

Padé approximants and g-2 for the muon

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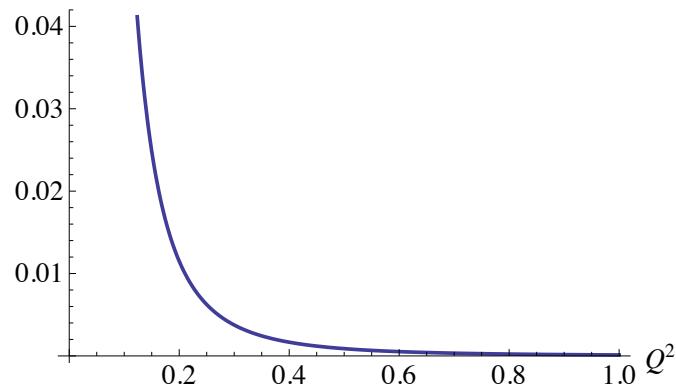
New Horizons for Lattice
Computations with Chiral Fermions
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Contribution from lowest-order hadronic vacuum polarization (HLO)

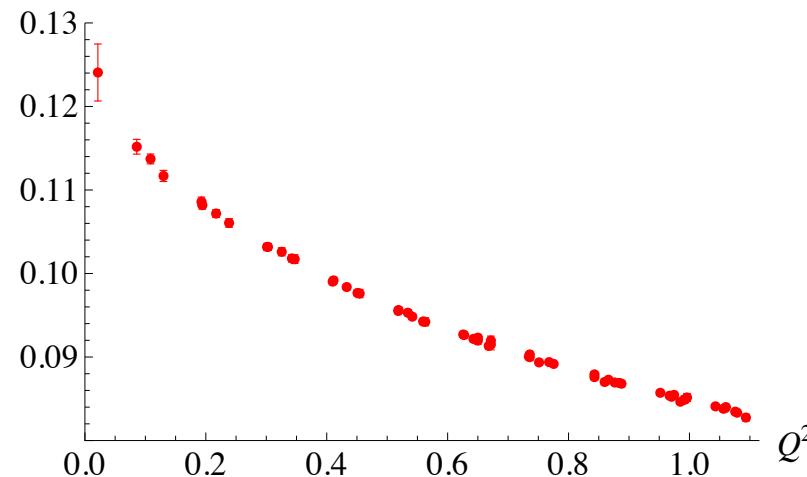
$$a_{\mu}^{\text{HLO}} = 4\alpha^2 \int_0^{\infty} dQ^2 f(Q^2) (\Pi(0) - \Pi(Q^2))$$

(Lautrup,
Peterman,
de Rafael 1971,
Blum 2002)

$$f(Q^2) =$$



Data for $\Pi(Q^2)$:



Need to fit data
to compute
integral;
VMD introduces
model dependence

Multi-point Padé approximants:

Write $(\Pi(0) - \Pi(Q^2)) / Q^2 = \int_{4m_\pi^2}^{\infty} dt \frac{\rho(t)}{t(t + Q^2)} \equiv \Phi(Q^2)$, $\rho(t) \geq 0$

This integral is a **Stieltjes** function, analytic everywhere except cut $(-\infty, -4m_\pi^2]$.

Theorem: Given P points $(Q_i^2, \Phi(Q_i^2))$ a sequence of PAs can be constructed which converge to $\Phi(Q^2)$ on any closed, bounded region of the complex plane excluding the cut, in the limit $P \rightarrow \infty$. (Baker 1969, Barnsley 1973)

$$\text{Construction: } \Phi(Q^2) = \frac{\Psi_0}{1 + \frac{(Q^2 - Q_1^2)\Psi_1}{1 + \frac{\ddots}{1 + \frac{(Q^2 - Q_{P-2}^2)\Psi_{P-2}}{(Q^2 - Q_{P-1}^2)\Psi_{P-1}}}}}$$

with Ψ_i related to $\Phi(Q_{j \leq i+1}^2)$ ($\Psi_0 = \Phi(Q_1^2)$, etc.), yields a $[(P-1)/2], [P/2]$ PA.

Parametrization and strategy

Furthermore, can prove that (Baker, Barnsley)

$$\Pi(Q^2) = \Pi(0) - Q^2 \left(a_0 + \sum_{n=1}^{[P/2]} \frac{a_n}{b_n + Q^2} \right)$$

with $a_n > 0$ (positive residues) and $b_{[P/2]} > \dots > b_1 > 4m_\pi^2$ (all poles on cut),
 $a_0 = 0$ for P even .

Fit this form for $P = 2, 3, 4, 5$; yields $[0, 1], [1, 1], [1, 2], [2, 2]$ PAs.

Compute $a_\mu^{\text{HLO}, Q^2 \leq 1} = 4\alpha^2 \int_0^{1 \text{ GeV}^2} dQ^2 f(Q^2) (\Pi(0) - \Pi(Q^2))$

Note: VMD is same as $[0, 1]$ PA with $b_1 = m_\rho^2$ fixed; NOT a (valid) PA!

