# Towards exact worldline models of lattice gauge theory at finite density

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# Lattice QCD at finite density

Consider **lattice QCD with staggered fermions** at finite density $\mu$:

$$Z(\beta) = \int [dU d\chi d\bar{\chi}]\, e^{-S_G(U) - S_F(\bar{\chi},\chi,U)} = \int [dU]\, e^{-S_G(U)} \det \not{D}(\mu, m; U)$$

$$S_G = \beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{N}\mathrm{ReTr}(U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu})\right)$$

$$S_F = \sum_{x,\mu,\alpha} \eta_{x\mu} (e^{\mu_\alpha a_\tau \delta_{\mu\tau}} \bar{\chi}^\alpha_x U_{x,\mu} \chi^\alpha_{x+\hat{\mu}} - e^{-\mu_\alpha a_\tau \delta_{\mu\tau}} \bar{\chi}^\alpha_{x+\hat{\mu}} U^\dagger_{x,\mu} \chi^\alpha_x) + \sum_{x,\alpha} a m_\alpha \bar{\chi}^\alpha_x \chi^\alpha_x$$

**Worldline approach**

- Integrate out $U$'s **before** $\chi$'s $\implies$ worldlines of color-neutral objects.
- Integration over $U$'s only possible for $\beta = 0$

# Lattice QCD at finite density

Consider **lattice QCD with staggered fermions** at finite density μ ($N_f = 1$):

$$Z(0) = \int [dU d\chi d\bar{\chi}]\, e^{-S_F(\bar{\chi},\chi,U)} = \int [d\chi d\bar{\chi}]\, e^{am\bar{\chi}\chi} \prod_{x,\mu} \int dU\, e^{\mathrm{Tr}(K^\dagger_{x,\mu} U) + \mathrm{Tr}(K_{x,\mu} U^\dagger)}$$

$$\begin{aligned} K_{x,\mu} &\equiv e^{+\mu a_\tau \delta_{\mu\tau}} \eta_{x\mu} \chi_x \bar{\chi}_{x+\hat{\mu}} \\ K^\dagger_{x,\mu} &\equiv e^{-\mu a_\tau \delta_{\mu\tau}} \eta_{x\mu} \chi_{x+\hat{\mu}} \bar{\chi}_x \end{aligned}$$

**Worldline approach**

- Integrate out $U$'s **before** $\chi$'s $\Longrightarrow$ worldlines of color-neutral objects.
- Integration over $U$'s only possible for $\beta = 0$ $\Longrightarrow$ **fermionic one-link integrals**

$$\int_{SU(N)} dU\, e^{\mathrm{Tr}(K^\dagger_{x,\mu} U) + \mathrm{Tr}(K_{x,\mu} U^\dagger)} = \sum_{k=0}^{N} \frac{(N-k)!}{N!k!} (M_x M_{x+\hat{\mu}})^k + e^{+N\mu a_\tau \delta_{\tau\mu}} \bar{B}_x B_{x+\hat{\mu}} + (-1)^N e^{-N\mu a_\tau \delta_{\tau\mu}} \bar{B}_{x+\hat{\mu}} B_x$$

# MDP models in the strong coupling limit

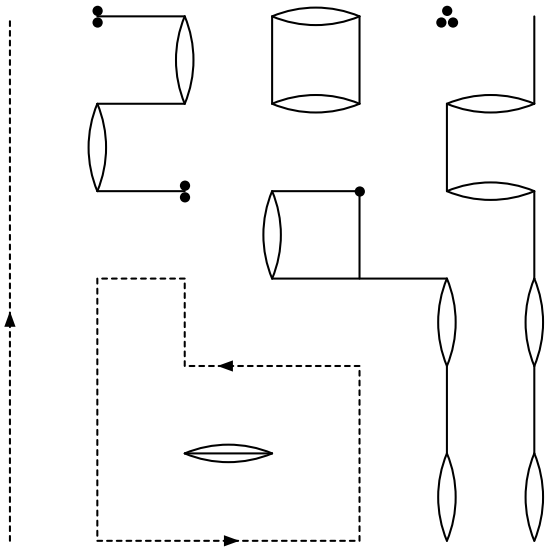
Integrating out $\bar{\chi}, \chi$ results in a **Monomer-Dimer-Polymer model (MDP)**:

