehausen and A. Wipf, *The phase diagram of a gauge theory with fermionic baryons*, Phys.Rev. D86 (2012).

A. Maas, L. von Smekal, B. H. Wellegehausen and A. Wipf, *Hadron masses and baryonic scales in G_2-QCD at finite density*, Phys.Rev. D89 (2014).

1. Transitions at finite density
2. Diquark sources and Majorana fermions
3. Results

Transitions at finite density

Heavy Ensemble

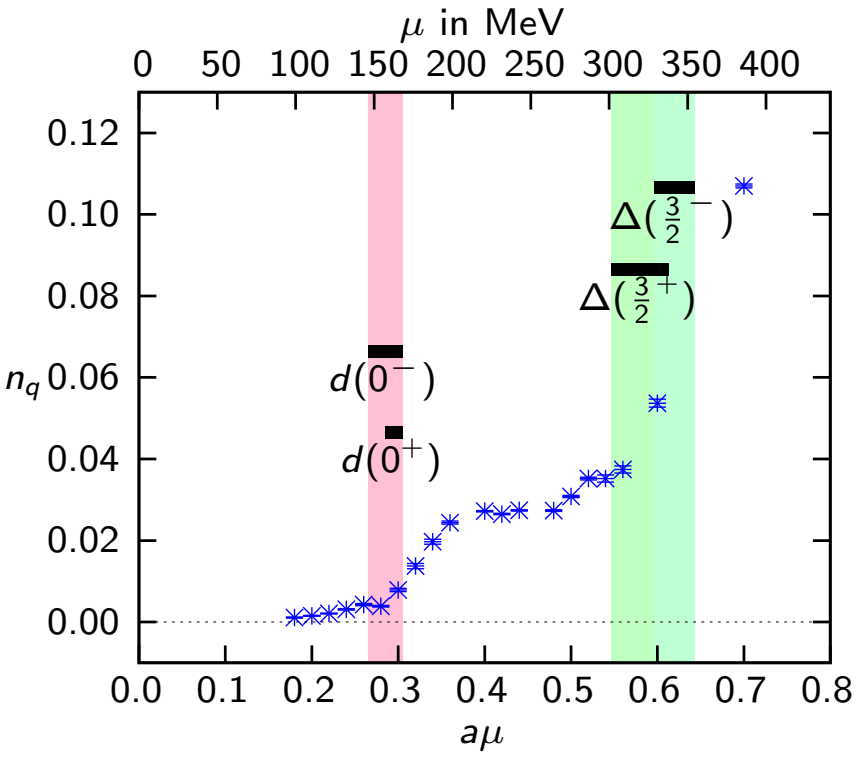
$\beta = 1.05$, $\kappa = 0.147$
Proton mass $m_N = 938\,\mathrm{MeV}$

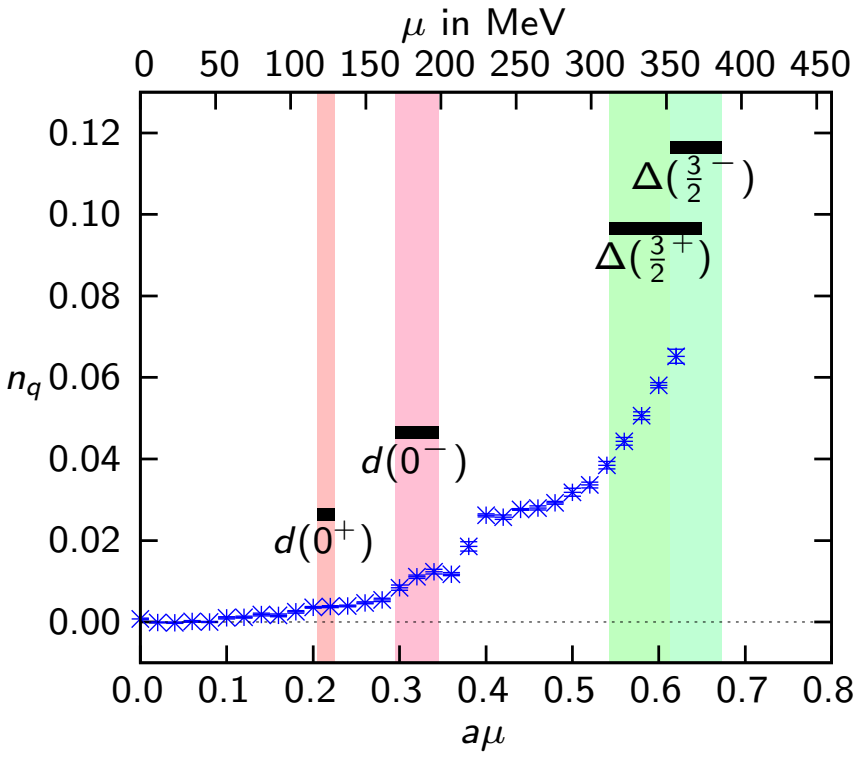
Diquark mass $m_{d(0^+)} = 326\,\mathrm{MeV}$
Lattice spacing
$a = 0.357\,\mathrm{fm} \sim (552\mathrm{MeV})^{-1}$

Light Ensemble

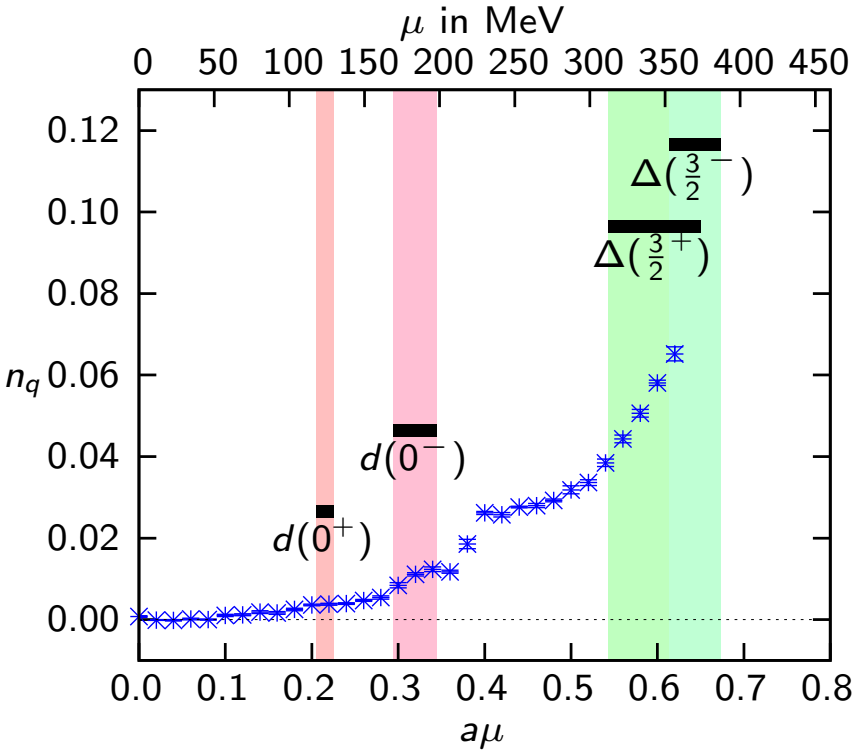
$\beta = 0.96$, $\kappa = 0.159$
Proton mass $m_N = 938\,\mathrm{MeV}$

