The leading disconnected contribution to the anomalous magnetic moment of the muon

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lattice 14

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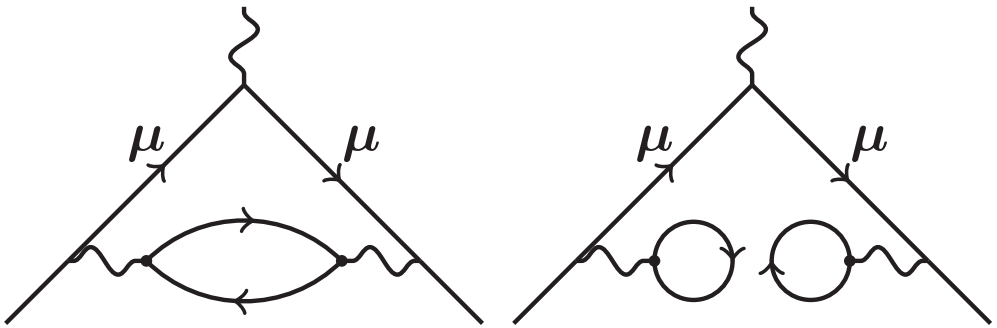
Hadronic Vacuum Polarisation

- leading order hadronic contribution to $\mathbf{a}_\mu$
- connected and disconnected diagram



- mixed time-momentum representation vector correlator

$$G^{\gamma\gamma}(x_0) = -\int d^3x \, \langle j_k^\gamma(x) j_k^\gamma(0) \rangle \quad \text{with} \quad j_k^\gamma = \frac{2}{3}\bar{u}\gamma_k u - \frac{1}{3}\bar{d}\gamma_k d + \ldots$$

- hadronic vacuum polarization [1107.4388]

$$\hat{\Pi}(Q^2) = 4\pi^2 \int_0^\infty dx_0 \, G^{\gamma\gamma}(x_0) \left[x_0^2 - \frac{4}{Q^2} \sin^2\left(\frac{1}{2} Q x_0\right) \right]$$

light quark contribution to the Vector Correlator

- electro-magnetic current for the light quarks

$$j_\mu^\ell = \underbrace{\frac{1}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right)}_{I=1,\ \ j_\mu^\rho} + \underbrace{\frac{1}{6}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right)}_{I=0,\ \ \frac{1}{3}j_\mu^\omega}$$

- only isoscalar state has a disconnected contribution
- definitions

$$G^{\rho\rho}(x_0) = \frac{1}{2}\, G^\ell_{\text{con}}(x_0) = -\int d^3x\, \langle j_k^\rho(x) j_k^\rho(0)\rangle$$

$$G^\ell_{\text{disc}}(x_0) = -\int d^3x\, \left\langle j_k^\ell(x) j_k^\ell(0)\right\rangle_{\text{disc}}$$

- vector correlator

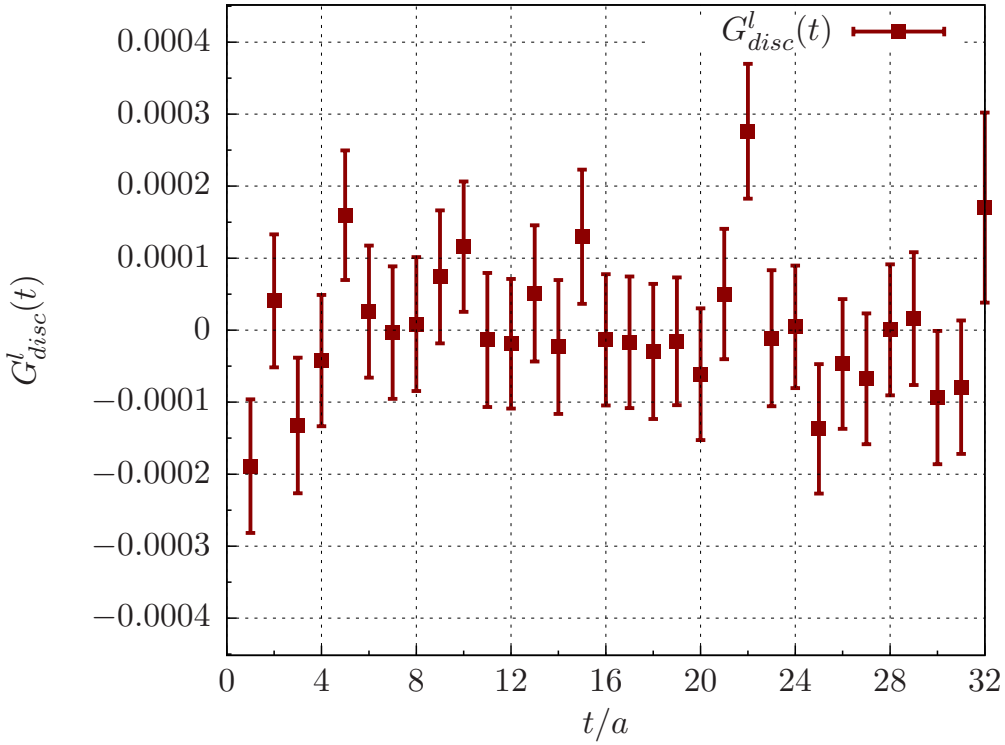
$$G^{\gamma\gamma}(x_0) = \frac{5}{9} G^\ell_{\text{con}}(x_0) + \frac{1}{9} G^\ell_{\text{disc}}(x_0)$$

The disconnected contribution for the light quarks

- on the lattice we calculate

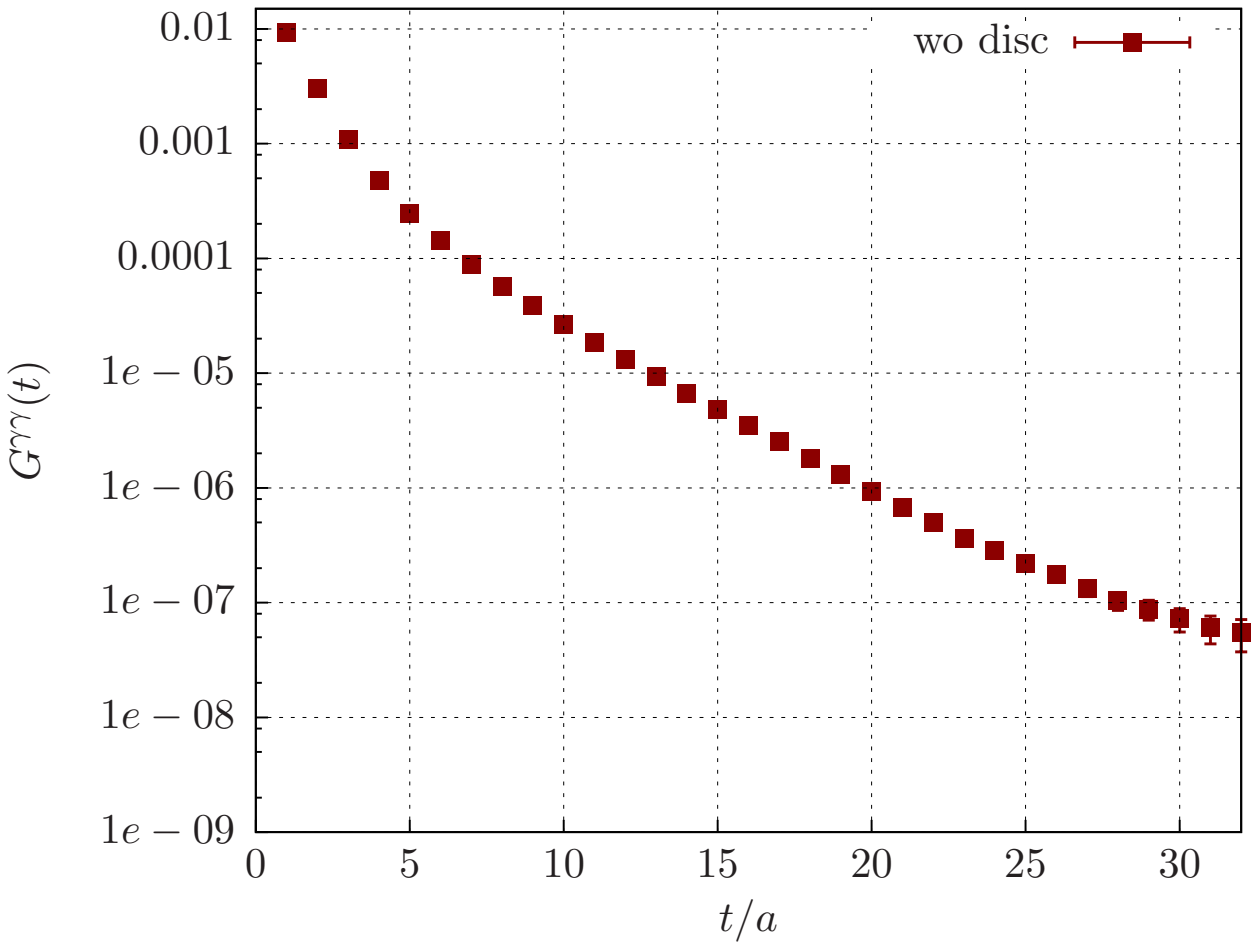
$$G^{\ell}_{\text{disc}}(x_0 - y_0) = -\frac{Z_V^2}{L^3}\left\langle\left(\sum_{\vec{x}} \text{Tr}\left[\gamma_k D^{-1}(x,x)\right]\right)\left(\sum_{\vec{y}} \text{Tr}\left[\gamma_k D^{-1}(y,y)\right]\right)\right\rangle$$