[Rossi & Wolff 1984] [Karsch & Mütter 1989]

$$Z(0) = \sum_{\{n,k,\ell\}} \left( \prod_x \frac{N!}{n_x!} (am)^{n_x} \right) \left( \prod_l \frac{(N-k_l)!}{N!k_l!} \right) \left( \frac{\sigma(\ell)\, e^{r(\ell)N\mu/T}}{N!^{L(\ell)}} \right)$$

**Degrees of freedom**

- **Monomers:** $n_x$ — $x$ ●
- **Mesons:** $k_{x,\mu}$ — $x$ ——— $x+\hat{\mu}$
- **Baryons:** $b_{x,\mu}$ — $x$ - - - ▶ - - - $x+\hat{\mu}$
  $\bar{b}_{x,\mu}$ — $x$ - - - ◀ - - - $x+\hat{\mu}$



**Grassmann constraints** $N$ $\bar{\chi}, \chi$'s saturate each site:

$$\bar{\chi}: \quad n_x + \sum_\mu (k_{x,\mu} + k_{x,-\mu} + N b_{x,\mu}) = N$$

$$\chi: \quad n_x + \sum_\mu (k_{x,\mu} + k_{x,-\mu} + N \bar{b}_{x,\mu}) = N$$

$$\Longrightarrow \sum_\mu (b_{x,\mu} - \bar{b}_{x,\mu}) = 0$$ **self-avoiding baryon loops**

e.g. $SU(3), N_f = 1$

# MDP models beyond the strong coupling limit

Recent numerical simulations mapped the phase diagram of $N_f = 1$, $SU(3)$ QCD:

- at $\beta = 0$ [de Forcrand & Fromm 2010] [Unger & de Forcrand 2012]
- including $O(\beta)$ corrections [de Forcrand *et al* 2014]

... but it is hard to go further in the strong-coupling expansion.

**Challenge:** How to approach the regime of continuum physics via MDP models?

Ideally, we would like to know:

$$Z(\beta) = \int [d\chi d\bar{\chi}] \int [dU]\, e^{-S_G(U) - S_F(\bar{\chi},\chi,U)} = \int [d\chi d\bar{\chi}]\, e^{am\bar{\chi}\chi}\, F(\beta;\bar{\chi},\chi)$$

but the **four coupled link variables** around a plaquette in the gauge action prevents the exact integration of the gauge field at $\beta > 0$.

**Use auxiliary bosonic fields to decouple the link variables**

# Gaussian auxiliary fields

Let us multiply partition functions by "1" = Gaussian integral over $X \in \mathrm{Mat}(N, \mathbb{C})$:

$$Z = \int du\, e^{\beta |Y|^2} \times \underbrace{\int \gamma_1(X)}_{1}, \qquad \gamma_a(X) = \frac{a}{2\pi} dX d\overline{X}\, e^{-\frac{a}{2}|X|^2}$$

and use **Hubbard-Stratonovich (HS) transformations** to simplify the action:

$$\textbf{HS}: \quad X \longmapsto \sqrt{\beta}\,(X - Y)$$

$$\textbf{action}: \quad -\frac{1}{2}|X|^2 + \frac{\beta}{2}|Y|^2 \longmapsto -\frac{\beta}{2}|X|^2 + \beta\,\mathrm{Re}(\overline{Y}X)$$

$$\textbf{measure}: \quad \frac{1}{2\pi} dX d\overline{X} \longmapsto \frac{\beta}{2\pi} dX d\overline{X}$$

$$\textbf{partition function}: \quad Z \longmapsto \int \gamma_\beta(X)\, du\, e^{\beta \mathrm{Re}(\overline{Y}X)}$$