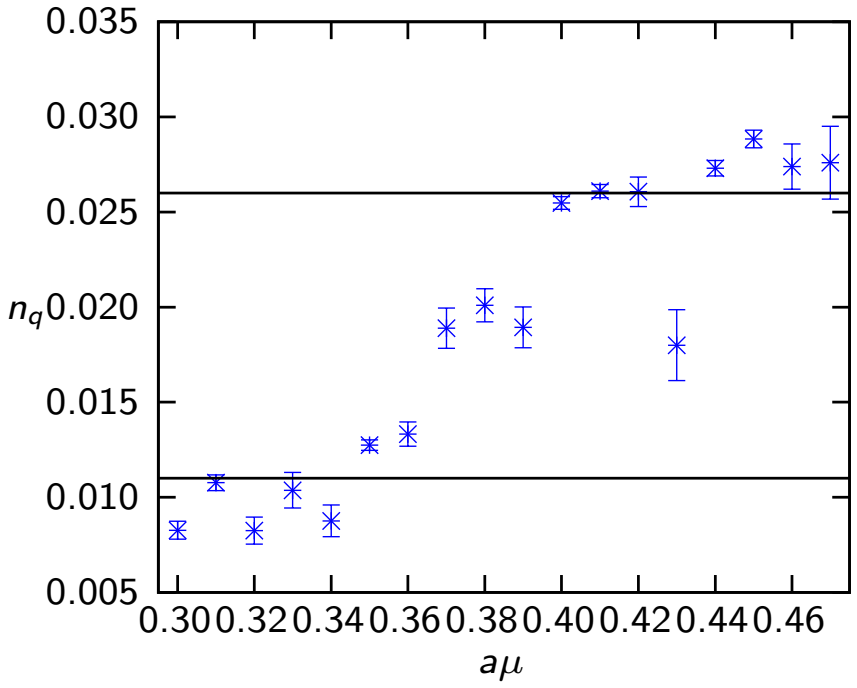
Diquark mass $m_{d(0^+)} = 247\,\mathrm{MeV}$
Lattice spacing
$a = 0.343\,\mathrm{fm} \sim (575\mathrm{MeV})^{-1}$





Quark number density $n_q = \frac{1}{V}\frac{\partial \ln Z}{\partial \mu}$





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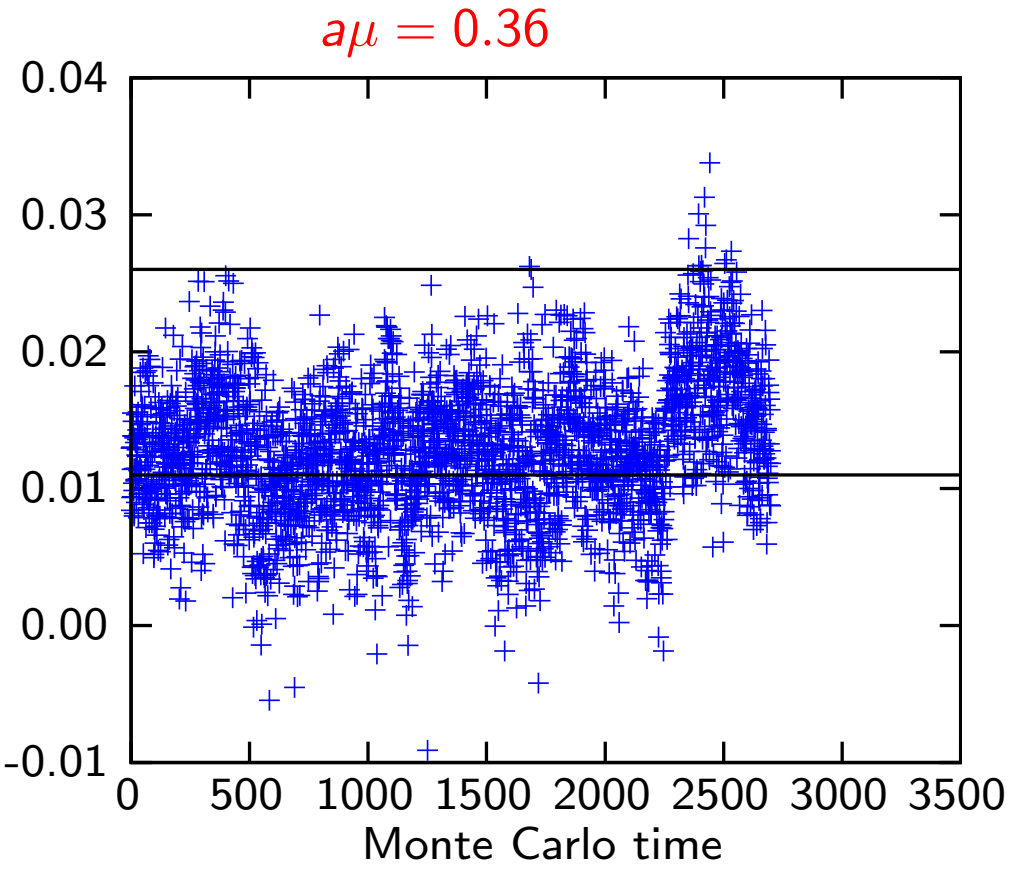
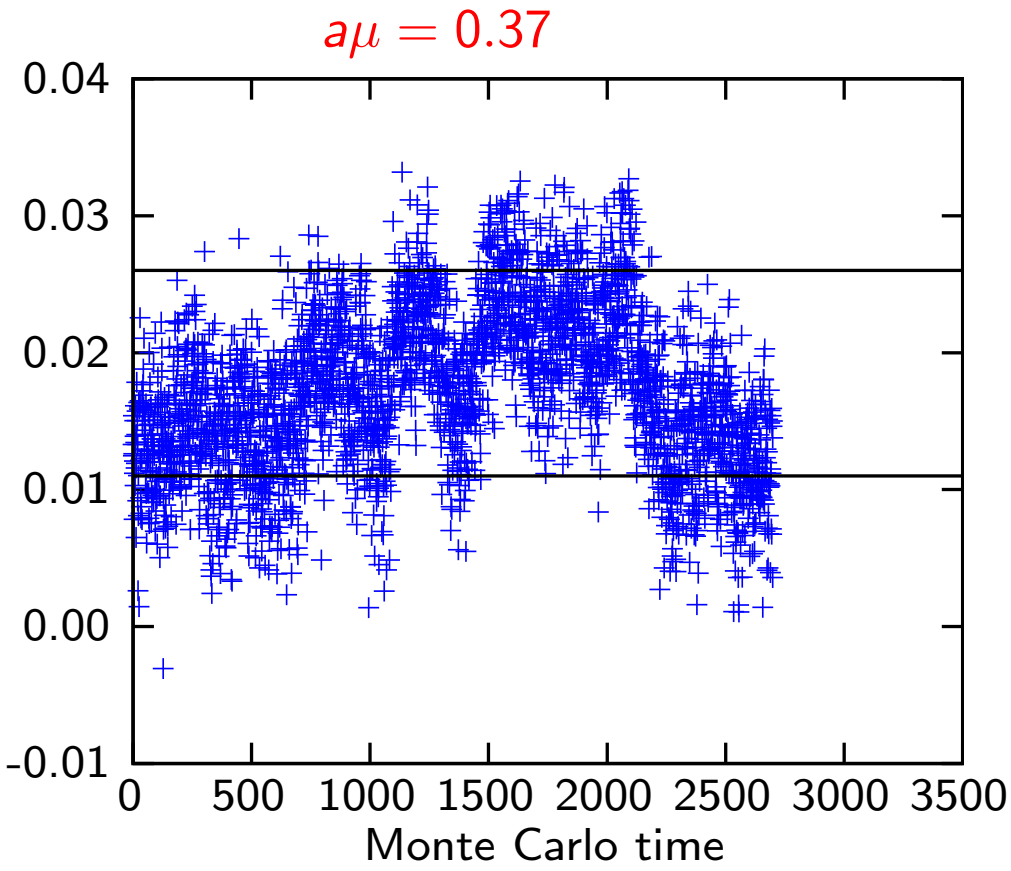
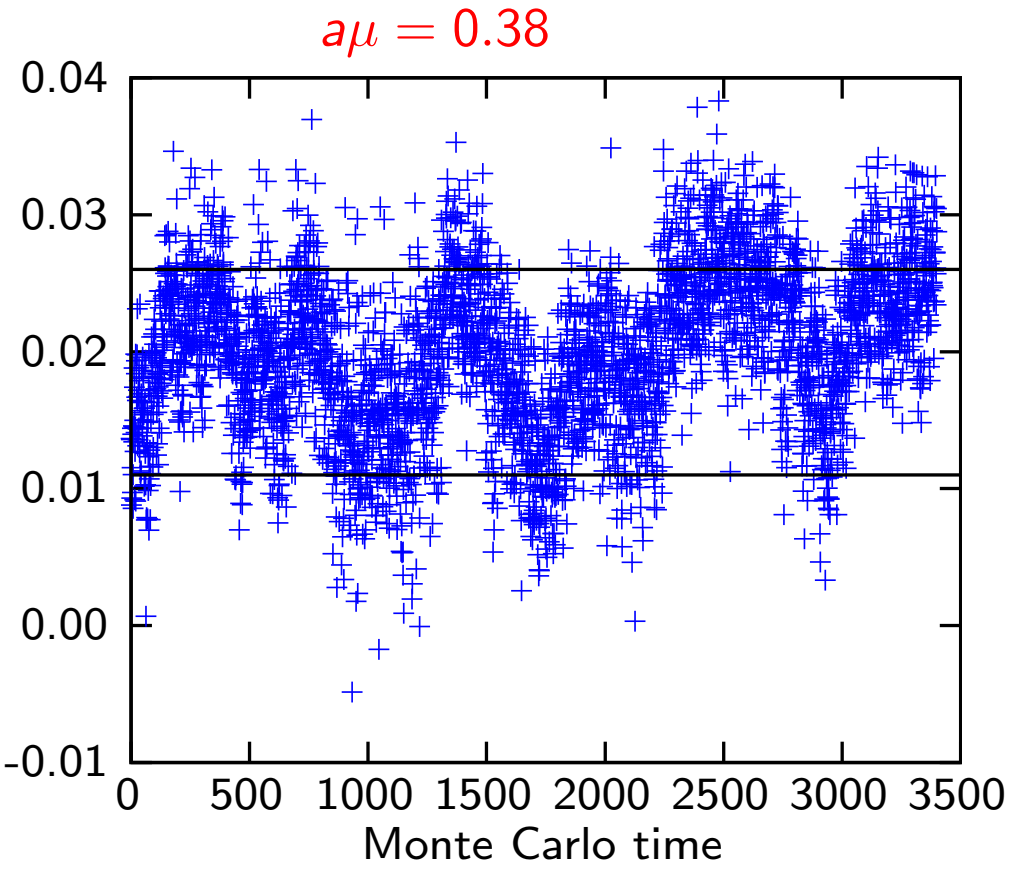