- all-to-all propagator with **3** stochastic sources and generalized hopping parameter expansion [0910.3970, 1309.2104]
- $\mathcal{O}(a)$-improved Wilson action with $N_f = 2$ (CLS)
- 64×32^3 lattice with $m_\pi \approx 450$ MeV and $a = 0.063$ fm



The total light quark vector Correlator

- $G^{\gamma\gamma}(t) = \frac{5}{9} G^{\ell}_{\text{con}}(t) + \frac{1}{9} G^{\ell}_{\text{disc}}(t)$



15a ≈ 1 fm

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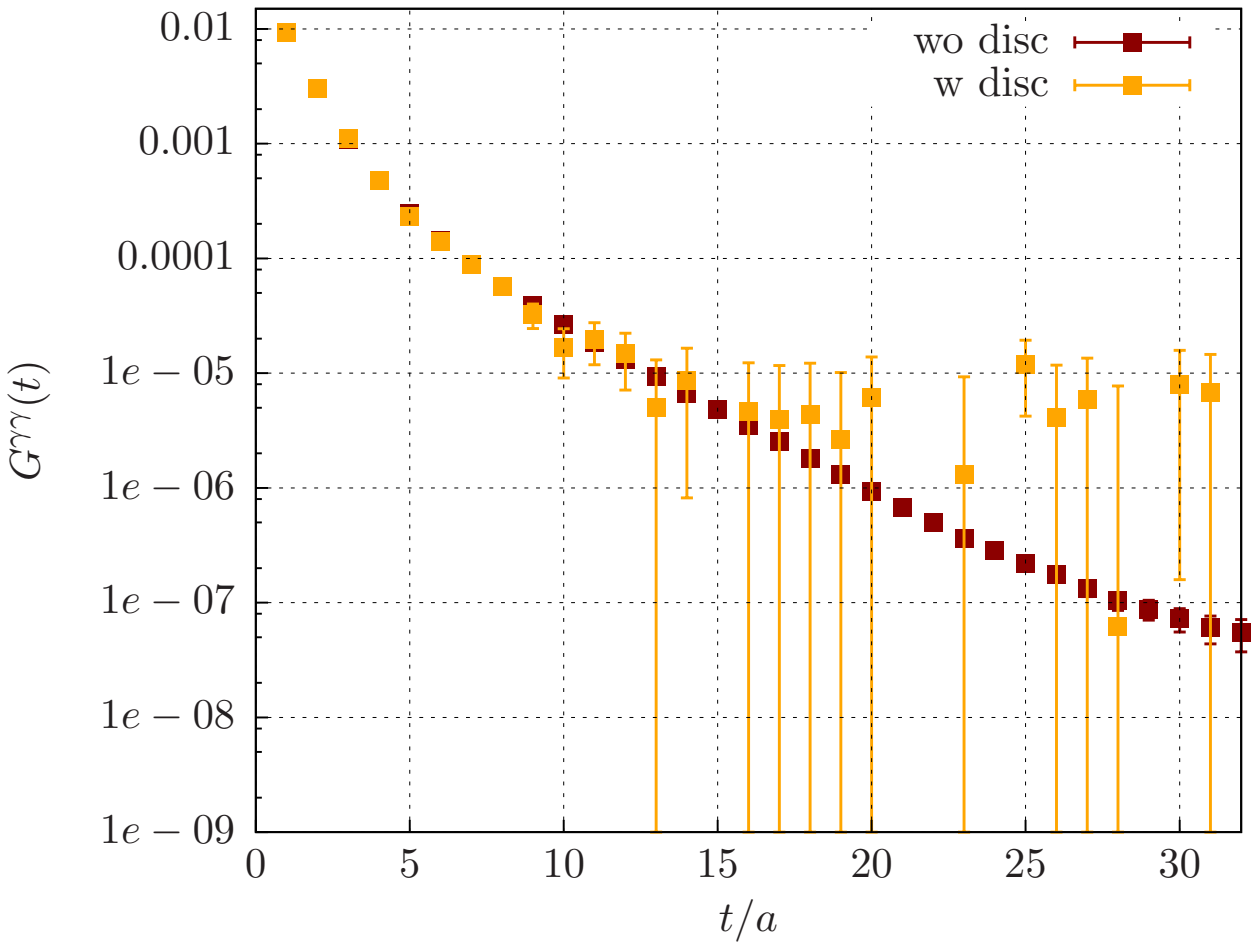