## 4-link action $\longmapsto$ 2-link action $\longmapsto$ 1-link action $\longmapsto$ 0-link action

Start with the usual partition function of pure lattice gauge theory:

$$Z(\beta) = \int [dU]\, e^{-\beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{N}\mathrm{ReTr}(U_{x,\mu} U_{x+\hat{\mu},\nu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu})\right)}$$

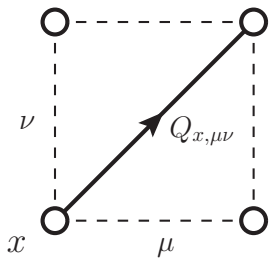
# 4-link action $\longmapsto$ 2-link action $\longmapsto$ 1-link action $\longmapsto$ 0-link action

Multiply the partition function of pure lattice gauge theory by "1": [Fabricius & Haan 1984]

$$Z(\beta) = \int [dU]\, e^{-\beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{N}\mathrm{ReTr}(U_{x,\mu} U_{x+\hat{\mu},\nu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu})\right)} \times \int \gamma_1[Q']$$

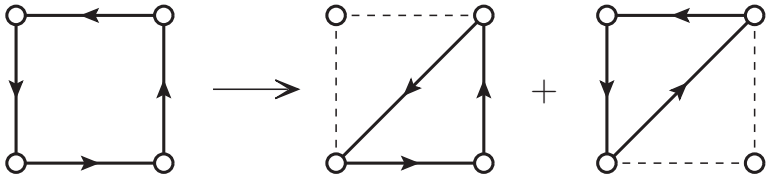
HS-transform the new auxiliary variable $Q'$:

$$Q'_{\mu\nu,x} = \sqrt{\frac{\beta}{N}} \left(Q_{\mu\nu,x} - U_{\mu,x} U_{\nu,x+\hat{\mu}} - U_{\nu,x} U_{\mu,x+\hat{\nu}}\right)$$




The partition function becomes:

$$Z(\beta) = \int \gamma_{\frac{\beta}{N}}[Q] \int [dU]\, e^{-\beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{N}\mathrm{ReTr}(Q^{\dagger}_{x,\mu\nu} U_{x,\mu} U_{x+\hat{\mu},\nu})\right)}$$



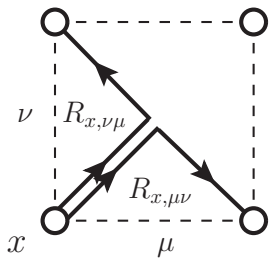
# 4-link action $\longmapsto$ 2-link action $\longmapsto$ 1-link action $\longmapsto$ 0-link action

Multiply the partition function of pure lattice gauge theory by "1" (again):

$$Z(\beta) = \int \gamma_{\frac{\beta}{N}}[Q] \int [dU]\, e^{-\beta \sum_{x,\mu<\nu}\left(1-\frac{1}{N}\mathrm{ReTr}(Q^{\dagger}_{x,\mu\nu}U_{x,\mu}U_{x+\hat{\mu},\nu})\right)} \times \int \gamma_1[R']$$

HS-transform the new auxiliary variable $Q'$:

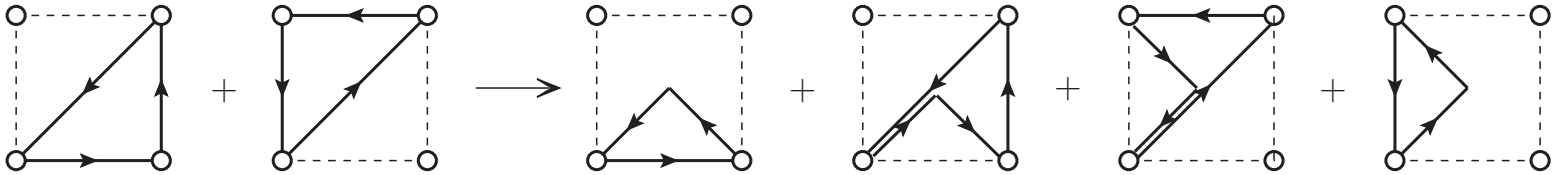
$$R'_{x,\mu\nu} = \sqrt{\frac{\beta}{N}}\left(R_{x,\mu\nu} - Q_{x,\mu\nu}U^{\dagger}_{x+\hat{\mu},\nu} - U_{x,\mu}\right)$$