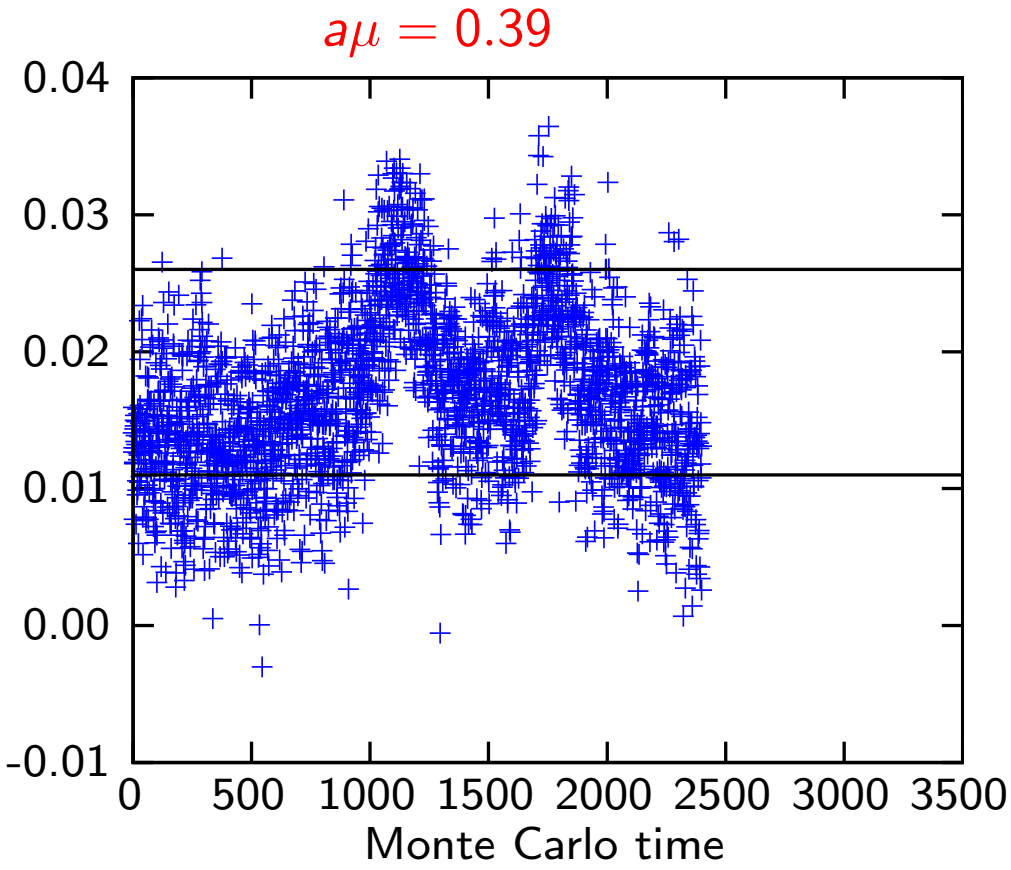
Quark number density $n_q = \frac{1}{V}\frac{\partial \ln Z}{\partial \mu}$



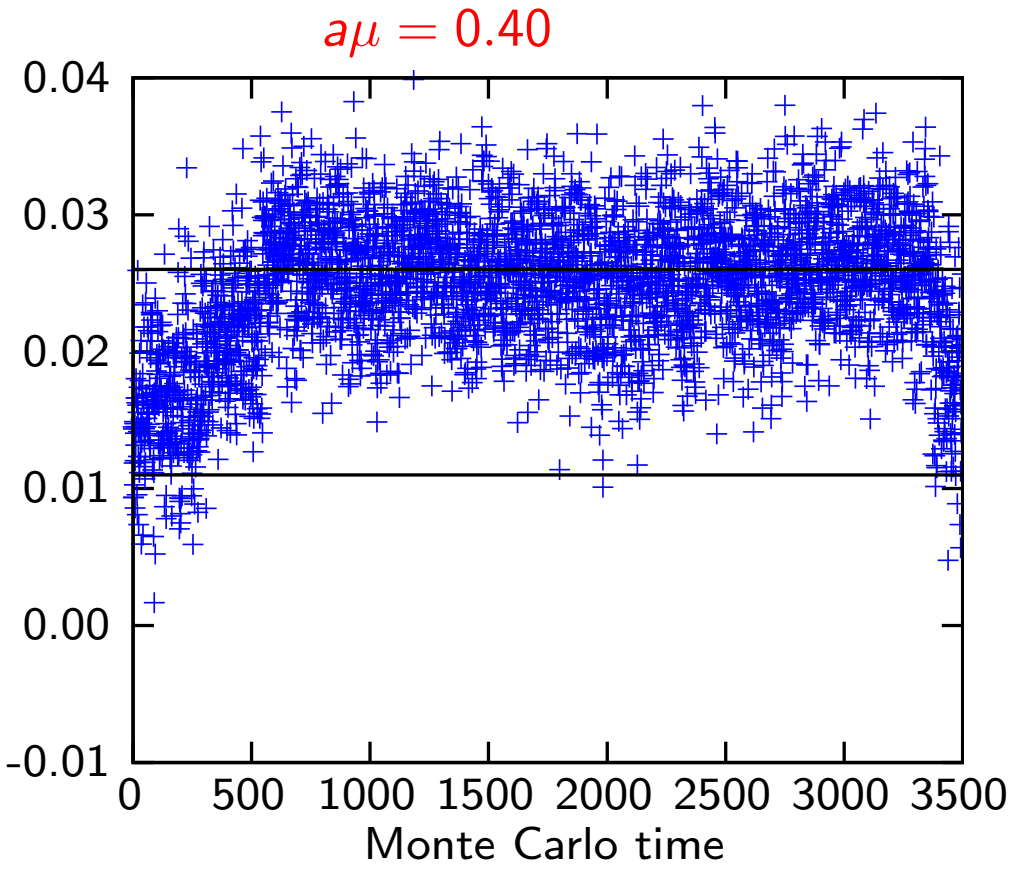
Quark number density $n_q = \frac{1}{V}\frac{\partial \ln Z}{\partial \mu}$