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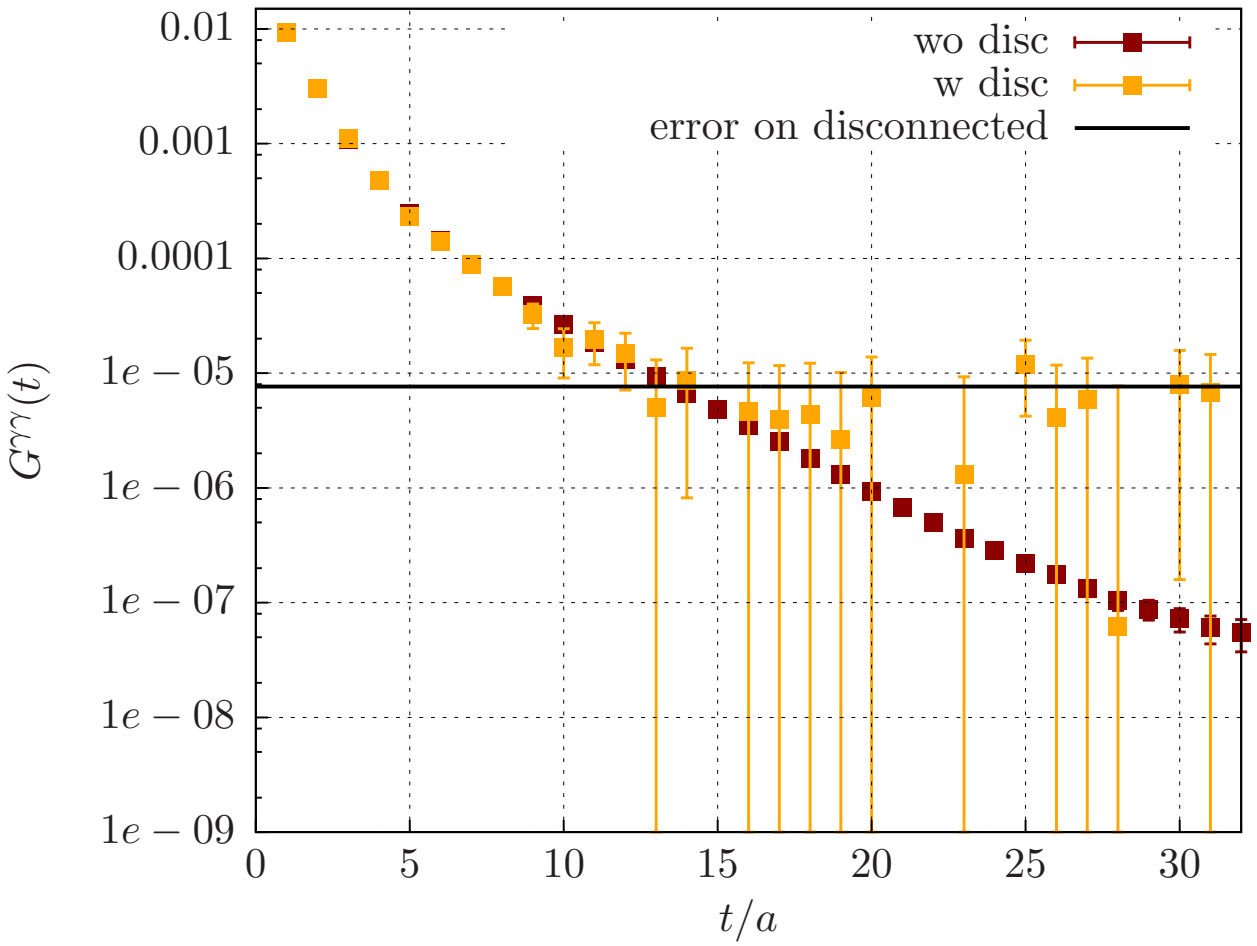


15a ≈ 1 fm

- for **t** $\gtrsim$ **0.8** fm the vector correlator is dominated by the error on the disconnected contribution

light and strange contribution to the Vector Correlator

- electro-magnetic current

$$j_\mu^{\ell s} = j_\mu^\ell + j_\mu^s = \underbrace{\frac{1}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right)}_{I=1,\ \ j_\mu^\rho} + \underbrace{\frac{1}{6}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d - 2\bar{s}\gamma_\mu s\right)}_{I=0}$$

- definitions

$$G_{con}^s(x_0) = -\int d^3x \left\langle j_k^s(x) j_k^s(0)\right\rangle_{con}$$

$$G_{disc}^{\ell s}(x_0) = -\int d^3x \left\langle j_k^{\ell s}(x) j_k^{\ell s}(0)\right\rangle_{disc}$$

- vector correlator

$$G^{\gamma\gamma}(x_0) = \frac{5}{9} G_{con}^\ell(x_0) + \frac{1}{9} G_{con}^s(x_0) + \frac{1}{9} G_{disc}^{\ell s}(x_0)$$

The disconnected contribution for light and strange quarks

- one can rewrite

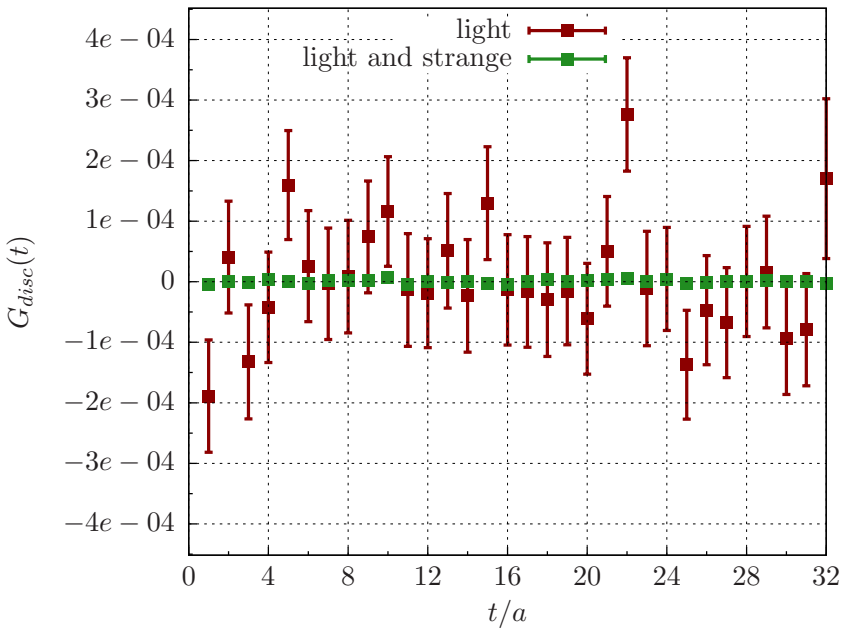
$$G^{\ell s}_{\text{disc}}(x_0) = -\int d^3x \left\langle j^{\ell s}_k(x) j^{\ell s}_k(0) \right\rangle_{\text{disc}}$$
$$= -\int d^3x \left\langle \left(j^{\ell}_k(x) - j^{s}_k(x)\right)\left(j^{\ell}_k(0) - j^{s}_k(0)\right) \right\rangle_{\text{disc}}$$

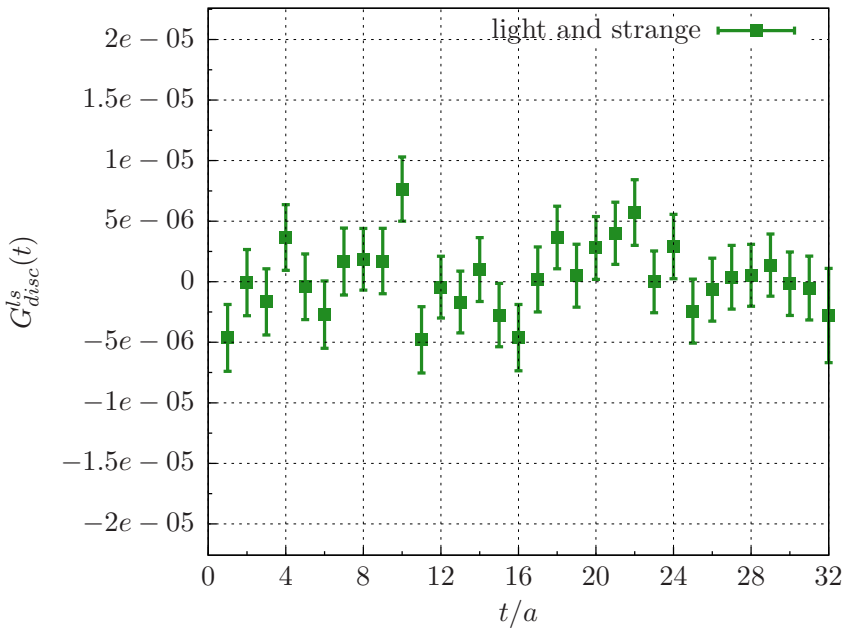
- on the lattice

$$G^{\ell s}_{\text{disc}}(x_0 - y_0) = -\frac{Z_V^2}{L^3}\left\langle \left(\sum_{\vec{x}} \text{Tr}\left[\gamma_k D_\ell^{-1}(x,x) - \gamma_k D_s^{-1}(x,x)\right] \right) \times \right.$$
$$\left. \left(\sum_{\vec{y}} \text{Tr}\left[\gamma_k D_\ell^{-1}(y,y) - \gamma_k D_s^{-1}(y,y)\right] \right) \right\rangle$$