**The integrand <u>factorizes</u> as a product of (solvable) bosonic one-link integrals:**

$$Z(\beta) = \int \gamma_{\frac{3\beta}{N}}[Q]\, \gamma_{\frac{\beta}{N}}[R] \prod_{x,\mu} \int dU\, e^{\frac{\beta}{N}\mathrm{ReTr}(J^{\dagger}_{x,\mu}U)}$$

$$J_{x,\mu} = \sum_{\nu\neq\mu}\left(R^{\dagger}_{x-\hat{\nu},\nu\mu}Q_{x-\hat{\nu},\nu\mu} + R_{x,\mu\nu}\right)$$



# 4-link action $\longmapsto$ 2-link action $\longmapsto$ 1-link action $\longmapsto$ 0-link action

We obtain an **exact representation** of the partition function of pure lattice gauge theory **without link variables**, in terms of bosonic one-link integrals:

$$Z(\beta) = \int \gamma_{\frac{3\beta}{N}}[Q]\,\gamma_{\frac{\beta}{N}}[R] \prod_{x,\mu} \mathscr{I}_G\left(\frac{\beta}{2N}J_{x,\mu}, \frac{\beta}{2N}J^\dagger_{x,\mu}\right)$$

$$\mathscr{I}_G(A,B) = \int_G dU\, e^{\mathrm{Tr}(AU^\dagger)+\mathrm{Tr}(BU)}$$

The bosonic one-link integral can be solved analytically. [Creutz 1978] [Eriksson *et al* 1981]

**Examples of 0-link partition functions**

$$U(1): \quad Z(\beta) = \int \gamma_{3\beta}[Q]\,\gamma_\beta[R] \prod_l I_0(\beta|J_l|)$$

$$SU(2): \quad Z(\beta) = \int \gamma_{\frac{3\beta}{2}}[Q]\,\gamma_{\frac{\beta}{2}}[R] \prod_l \frac{2I_1(z_l)}{z_l}, \qquad z_l^2 = \frac{\beta^2}{4}\left[\mathrm{Tr}(J_lJ_l^\dagger)+\det(J_l)+\det(J_l^\dagger)\right]$$

$$SU(3): \quad Z(\beta) = \int \gamma_\beta[Q]\,\gamma_{\frac{\beta}{3}}[R] \prod_l \oint_0 \frac{dx}{2\pi i}\, e^{x\frac{\beta^2}{18}\mathrm{Re}\det(J_l)}\, \frac{I_1(2z_l(x))}{z_l(x)}, \qquad z_l^2(x) = x^{-1}\det\left(1+x\frac{\beta^2}{36}J_lJ_l^\dagger\right)$$

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$$\mathscr{I}_G(A,B) = \int_G dU\, e^{\mathrm{Tr}(AU^\dagger)+\mathrm{Tr}(BU)}$$

**Gauge-invariant loop operators** are constructed using **effective links**, $\widetilde{U}^{ij}_l = \langle U^{ij}\rangle$:

$$U(1): \quad \widetilde{U}_l = \frac{J_l}{|J_l|}\frac{I_1(\beta|J_l|)}{I_0(\beta|J_l|)}$$

$$\langle W(C)\rangle = \left\langle \frac{1}{N}\mathrm{Tr}\left(\mathcal{P}\prod_{l\in C} U_l\right)\right\rangle = \left\langle \frac{1}{N}\mathrm{Tr}\left(\mathcal{P}\prod_{l\in C} \widetilde{U}_l\right)\right\rangle$$

$$SU(2): \quad \widetilde{U}_l = \frac{1}{z_l}\frac{I_2(z_l)}{I_1(z_l)}\left[\, J_l - J_l^\dagger + \mathrm{Tr}(J_l^\dagger)\,\right]$$