Quark number density $n_q = \frac{1}{V}\frac{\partial \ln Z}{\partial \mu}$



Quark number density $n_q = \frac{1}{V}\frac{\partial \ln Z}{\partial \mu}$



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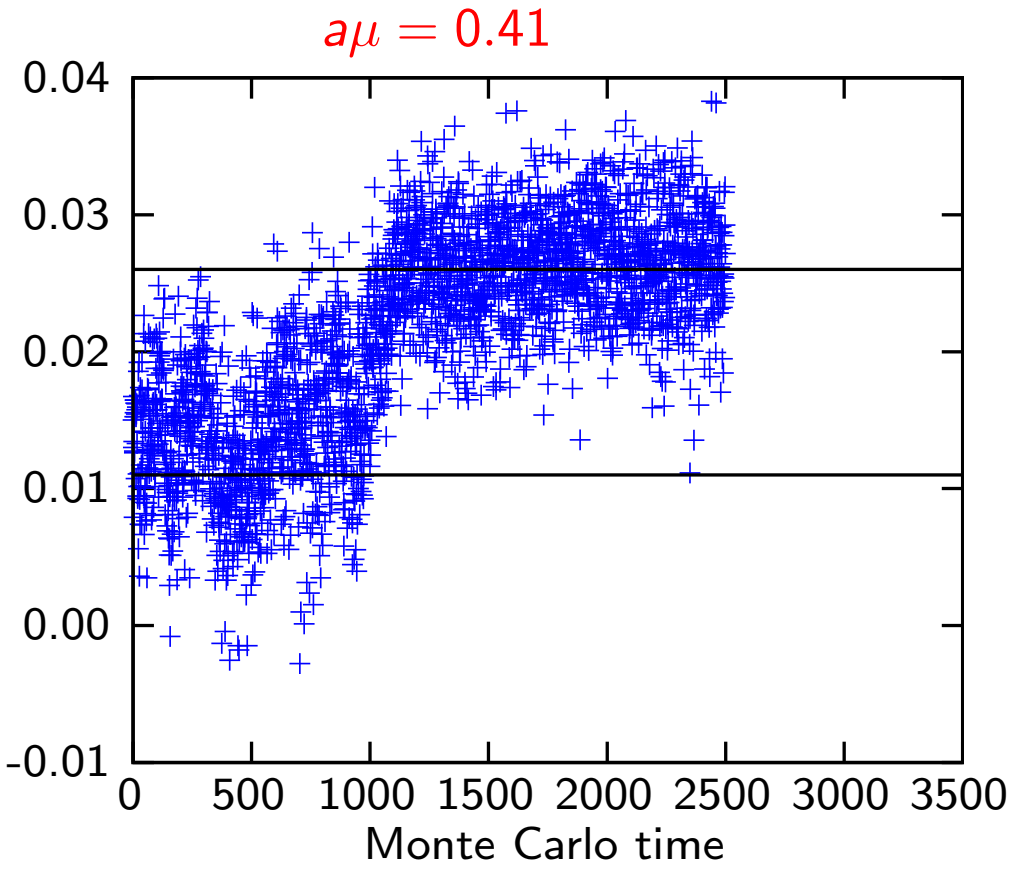