- we need only differences of light- and strange propagator
- **idea:** calculate light- and strange propagator with the same stochastic sources to cancel stochastic noise

Results for the disconnected contribution

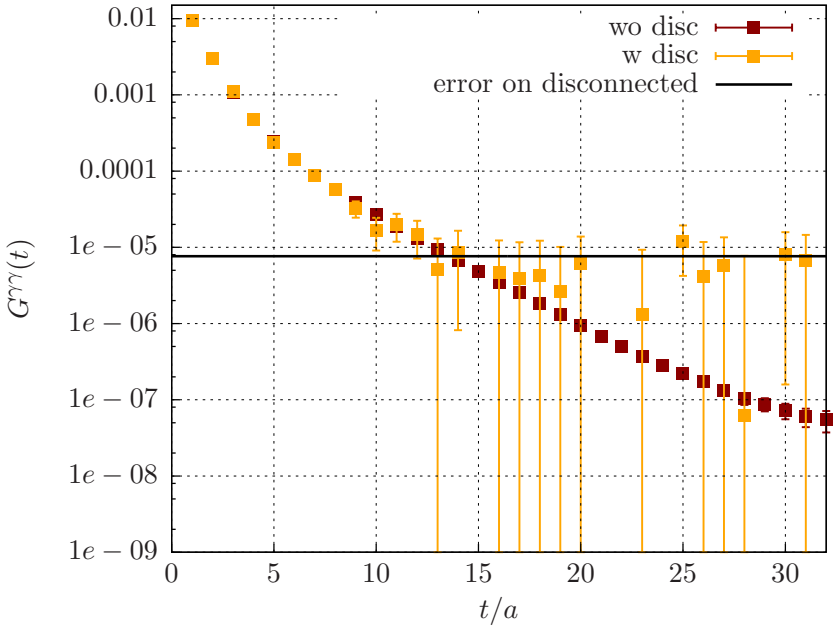




- reduction of the error $\approx$ **95%** compared to the individual light/strange quark contribution
- $\mathbf{G^{\ell s}_{disc}(t)}$ consistent with zero

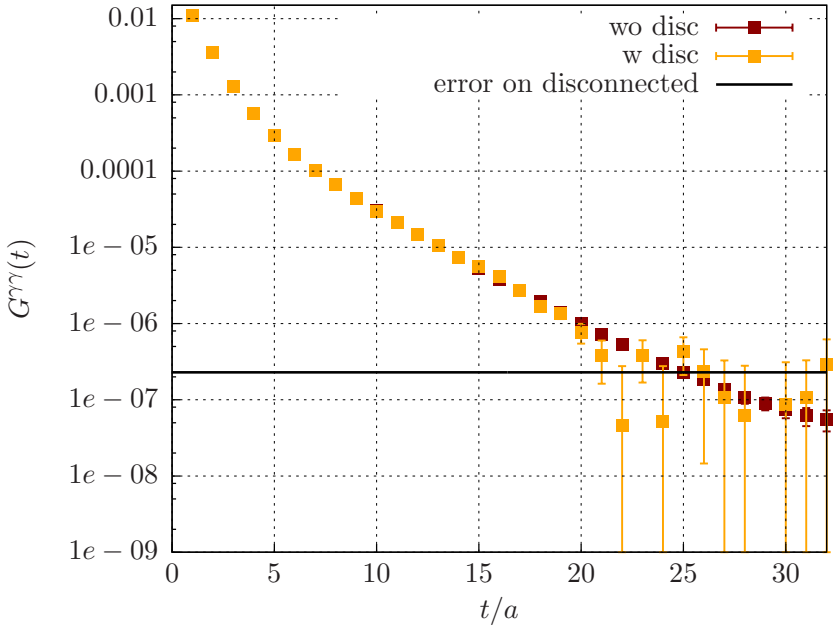
The total vector Correlator

- light quarks
- $G^{\gamma\gamma}(t) = \frac{5}{9} G^{\ell}_{con}(t) + \frac{1}{9} G^{\ell}_{disc}(t)$



- error on disconnected dominates for $t \gtrsim 0.8$ fm

- light and strange quarks
- $G^{\gamma\gamma}(t) = \frac{5}{9} G^{\ell}_{con}(t) + \frac{1}{9} G^{s}_{con}(t) + \frac{1}{9} G^{\ell s}_{disc}(t)$



- error on disconnected dominates for $t \gtrsim 1.5$ fm

Disconnected Contribution for $t \to \infty$

- vector correlator

$$G^{\gamma\gamma}(t) = \frac{5}{9} G^{\ell}_{\text{con}}(t) + \frac{1}{9} G^{s}_{\text{con}}(t) + \frac{1}{9} G^{\ell s}_{\text{disc}}(t) \qquad \text{with} \quad G^{\ell}_{\text{con}}(t) = 2\, G^{\rho\rho}(t)$$

- rewrite:

$$\frac{1}{9} \frac{G^{\ell s}_{\text{disc}}(t)}{G^{\rho\rho}(t)} = \frac{G^{\gamma\gamma}(t) - G^{\rho\rho}(t)}{G^{\rho\rho}(t)} - \frac{1}{9} \left(1 + 2 \frac{G^{s}_{\text{con}}(t)}{G^{\ell}_{\text{con}}(t)} \right)$$

- [1306.2532] for large t, the isovector state dominates, i.e. the ρ

$$G^{\gamma\gamma}(t) = G^{\rho\rho}(t) \left(1 + \mathcal{O}(e^{-m_\pi t}) \right)$$

Disconnected Contribution for $t \to \infty$

- vector correlator

$$G^{\gamma\gamma}(t) = \frac{5}{9} G^{\ell}_{\text{con}}(t) + \frac{1}{9} G^{s}_{\text{con}}(t) + \frac{1}{9} G^{\ell s}_{\text{disc}}(t) \qquad \text{with} \quad G^{\ell}_{\text{con}}(t) = 2\, G^{\rho\rho}(t)$$

- rewrite:

$$\frac{1}{9} \frac{G^{\ell s}_{\text{disc}}(t)}{G^{\rho\rho}(t)} = \underbrace{\frac{G^{\gamma\gamma}(t) - G^{\rho\rho}(t)}{G^{\rho\rho}(t)}}_{\to 0 \quad \text{for } t \to \infty} - \frac{1}{9} \underbrace{\left(1 + 2 \frac{G^{s}_{\text{con}}(t)}{G^{\ell}_{\text{con}}(t)}\right)}_{\to 1 \quad \text{for } t \to \infty}$$

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Disconnected Contribution for $t \to \infty$