$$SU(3): \quad \widetilde{U}_l = \frac{\beta}{6}\mathrm{adj}(J_l^\dagger)\oint_0 \frac{dx}{2\pi i}\, e^{x\frac{\beta^2}{18}\mathrm{Re}\det(J_l)}\,\frac{x\, I_1(2z_l(x))}{z_l(x)}$$

$$+\frac{\beta}{6}\oint_0 \frac{dx}{2\pi i}\, e^{x\frac{\beta^2}{18}\mathrm{Re}\det(J_l)}\,\frac{I_2(2z_l(x))}{z_l(x)}\left[J_l + x\frac{\beta}{6}\mathrm{Tr}(J_lJ_l^\dagger) - x\frac{\beta^2}{36}J_lJ_l^\dagger J_l + x^2\mathrm{adj}(J_l)\right]$$

[de Forcrand & Roiesnel 1985]

# 4-link action $\longmapsto$ 2-link action $\longmapsto$ 1-link action $\longmapsto$ 0-link action

$$
\begin{aligned}
Z(\beta) &= \int [dU]\, e^{-\beta \sum_{x,\mu<\nu} \left(1-\frac{1}{N}\mathrm{ReTr}(U_{x,\mu}U_{x+\hat{\mu},\nu}U^{\dagger}_{x+\hat{\nu},\mu}U^{\dagger}_{x,\nu})\right)} \\
&= \int \gamma_{\frac{\beta}{N}}[Q] \int [dU]\, e^{-\beta \sum_{x,\mu<\nu} \left(1-\frac{1}{N}\mathrm{ReTr}(Q^{\dagger}_{x,\mu\nu}U_{x,\mu}U_{x+\hat{\mu},\nu})\right)} \\
&= \int \gamma_{\frac{3\beta}{N}}[Q]\gamma_{\frac{\beta}{N}}[R] \prod_{x,\mu} \int dU\, e^{\frac{\beta}{N}\mathrm{ReTr}(J^{\dagger}_{x,\mu}U)} \\
&= \int \gamma_{\frac{3\beta}{N}}[Q]\gamma_{\frac{\beta}{N}}[R] \prod_{x,\mu} \mathscr{I}\left(\frac{\beta}{2N}J_{x,\mu}, \frac{\beta}{2N}J^{\dagger}_{x,\mu}\right)
\end{aligned}
$$

**Gauge covariance** is preserved:

$$
\begin{aligned}
Q_{x,\mu\nu} &\longmapsto \Omega_x\, Q_{x,\mu\nu}\, \Omega^{\dagger}_{x+\hat{\mu}+\hat{\nu}} & U_{x,\mu} &\longmapsto \Omega_x\, U_{x,\mu}\, \Omega^{\dagger}_{x+\hat{\mu}} \\
R_{x,\mu\nu} &\longmapsto \Omega_x\, R_{x,\mu\nu}\, \Omega^{\dagger}_{x+\hat{\mu}} & J_{x,\mu} &\longmapsto \Omega_x\, J_{x,\mu}\, \Omega^{\dagger}_{x+\hat{\mu}}
\end{aligned}
$$

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\begin{aligned}
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&= \int \gamma_{\frac{\beta}{N}}[Q] \int [dU]\, e^{-\beta \sum_{x,\mu<\nu} \left(1-\frac{1}{N}\mathrm{ReTr}(Q^{\dagger}_{x,\mu\nu}U_{x,\mu}U_{x+\hat{\mu},\nu})\right)} \\
&= \int \gamma_{\frac{3\beta}{N}}[Q]\gamma_{\frac{\beta}{N}}[R] \prod_{x,\mu} \int dU\, e^{\frac{\beta}{N}\mathrm{ReTr}(J^{\dagger}_{x,\mu}U)} \\
&= \int \gamma_{\frac{3\beta}{N}}[Q]\gamma_{\frac{\beta}{N}}[R] \prod_{x,\mu} \mathscr{I}\left(\frac{\beta}{2N}J_{x,\mu}, \frac{\beta}{2N}J^{\dagger}_{x,\mu}\right)
\end{aligned}
$$