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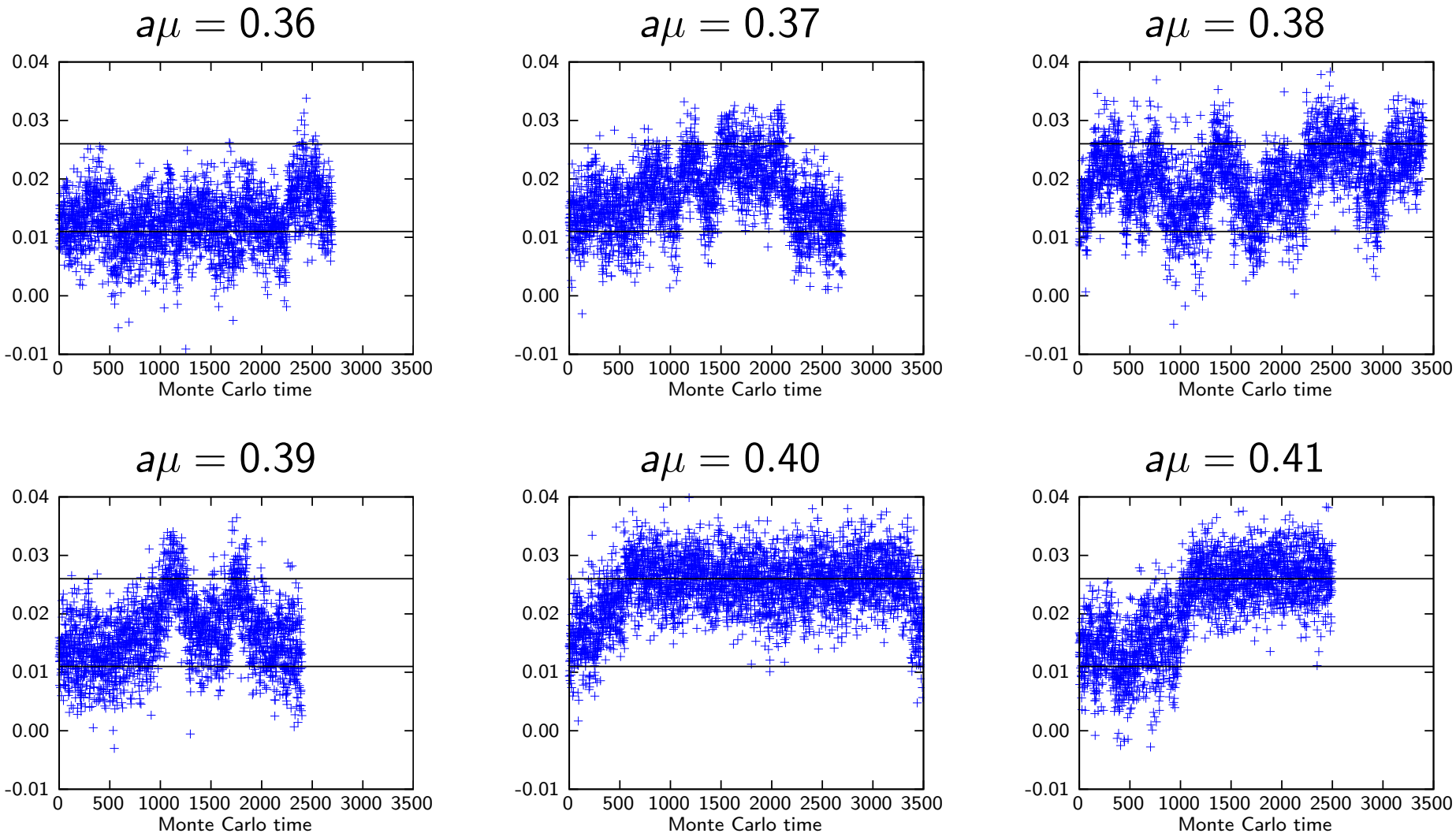
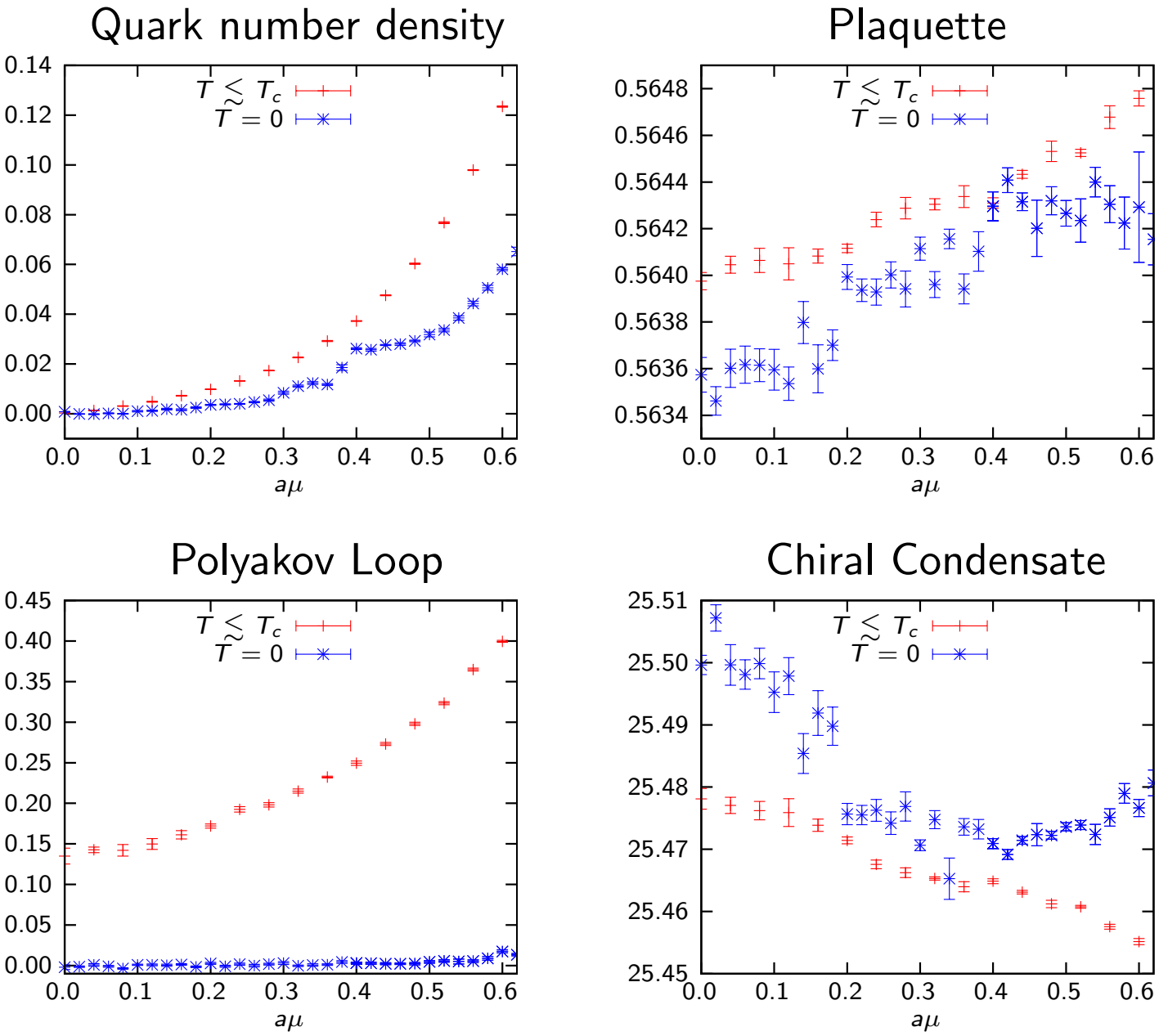



Quark number density
Plaquette
Polyakov Loop
Chiral Condensate
$T \lesssim T_c$
$T = 0$
$a\mu$

Diquark sources and Majorana fermions

$\mathcal{L} = \bar{\psi} D_0(m, m_5, \mu)\psi + \mathcal{L}_{\mathbb{1}}(J) + \mathcal{L}_{\gamma_5}(\tilde{J})$ for $N_f = 1$ Dirac spinor ψ with

$$D_0(m, m_5, \mu) = \not{D} - m - m_5\gamma_5 - \mu\gamma_0$$

$$\Rightarrow T = C\gamma_5 \quad \Rightarrow T\, D_0(m, m_5, \mu) = D_0^*(m, m_5, \mu)\, T \quad \Rightarrow \det D_0(m, m_5, \mu) \geq 0$$

Possible diquark sources

$$\mathcal{L}_{\mathbb{1}}(J) = \frac{1}{2}\left(J\bar{\psi}^C\psi + J^*\bar{\psi}\psi^C\right) \qquad \text{negative parity source}$$

$$\mathcal{L}_{\gamma_5}(\tilde{J}) = \frac{1}{2}\left(\tilde{J}\bar{\psi}^C\gamma_5\psi - \tilde{J}^*\bar{\psi}\gamma_5\psi^C\right) \qquad \text{positive parity source}$$

2 Majorana fermions $\lambda = (\chi, \eta)$ with $\lambda^C = \lambda$ and $\psi = \chi + \mathrm{i}\,\eta$

$$\mathcal{L} = \bar{\lambda} D(m, m_5, \mu, J, \tilde{J})\lambda$$

$$D(m, m_5, \mu, J, \tilde{J}) = D_0(m, m_5, \mu) + \left(\mathrm{i}\tilde{J}_1\gamma_5 - J_2\right)\sigma_1 + \left(J_1 + \mathrm{i}\tilde{J}_2\gamma_5\right)\sigma_3$$

$$D_0(m, m_5, \mu) = \left(\not{D} - m - m_5\gamma_5\right) - \mu\gamma_0\sigma_2$$

Chiral symmetry without diquark sources for $N_f = 1$

$$SU(2) \to U(1)$$

Goldstone bosons:

$$d(0^{++}) \sim \bar{\psi}^{\mathrm{C}}\gamma_5\psi + \bar{\psi}\gamma_5\psi^{\mathrm{C}} \quad \text{and} \quad d(0^{+-}) \sim \bar{\psi}^{\mathrm{C}}\gamma_5\psi - \bar{\psi}\gamma_5\psi^{\mathrm{C}}$$

Massive state:

$$f(0^{++}) \sim \bar{\psi}\psi$$

$$\begin{tikzcd}
SU(2)_{\mathrm{L=R^*}} \arrow[r, "m"] \arrow[d, "\mu"'] & U(1)_{\mathrm{B}} \arrow[d, "\mu"] \\
U(1)_{\mathrm{B}} \arrow[r, "m"] & U(1)_{\mathrm{B}}
\end{tikzcd}$$

$SU(2)$ generators for the chiral transformation: $T_{\rm V} = \sigma_2$ and $T_{\rm A} = \gamma_5\{\sigma_1, \sigma_3\}$

$$O_{\rm A,1/3}\,\lambda = e^{{\rm i}\,\alpha\,T_{\rm A,1/3}}\,\lambda \quad \text{and} \quad \bar\lambda \to \bar\lambda\, e^{{\rm i}\,\alpha\,T_{\rm A,1/3}}$$
$$O_{\rm V,2}\,\lambda = e^{{\rm i}\,\alpha\,T_{\rm V}}\,\lambda \quad \text{and} \quad \bar\lambda \to \bar\lambda\, e^{-{\rm i}\,\alpha\,T_{\rm V}}$$