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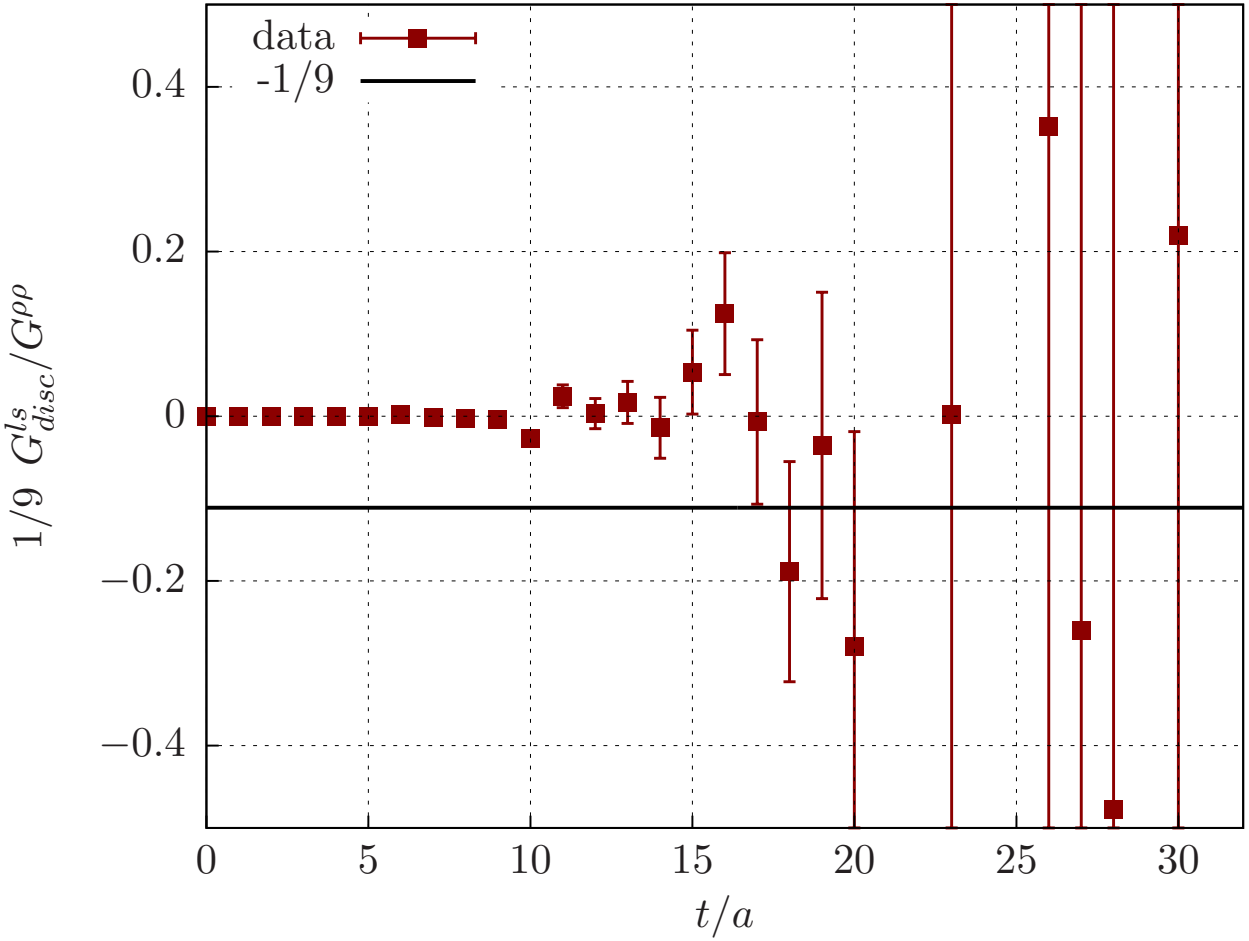
- [1306.2532] for large t, the isovector state dominates, i.e. the ρ

$$G^{\gamma\gamma}(t) = G^{\rho\rho}(t) \left(1 + \mathcal{O}(e^{-m_\pi t})\right)$$

- for $t \to \infty$

$$\frac{1}{9} \frac{G^{\ell s}_{\text{disc}}(t)}{G^{\rho\rho}(t)} \quad \longrightarrow \quad -\frac{1}{9}$$

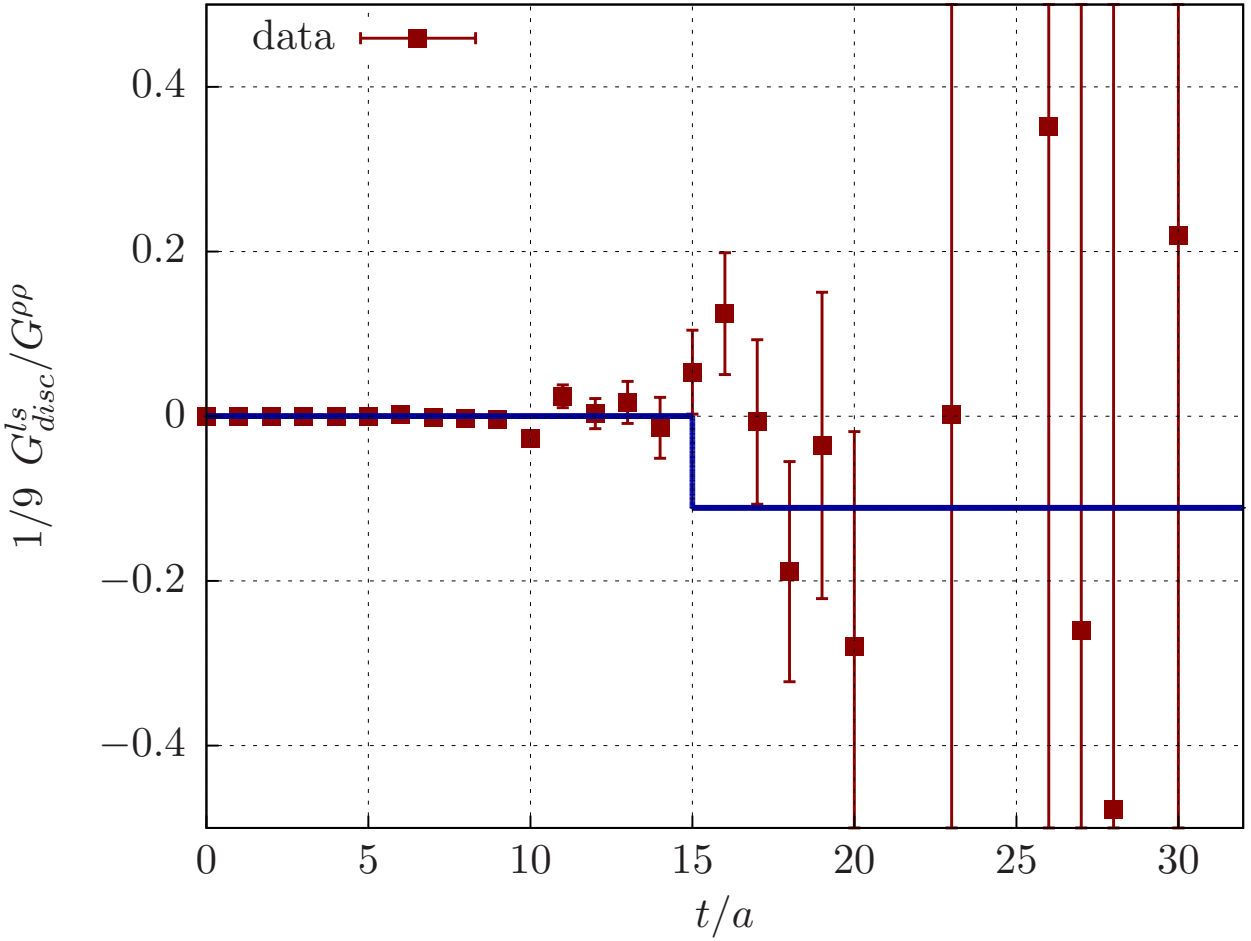
Disconnected Contribution for $\mathbf{t} \to \infty$



15a $\approx$ **1** fm

- up to $\mathbf{t} \approx \mathbf{15a}$ we can distinguish the ratio from $-\mathbf{1/9}$

Disconnected Contribution for $\mathbf{t} \to \infty$



15a ≈ 1 fm

- up to $\mathbf{t \approx 15a}$ we can distinguish the ratio from $\mathbf{-1/9}$
- **idea:** use $\frac{1}{9}\ \mathbf{G^{\ell s}_{disc}(t)/G^{\rho\rho}(t) = -1/9}$ for $\mathbf{t > 15}$ to give an upper bound for the magnitude of the disconnected contribution