**Expectation values of the plaquette operator**

| | $U(1)$ | | $SU(2)$ | | $SU(3)$ | |
|---|---|---|---|---|---|---|
| $n$-link action | $\beta$ | plaquette | $\beta$ | plaquette | $\beta$ | plaquette |
| 4 | 1.00 | 0.58570(23) | 2.25 | 0.586189(37) | 5.70 | 0.549190(51) |
| 2 | 1.00 | 0.58572(59) | 2.25 | 0.586161(65) | 5.70 | 0.549098(73) |
| 1 | 1.00 | 0.5864(11) | 2.25 | 0.58618(12) | 5.70 | 0.54917(15) |
| 0 | 1.00 | 0.5864(11) | 2.25 | 0.58618(12) | 5.70 | *in progress* |

# 0-link action: including fermions

Extending the 1-link partition function to include $N_f$ flavours of staggered fermions is straightforward, because $S_F$ is already linear with respect to the link variables:

$$\frac{\beta}{2N} J^{ij}_{x,\mu} \longmapsto \frac{\beta}{2N} J^{ij}_{x,\mu} + \sum_{\alpha=1}^{N_f} e^{+\mu_\alpha a_\tau \delta_{\mu\tau}} K^{\alpha ij}_{x,\mu}$$

$$\frac{\beta}{2N} \bar{J}^{ij}_{x,\mu} \longmapsto \frac{\beta}{2N} \bar{J}^{ij}_{x,\mu} - \sum_{\alpha=1}^{N_f} e^{-\mu_\alpha a_\tau \delta_{\mu\tau}} \bar{K}^{\alpha ij}_{x,\mu} \qquad K^{\alpha ij}_{x,\mu} = \eta_{x\mu} \chi^{\alpha i}_x \bar{\chi}^{\alpha j}_{x+\hat{\mu}}$$

$$Z(\beta) = \int [d\chi d\bar{\chi}]\, e^{am\bar{\chi}\chi} \int \gamma_{\frac{3\beta}{N}}[Q] \gamma_{\frac{\beta}{N}}[R] \prod_l \int_G dU\, e^{\mathrm{Tr}\left[\left(\frac{\beta}{2N} J_l^\dagger + \sum_\alpha \bar{\eta}_\alpha K_l^{\alpha\dagger}\right) U\right] + \mathrm{Tr}\left[\left(\frac{\beta}{2N} J_l - \sum_\alpha \eta_\alpha K_l^\alpha\right) U^\dagger\right]}$$

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$$\frac{\beta}{2N} \bar{J}^{ij}_{x,\mu} \longmapsto \frac{\beta}{2N} \bar{J}^{ij}_{x,\mu} - \sum_{\alpha=1}^{N_f} e^{-\mu_\alpha a_\tau \delta_{\mu\tau}} \bar{K}^{\alpha ij}_{x,\mu} \qquad K^{\alpha ij}_{x,\mu} = \eta_{x\mu} \chi^{\alpha i}_x \bar{\chi}^{\alpha j}_{x+\hat{\mu}}$$

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**Examples**

$G = U(1)$, $N_f = 1$:

$$\mathscr{I}^{(1)}_{U(1)} = I_0(\beta|J_l|) \left(1 + e^{+\mu a_\tau \delta_{\mu\tau}} \widetilde{U}_l^\dagger K_l + e^{-\mu a_\tau \delta_{\mu\tau}} \widetilde{U}_l K_l^\dagger + K_l K_l^\dagger\right)$$

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$$Z(\beta) = \int [d\chi d\bar{\chi}]\, e^{am\bar{\chi}\chi} \int \gamma_{\frac{3\beta}{N}}[Q] \gamma_{\frac{\beta}{N}}[R] \underbrace{\prod_l \int_G dU\, e^{\mathrm{Tr}\left[\left(\frac{\beta}{2N} J_l^\dagger + \sum_\alpha \bar{\eta}_\alpha K_l^{\alpha\dagger}\right) U\right] + \mathrm{Tr}\left[\left(\frac{\beta}{2N} J_l - \sum_\alpha \eta_\alpha K_l^\alpha\right) U^\dagger\right]}}_{\mathscr{I}_G^{(N_f)}}$$