All Possible bilinear bound states for a single Dirac flavour

$$d(0^{+-}) = \bar\lambda\gamma_5\sigma_1\lambda = \bar\chi\gamma_5\eta$$
$$d(0^{++}) = \bar\lambda\gamma_5\sigma_3\lambda = \bar\chi\gamma_5\chi - \bar\eta\gamma_5\eta$$
$$d(0^{--}) = \bar\lambda\sigma_1\lambda = \bar\chi\eta$$
$$d(0^{-+}) = \bar\lambda\sigma_3\lambda = \bar\chi\chi - \bar\eta\eta$$
$$f(0^{++}) = \bar\lambda\lambda = \bar\chi\chi + \bar\eta\eta$$
$$\eta(0^{-+}) = \bar\lambda\gamma_5\lambda = \bar\chi\gamma_5\chi + \bar\eta\gamma_5\eta$$

Under $SU(2)$ transformations they decompose as $2 \otimes 2 = 1 \oplus 3$

positive parity

$$3 \sim \begin{pmatrix} f(0^{++}) \\ d(0^{++}) \\ d(0^{+-}) \end{pmatrix}$$

negative parity

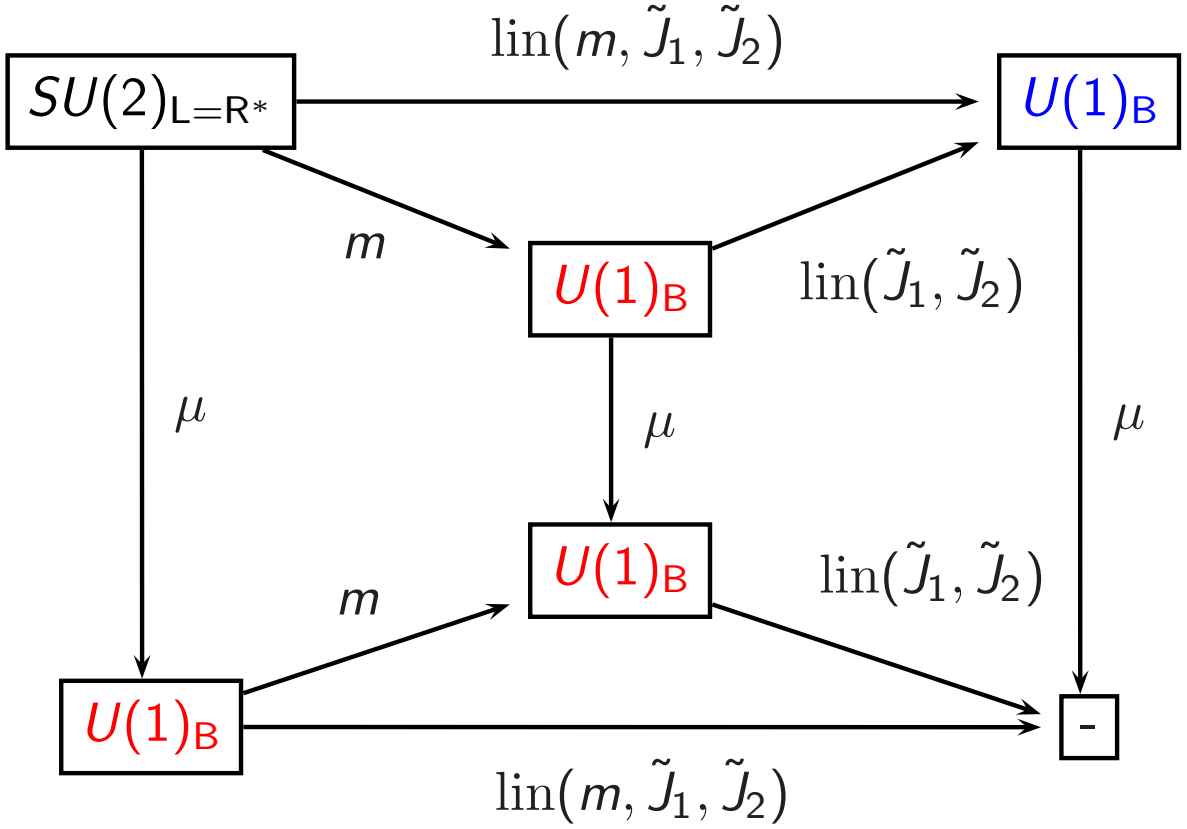
$$3 \sim \begin{pmatrix} \eta(0^{-+}) \\ d(0^{-+}) \\ d(0^{--}) \end{pmatrix}$$

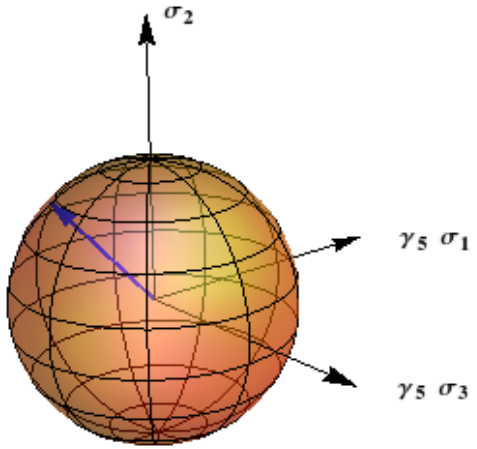
Operator	Parameter	$O_{A,1}$	$O_{A,3}$	$O_{V,2}$	Goldstone bosons	Massive state
$\bar{\lambda}\lambda$	m	✗	✗	✓	$d(0^{++})$, $d(0^{+-})$	$f(0^{++})$
$\bar{\lambda}\gamma_5\sigma_1\lambda$	$\tilde{J}_1$	✗	✓	✗	$d(0^{++})$, $f(0^{++})$	$d(0^{+-})$
$\bar{\lambda}\gamma_5\sigma_3\lambda$	$\tilde{J}_2$	✓	✗	✗	$d(0^{+-})$, $f(0^{++})$	$d(0^{++})$
$\bar{\lambda}\gamma_5\lambda$	m_5	✗	✗	✓	$d(0^{-+})$, $d(0^{--})$	$\eta(0^{-+})$
$\bar{\lambda}\sigma_1\lambda$	J_2	✗	✓	✗	$d(0^{-+})$, $\eta(0^{-+})$	$d(0^{--})$
$\bar{\lambda}\sigma_3\lambda$	J_1	✓	✗	✗	$d(0^{--})$, $\eta(0^{-+})$	$d(0^{-+})$
$\bar{\lambda}\gamma_0\sigma_2\lambda$	μ	✗	✗	✓	-	-

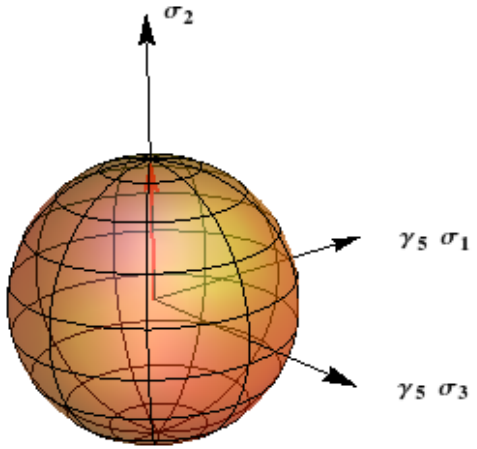
✓ means conserved, ✗ not conserved

μ	m_5	$J_{1,2}$	m	$\tilde{J}_{1,2}$	Symmetry	Generator
✓	✓	✓	✓	✓	$SU(2)$	$T = \{\gamma_5\sigma_1, \gamma_5\sigma_3, \sigma_2\}$
✓	✓	✓	×	✓	$U(1)$	$T = \sigma_2$
✓	✓	✓	✓	×	$U(1)$	$T = \gamma_5(\tilde{J}_2\sigma_1 - \tilde{J}_1\sigma_3)$
✓	✓	✓	×	×	$U(1)$	$T = \gamma_5(\tilde{J}_2\sigma_1 - \tilde{J}_1\,\sigma_3) + m\sigma_2$
×	✓	✓	✓	✓	$U(1)$	$T = \sigma_2$
×	✓	✓	×	✓	$U(1)$	$T = \sigma_2$
×	✓	✓	✓	×	-	-
×	✓	✓	×	×	-	-
✓	✓	×	✓	✓	$U(1)$	$T = \gamma_5(J_1\sigma_1 + J_2\sigma_3)$
✓	✓	×	×	✓	-	-
✓	✓	×	✓	×	$U(1)$ or -	$T = \gamma_5(A\sigma_1 + B\sigma_3)$
✓	✓	×	×	×	-	-
×	✓	×	✓	✓	-	-
×	✓	×	×	✓	-	-
×	✓	×	✓	×	-	-
×	✓	×	×	×	-	-