Vacuum Polarization with disconnected estimate

$$\hat{\Pi}(Q^2) = 4\pi^2 \int_0^\infty dt\, G^{\gamma\gamma}(t) \left[t^2 - \frac{4}{Q^2} \sin^2\left(\frac{1}{2} Q t\right) \right]$$

- for $t \leq 15a \approx 1$ fm, the vector correlator is well described by the connected part

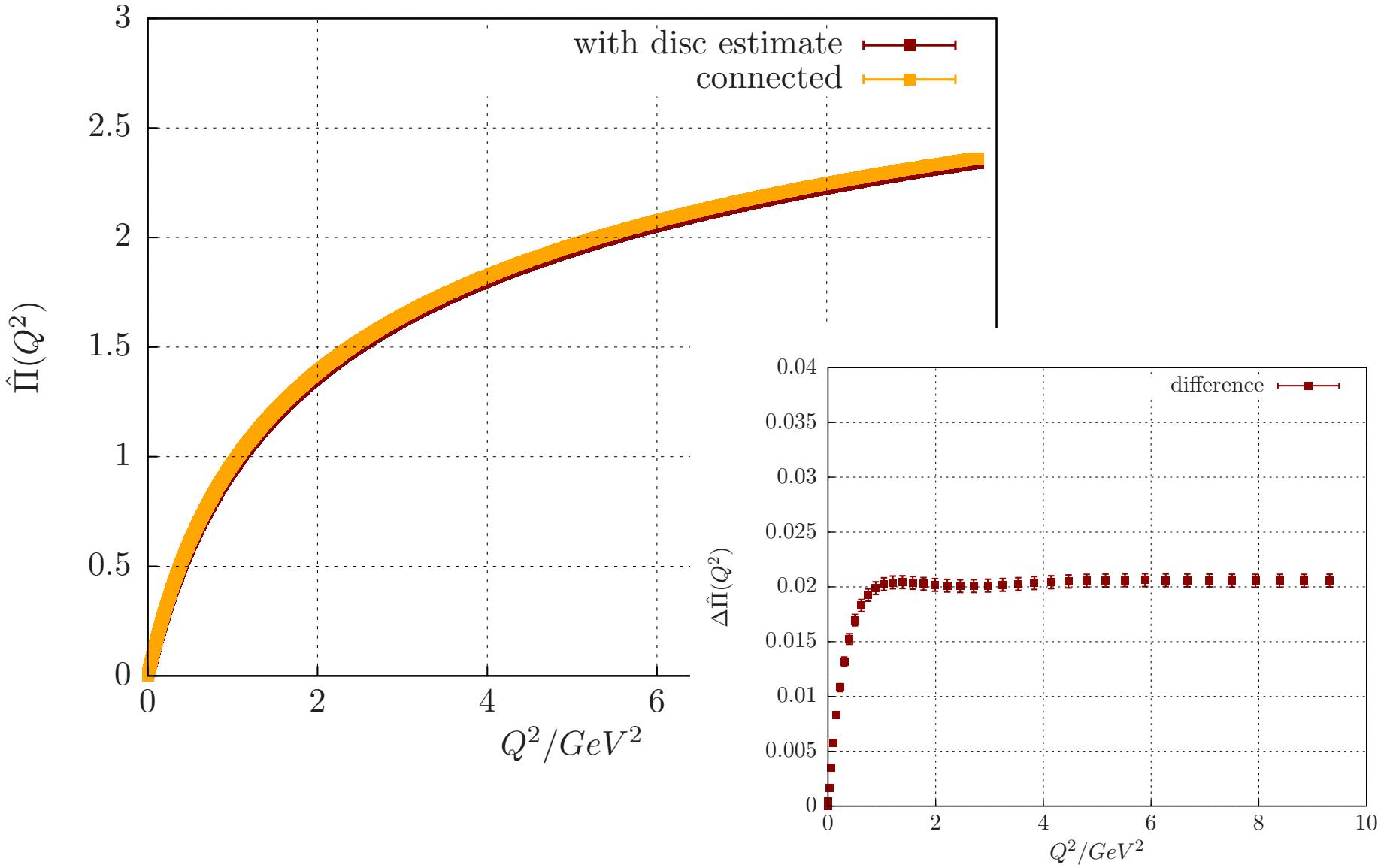
$$G^{\gamma\gamma}(t) = \frac{5}{9} G^{\ell}_{\text{con}}(t) + \frac{1}{9} G^{s}_{\text{con}}(t)$$

- for $t > 15a$ we use $\frac{1}{9}\ G^{\ell s}_{\text{disc}}(t)/G^{\rho\rho}(t) = -1/9$ as upper bound for disconnected part

$$G^{\gamma\gamma}(t) = \frac{5}{9} G^{\ell}_{\text{con}}(t) + \frac{1}{9} G^{s}_{\text{con}}(t) - \frac{1}{9} G^{\rho\rho}(t)$$