**Examples**

$G = U(1)$, arbitrary $N_f$:

$$\mathscr{I}^{(N_f)}_{U(1)} = I_0(\beta|J_l|) \sum_{\bar{q}_\alpha, q_\alpha = 0}^{1} \left(\frac{J_l^\dagger}{|J_l|}\right)^{\Delta q} \frac{I_{|\Delta q|}(\beta|J_l|)}{I_0(\beta|J_l|)} e^{a_\tau \delta_{\mu\tau} \sum_\alpha \mu_\alpha \Delta q_\alpha} K_l^{q_\alpha} K_l^{\dagger \bar{q}_\alpha}$$

$$\Delta q = \sum_\alpha \Delta q_\alpha, \quad \Delta q_\alpha = q_\alpha - \bar{q}_\alpha$$

# 0-link action: including fermions

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**Examples**

$G = SU(2)$, $N_f = 1$:

$$\mathscr{I}^{(1)}_{SU(2)} = \frac{2I_1(\frac{\beta}{2} z_l)}{\frac{\beta}{2} z_l} \sum_{\bar{q},q,m,\bar{b},b=0}^{2} \frac{e^{\mu \Delta q\, a_\tau \delta_{\mu\tau}}}{2^{2n+1}(\frac{\beta}{2} z_l)^n} \frac{I_{n+1}(\frac{\beta}{2} z_l)}{I_1(\frac{\beta}{2} z_l)} \frac{\mathrm{Tr}(\tilde{J}_l^\dagger K_l)^q}{q!} \frac{\mathrm{Tr}(\tilde{J}_l K_l^\dagger)^{\bar{q}}}{\bar{q}!} \frac{\mathrm{Tr}(K_l K_l^\dagger)^k}{k!} \frac{\det(K_l)^b}{b!} \frac{\det(K_l^\dagger)^{\bar{b}}}{\bar{b}!}$$

$$\Delta q = q - \bar{q}, \quad n = q + \bar{q} + k + b + \bar{b}, \quad \tilde{J}_l = J_l - J_l^\dagger + \mathrm{Tr}(J_l^\dagger)$$

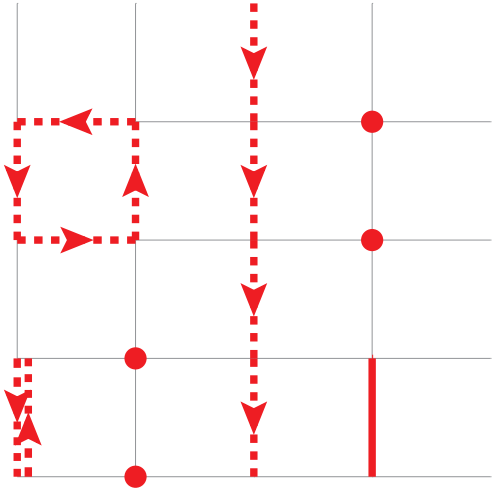
# MDP models at finite $\beta$: $U(1)$, $N_f = 1$

$$Z(\beta) = \int \gamma_{3\beta}[Q]\gamma_\beta[R] \prod_l I_0(\beta|J_l|) \sum_{\{n,k,\ell\}} \prod_{l\in\ell} \frac{\bar{J}_l}{|J_l|} \frac{I_1(\beta|J_l|)}{I_0(\beta|J_l|)} \cdot \prod_x (am)^{n_x} \cdot \sigma(\ell) e^{r(\ell)\mu/T}$$

**Degrees of freedom**

- **Monomers:** $n_x$ $\quad x$
- **Dimers:** $k_{x,\mu}$ $\quad x \text{ ——— } x+\hat{\mu}$
- **Electrons:** $q_{x,\mu}$ $\quad x \text{ - - - ▶ - - - } x+\hat{\mu}$