✓: no explicit or spontaneous breaking for this operator


$SU(2)_{L=R^*}$
$U(1)_B$
$U(1)_B$
$U(1)_B$
$U(1)_B$
-
$\mathrm{lin}(m, \tilde{J}_1, \tilde{J}_2)$
m
$\mathrm{lin}(\tilde{J}_1, \tilde{J}_2)$
μ
μ
μ
m
$\mathrm{lin}(\tilde{J}_1, \tilde{J}_2)$
$\mathrm{lin}(m, \tilde{J}_1, \tilde{J}_2)$


σ_2
$\gamma_5\ \sigma_1$
$\gamma_5\ \sigma_3$


σ_2
$\gamma_5\ \sigma_1$
$\gamma_5\ \sigma_3$

Monte-Carlo simulations with Majorana fermions

$$Z_f = \int \mathcal{D}\lambda e^{-\frac{1}{2}\bar{\lambda} D \lambda} = \mathrm{Pf}(CD) = \mathrm{sgn}(\mathrm{Pf})\sqrt{\det D}$$

- Set of unitary operators ($T_1 = T\sigma_1$, $T_2 = T\sigma_3$, $T_3 = -\sigma_2$) with $T_1 T_1^* = T_2 T_2^* = -\mathbb{1}$ and $T_3 T_3^* = \mathbb{1}$ and

$$T_1 D_0 = D_0^* T_1 , \quad T_2 D_0 = D_0^* T_2 , \quad T_3 D_0 = D_0 T_3$$

$$D_0 \psi = \lambda \psi \quad \Rightarrow \quad D_0(\psi, \chi, \eta, \xi) = (\lambda\psi, \lambda\chi, \lambda^*\eta, \lambda^*\xi)$$

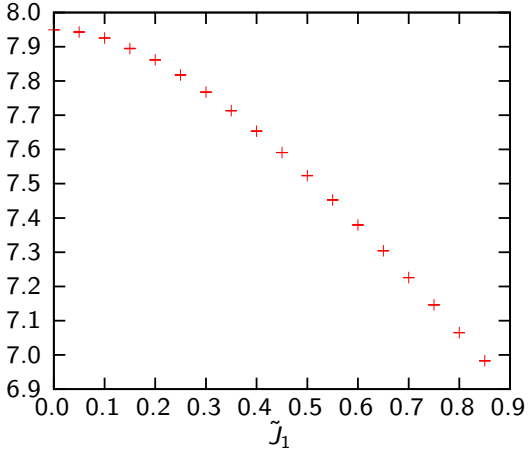
  with linearly independent eigenvectors (ψ, χ, η, ξ).
- eigenvalues together with its complex conjugate $\Rightarrow$ $\mathrm{Pf}(CD_0) \in \mathbb{R}$
- every eigenvalue is two-fold degenerate $\Rightarrow$ $\mathrm{Pf}(CD_0) \geq 0$
- For $\mu = 0$ and $m \neq \tilde{J}_1 \neq \tilde{J}_2 \neq 0$ $\Rightarrow$ $\mathrm{Pf}(CD) \geq 0$
- For $\mu \neq 0$ and $m \neq \tilde{J}_1 \neq \tilde{J}_2 \neq 0$ $\Rightarrow$ $\mathrm{Pf}(CD) \in \mathbb{R}$

Results

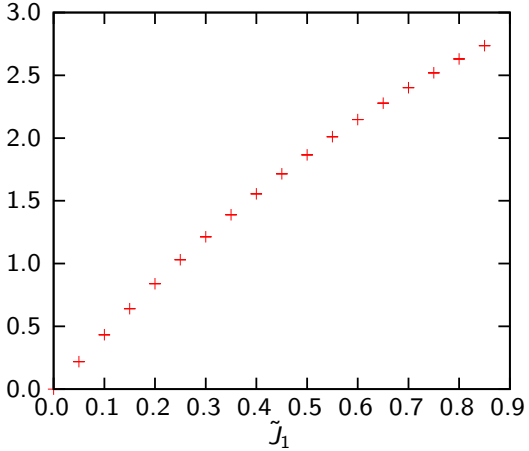
Lattice Setup

$N \times N_t = 8^3 \times 16$, $\beta = 0.96$, $\kappa = 0.151$ $\Rightarrow$ Heavy Quarks

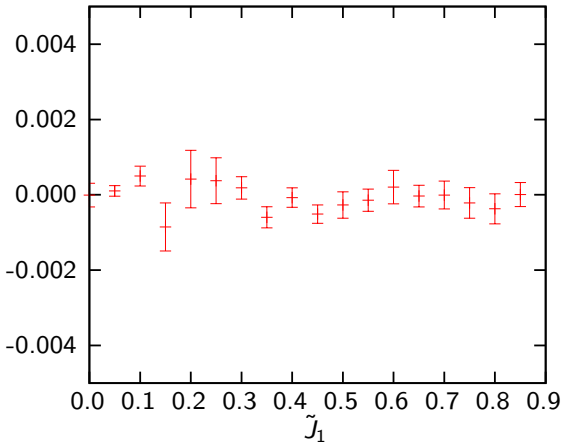
Chiral Condensate $\langle \bar{\lambda}\lambda \rangle$



Diquark Condensate $\langle \bar{\lambda}\gamma_5\sigma_1\lambda \rangle$



Quark number density $\langle \bar{\lambda}\gamma_0\sigma_2\lambda \rangle$

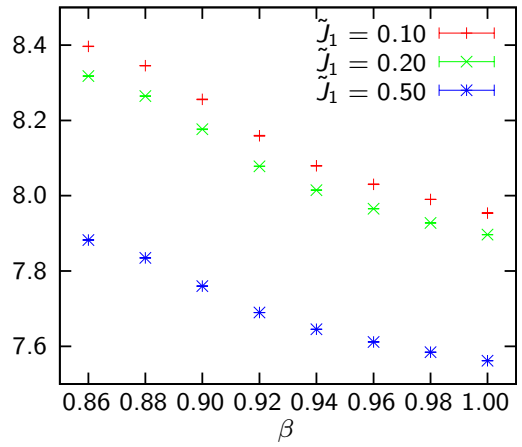


- Condensate changes from a Chiral towards a Diquark Condensate.
- Quark number density vanishes.