- give an upper bound for the magnitude of the disconnected contribution to a_μ

Vacuum Polarization with disconnected estimate



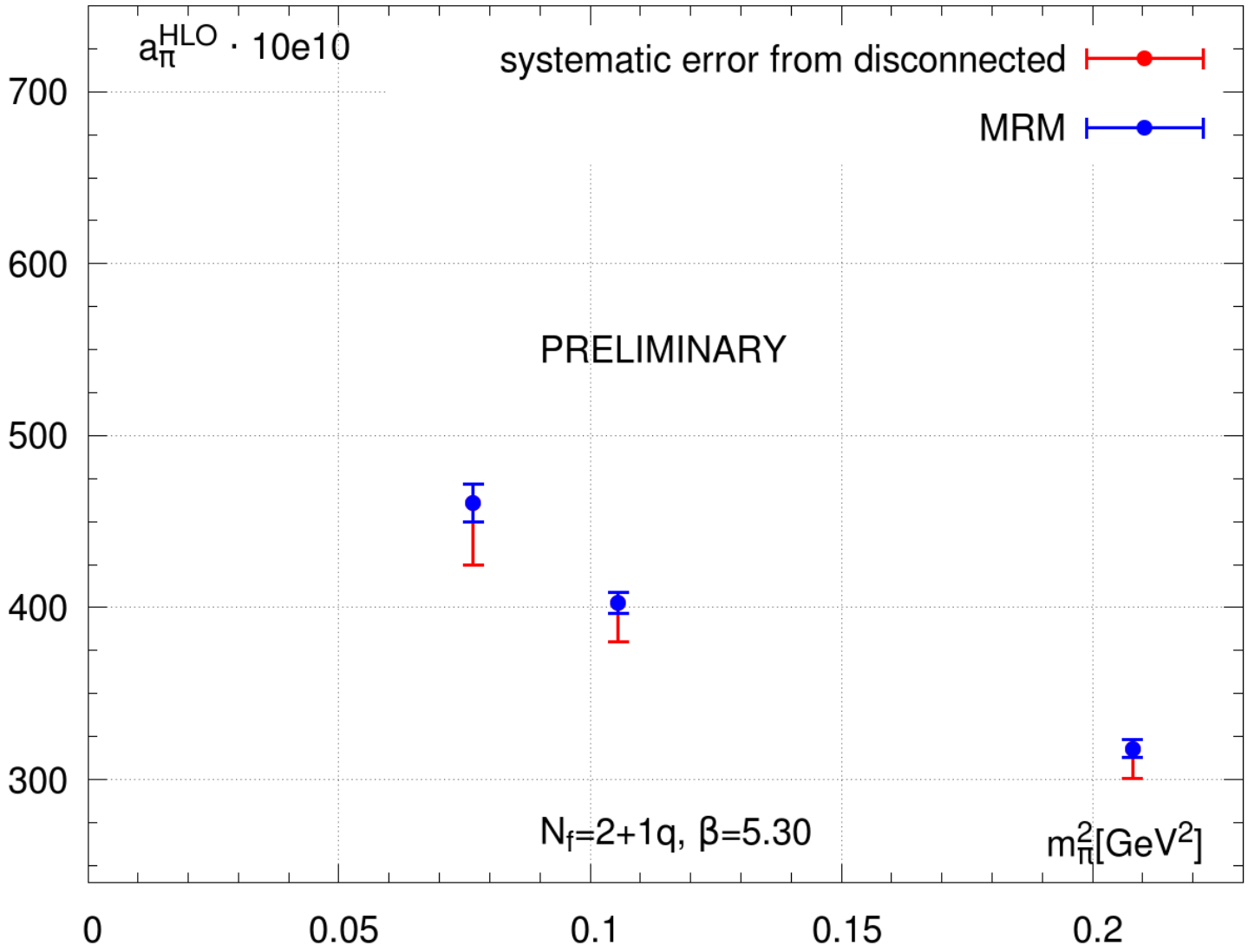
a_μ with disconnected estimate

$$a_\mu^{\text{had}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 \, \frac{1}{Q^2} K(Q^2) \, \hat{\Pi}(Q^2)$$

- with the disconnected estimate, a_μ is $\sim$ **4%** smaller then connected contribution only
- use **4%** as a conservative upper bound for a systematic error that arises from neglecting the disconnected contribution

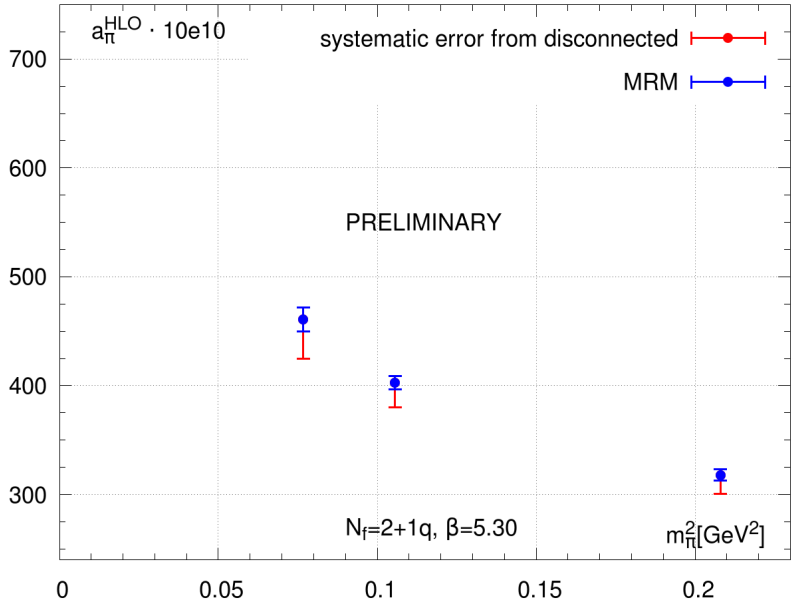
β	a[fm]	lattice	m_π[MeV]	$m_\pi L$	Label	N_{cnfg}
5.3	0.063	64×32^3	455	4.7	E5	1000
5.3	0.063	96×48^3	325	5.0	F6	300
5.3	0.063	96×48^3	280	4.3	F7	250

a_μ with disconnected estimate



Summary

- the disconnected contribution to the vector correlator for light and strange quarks depends only on difference of light and strange propagators
  $\rightarrow$ error is reduced when using the same stochastic sources
- disconnected contribution to $\mathbf{G^{\gamma\gamma}(t)}$ from our lattice calculation consistent with zero within the errors
- use asymptotic behavior of $\mathbf{G^{\ell s}_{disc}(t)}$ to give a conservative upper bound for the disconnected contribution to $\mathbf{a_\mu}$
  $\rightarrow$ disconnected contribution $< \mathbf{4\% - 5\%}$ of the connected one



Backup

the mixed time-momentum representation method

- hadronic vacuum polarization

$$\Pi_{kk}(\omega, q = 0) = \int d^4x e^{i\,Q\cdot x} \langle j_k^\gamma(x)\; j_k^\gamma(0)\rangle = -\int dt\; e^{i\omega t}\, G^{\gamma\gamma}(t)$$

- vector correlator

$$G^{\gamma\gamma}(t) = -\int d^3x\, \langle j_k^\gamma(x) j_k^\gamma(0)\rangle \qquad \text{with} \quad j_k^\gamma = \frac{2}{3}\bar{u}\gamma_k u - \frac{1}{3}\bar{d}\gamma_k d + \ldots$$

- tensor structure of the vacuum polarization

$$\Pi_{kk}(\omega, q = 0) = (Q_k Q_k - \delta_{kk} Q^2)\, \Pi(Q^2) \overset{Q^2 = \omega^2}{=} -\omega^2\, \Pi(\omega^2)$$

- subtracted vacuum polarization after Taylor expansion at $Q^2 = 0$

$$\hat{\Pi}(\omega^2) = 4\,\pi^2 \left[\Pi(\omega^2) - \Pi(0)\right] = 4\,\pi^2 \int_{-\infty}^{\infty} dt\, G^{\gamma\gamma}(t) \left[\frac{e^{-i\omega t} - 1}{\omega^2} + \frac{t^2}{2}\right]$$

$$= 4\pi^2 \int_0^{\infty} dt\, G^{\gamma\gamma}(t) \left[t^2 - \frac{4}{\omega^2} \sin^2\left(\frac{1}{2}\omega t\right)\right]$$

generalized Hopping Parameter Expansion

cf. [Bali et al. arXiv:0910.3970]

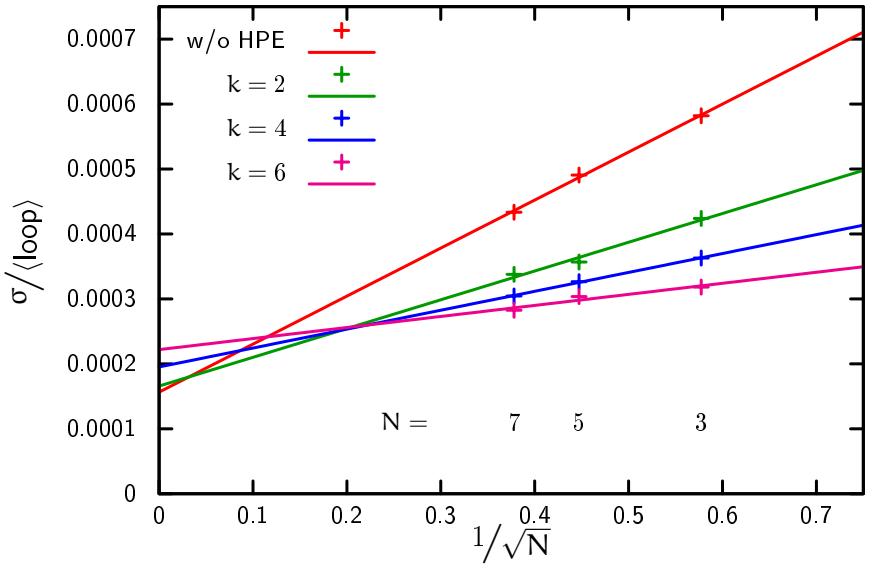
- $\mathcal{O}(a)$-improved Wilson-Dirac operator

$$D_{sw} = \frac{1}{2\kappa}\mathbb{1} + c_{sw}B - \frac{1}{2}H \quad = \quad A - \frac{1}{2}H \quad = \quad A\left(\mathbb{1} - \frac{1}{2}A^{-1}H\right)$$