  $\bar{q}_{x,\mu}$ $\quad x \text{ - - - ◀ - - - } x+\hat{\mu}$



**Grassmann constraints**

$$\bar{\chi}: \quad n_x + \sum_\mu (k_{x,\mu} + k_{x,-\mu} + q_{x,\mu}) = 1$$

$$\chi: \quad n_x + \sum_\mu (k_{x,\mu} + k_{x,-\mu} + \bar{q}_{x,\mu}) = 1$$

$$\Longrightarrow \sum_\mu (q_{x,\mu} - \bar{q}_{x,\mu}) = 0$$

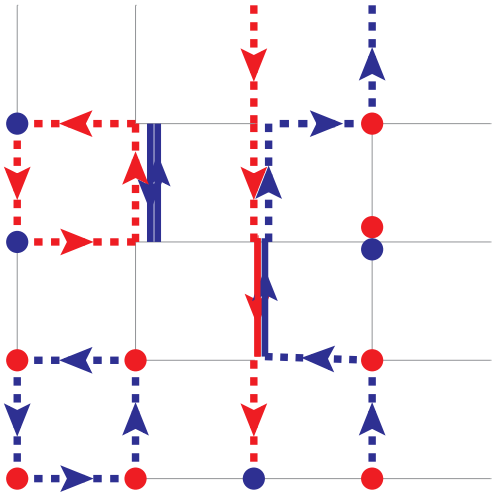
# MDP models at finite $\beta$: $U(1)$, arbitrary $N_f$

$$Z(\beta) = \int \gamma_{3\beta}[Q]\gamma_\beta[R] \prod_l I_0(\beta|J_l|) \sum_{\{n,k,\ell\}} \prod_{l\in\ell} \left(\frac{\bar{J}_l}{|J_l|}\right)^{\Delta q_l} \frac{I_{|\Delta q_l|}(\beta|J_l|)}{I_0(\beta|J_l|)} \prod_{\alpha,x} (am_\alpha)^{n_x^\alpha} \prod_\alpha \sigma(\ell_\alpha) e^{r(\ell_\alpha)\mu_\alpha/T}$$

**Degrees of freedom**

- **Monomers:** $n_x^\alpha$ $\quad x$ ●
- **Dimers:** $k_{x,\mu}^{\alpha\beta}$ $\quad x$ ——— $x+\hat{\mu}$
- **Electrons:** $q_{x,\mu}^\alpha$ $\quad x$ - - - ▶ - - - $x+\hat{\mu}$

  $\bar{q}_{x,\mu}^\alpha$ $\quad x$ - - - ◀ - - - $x+\hat{\mu}$



**Grassmann constraints**

$$\bar{\chi}^\alpha: \quad n_x^\alpha + \sum_\mu \left( \sum_\beta (k_{x,\mu}^{\alpha\beta} + k_{x,-\mu}^{\beta\alpha}) + \bar{q}_{x,\mu}^\alpha \right) = 1$$

$$\chi^\alpha: \quad n_x^\alpha + \sum_\mu \left( \sum_\beta (k_{x,\mu}^{\beta\alpha} + k_{x,-\mu}^{\alpha\beta}) + q_{x,\mu}^\alpha \right) = 1$$

$$\Longrightarrow \sum_\mu \left[ q_{x,\mu}^\alpha - \bar{q}_{x,\mu}^\alpha + \sum_\beta (k_{x,\mu}^{\beta\alpha} - k_{x,\mu}^{\alpha\beta}) + \sum_\beta (k_{x,-\mu}^{\alpha\beta} - k_{x,-\mu}^{\beta\alpha}) \right] = 0$$

# Conclusion

- We decouple all link variables in the gauge action using auxiliary bosonic fields.
- We obtain a sequence of $n$-link partition functions ($n$ = 4,2,1,0) that encode the same physics. In particular, we **integrate out all the link variables exactly for arbitrary** $\beta$. Numerical simulations of pure gauge theory are under control.
- The extension of the 0-link action to full QCD with $N_f$ staggered flavours at finite density is straightforward.
- By integrating out the lattice fermions a posteriori, we construct MDP models of lattice gauge theories for arbitrary $\beta$.

**Challenges**

- Find a suitable resummation of worldlines that makes the sign problem milder.
- Perform numerical simulations of the MDP models at finite $\beta$.