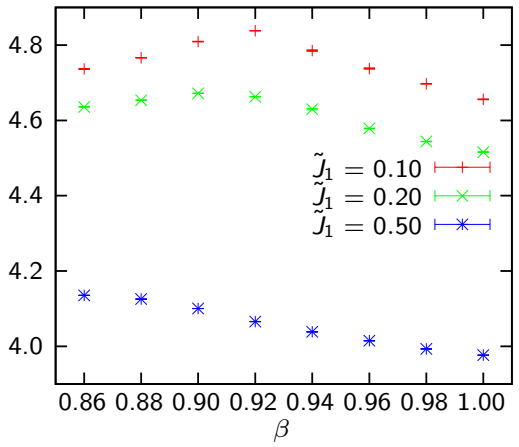
Lattice Setup

$N \times N_t = 12^3 \times 6$, $\beta = 0.96$, $\kappa = 0.156$

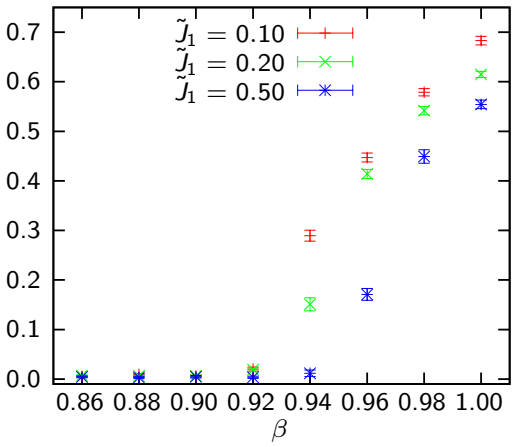
Chiral Condensate $\langle \bar{\lambda}\lambda \rangle$



Diquark Condensate $\langle \bar{\lambda}\gamma_5\sigma_1\lambda \rangle / \tilde{J}_1$



Polyakov Loop

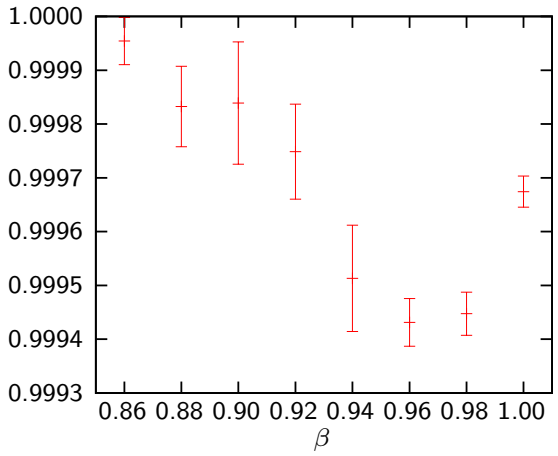


- Deconfinement transition
- Almost no Chiral transition visible in the chiral condensate
  $\Rightarrow$ Renormalized condensates.

Renormalized Condensates

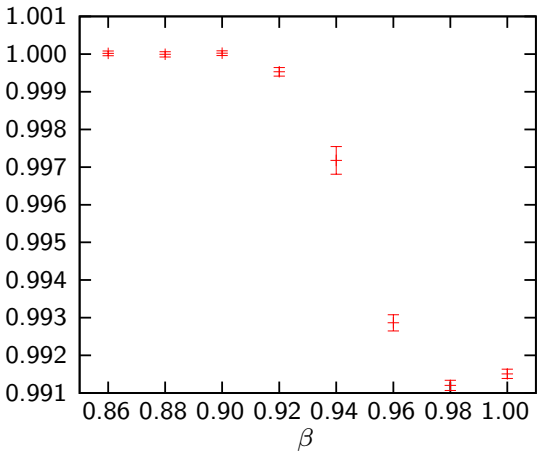
Chiral Condensate

$\langle \bar{\lambda}\lambda \rangle_T / \langle \bar{\lambda}\lambda \rangle_{T=0}$

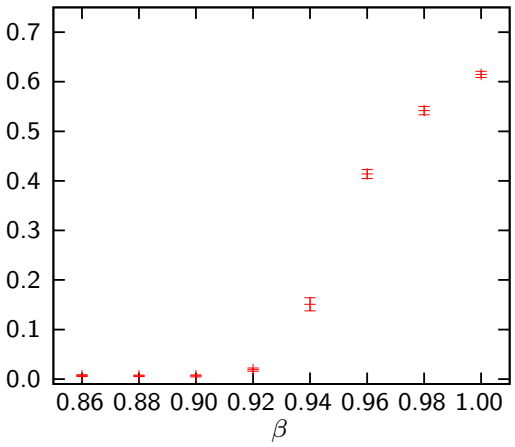


Diquark Condensate

$\langle \bar{\lambda}\gamma_5\sigma_1\lambda \rangle_T / \langle \bar{\lambda}\gamma_5\sigma_1\lambda \rangle_{T=0}$



Polyakov Loop



- Chiral transition shows up in the Chiral and in the Diquark Condensate
- Better signal for Chiral transition in the Diquark Condensate
- Deconfinement and Chiral transition temperature agree

Lattice Setup

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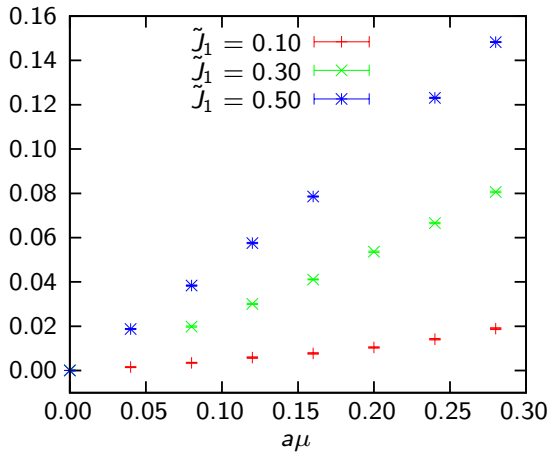
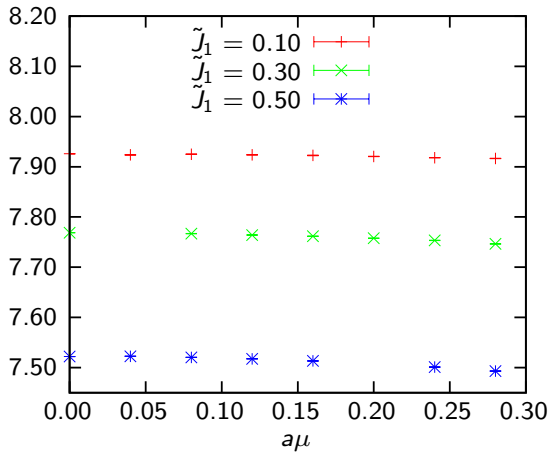
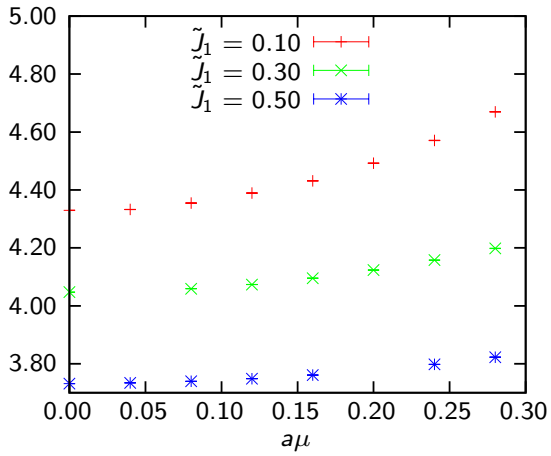
Quark Number Density



Chiral Condensate $\langle \bar{\lambda}\lambda \rangle$



Diquark Condensate $\langle \bar{\lambda}\gamma_5\sigma_1\lambda \rangle / \tilde{J}_1$



Conclusions

- Evidence for a first order nuclear matter transition
  - ⇒ Finite size scaling
  - ⇒ Nucleon mass dependence on μ
  - ⇒ Other observables, EoS (pressure)

- First simulation results with diquark sources
  - ⇒ "Improved" Chiral properties at finite lattice spacing, very similar to Twisted-Mass fermions
  - ⇒ Spectroscopy at finite diquark source
  - ⇒ Critical slowing down at nuclear matter transition due to small eigenvalues, extrapolation with finite diquark sources?