- generalized hopping parameter expansion

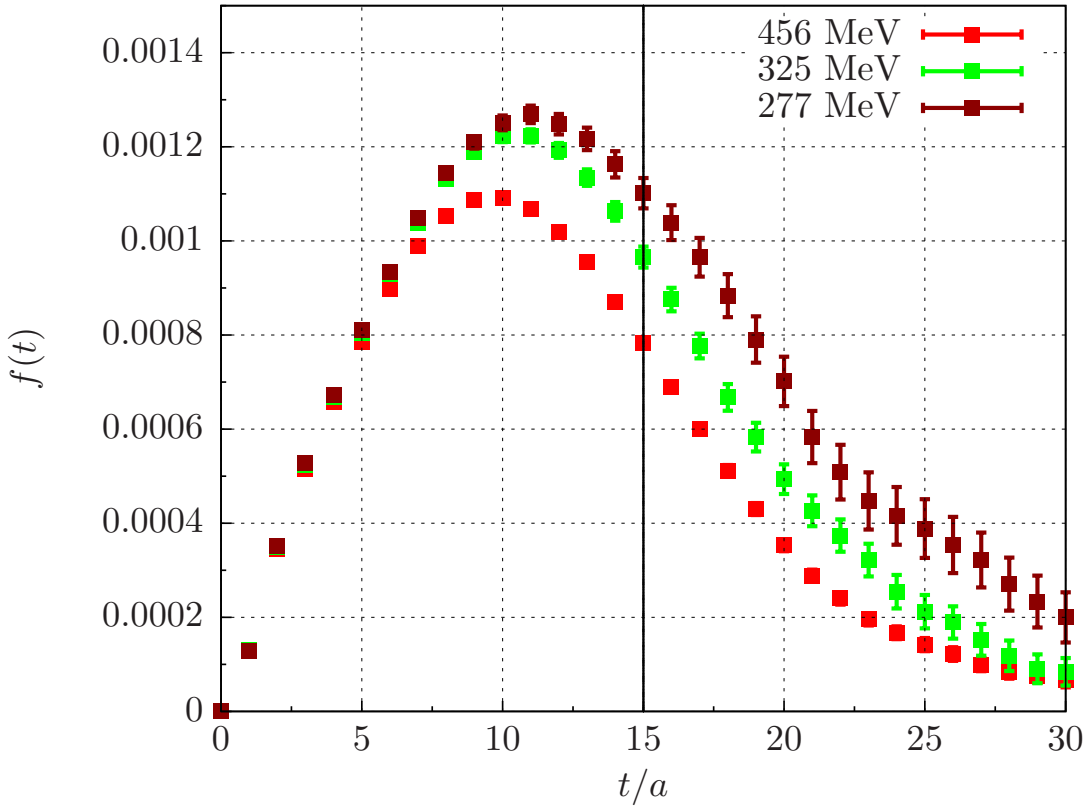
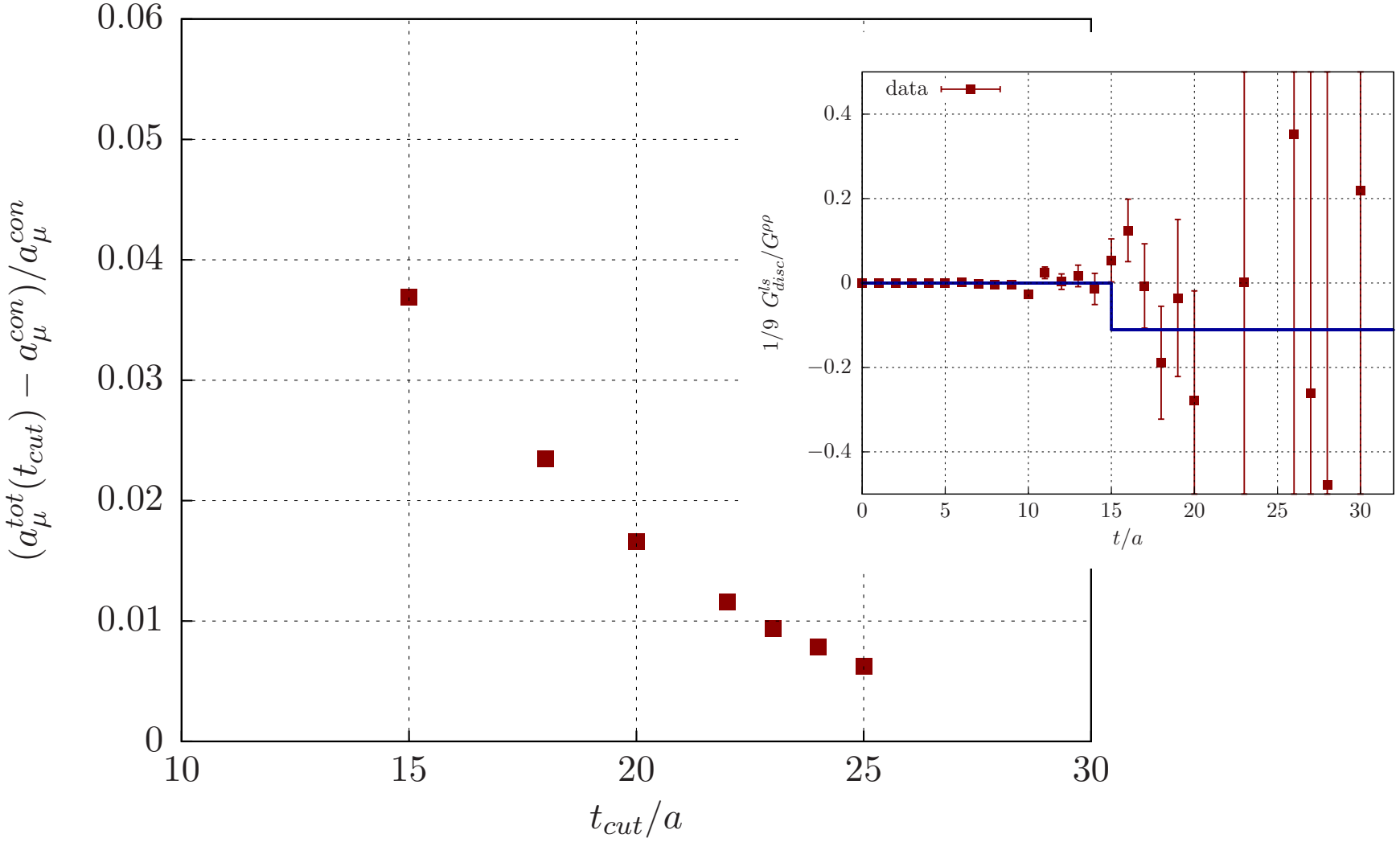
$$D_{sw}^{-1} = \sum_{i=0}^{k-1}\left(\frac{1}{2}A^{-1}H\right)^{i} A^{-1} + \left(\frac{1}{2}A^{-1}H\right)^{k} D_{sw}^{-1}$$

- D_{sw}^{-1} on the right hand side estimated using stochastic sources
- $\langle \text{loop} \rangle = \left\langle \sum_{\vec{x}} \mathrm{Tr}\left(D^{-1}(x,x)\right)\right\rangle_G$
- choose $N = 3$ sources with order $k = 6$ of the generalized HPE



The integrand for the vacuum polarization

- $$\hat{\Pi}(Q^2) = 4\pi^2 \int_0^\infty dt \underbrace{G^{\gamma\gamma}(t)\left[t^2 - \frac{4}{Q^2}\sin^2\left(\frac{1}{2}Qt\right)\right]}_{f(t)}$$



Can we resolve the disconnected to be $\lesssim$ **1%**