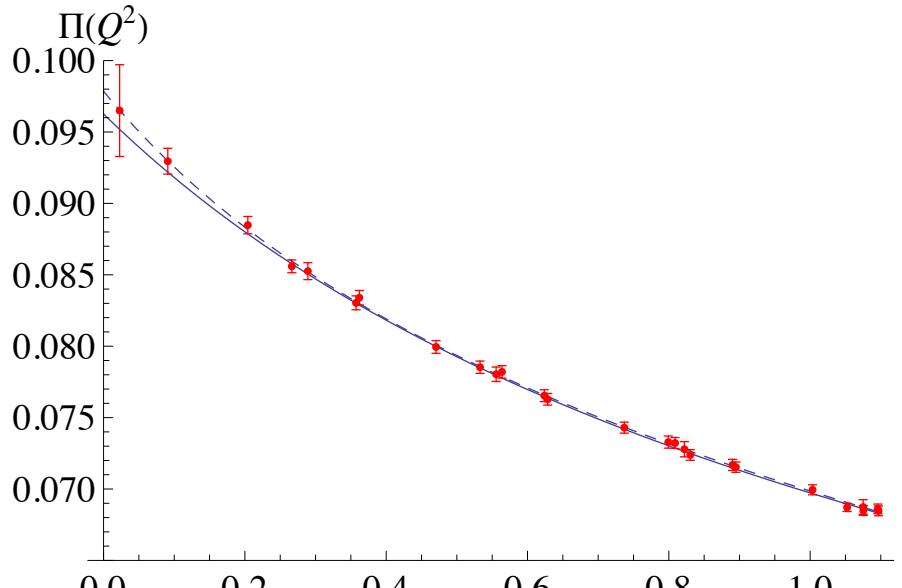
Test on MILC lattices with $a = 0.09$ fm , $m_\pi = 480$ MeV

		correlated interval $0 < Q^2 \leq 0.6$ GeV 2		uncorrelated interval $0 < Q^2 \leq 1$ GeV 2	
PA	# parameters	χ^2/dof	$10^{10} a_\mu^{\text{HLO}, Q^2 \leq 1}$	χ^2/dof	$10^{10} a_\mu^{\text{HLO}, Q^2 \leq 1}$
VMD	2	5.86/3*	363(7)	4.37/18	413(8)
[0, 1]	3	11.4/8	338(6)	3.58/17	373(37)
[1, 1]	4	7.49/7	350(8)	3.36/16	424(116)
[1, 2]	5	7.49/6	350(8)	3.35/15	443(293)
[2, 2]	6	7.49/5	350(7)	3.35/14	445(432)

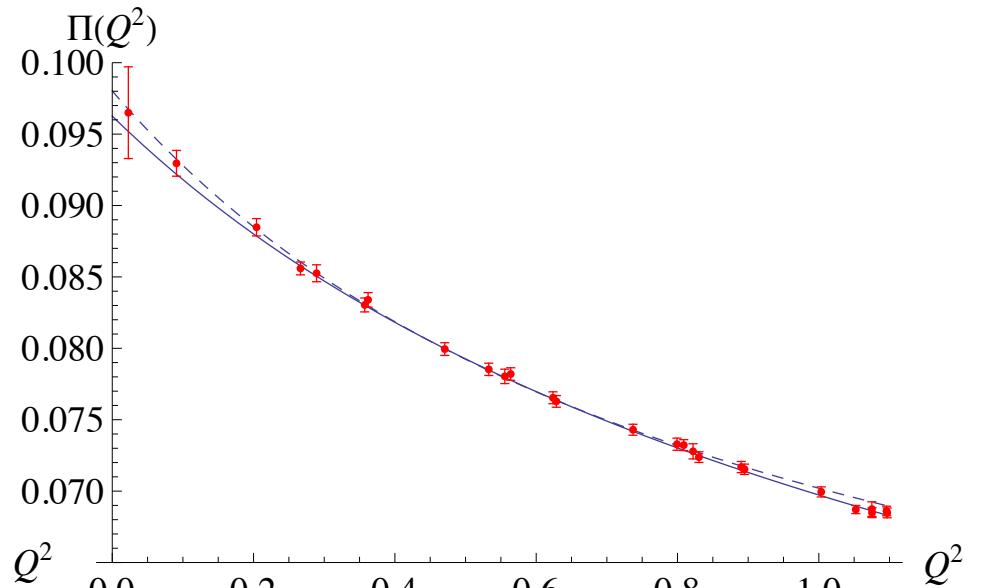
* interval $0 < Q^2 \leq 0.35$ GeV 2

uncorrelated VMD fit agrees with Aubin and Blum, 2007

- Correlated: VMD bad, clear improvement with addition of parameters
- Difficult to determine 2nd pole, but a_μ insensitive to higher poles
- Internal consistency, **except uncorr. VMD** (unknown systematic error!) and correlated PAs



[1,1] corr. (solid) and uncorr. (dashed)

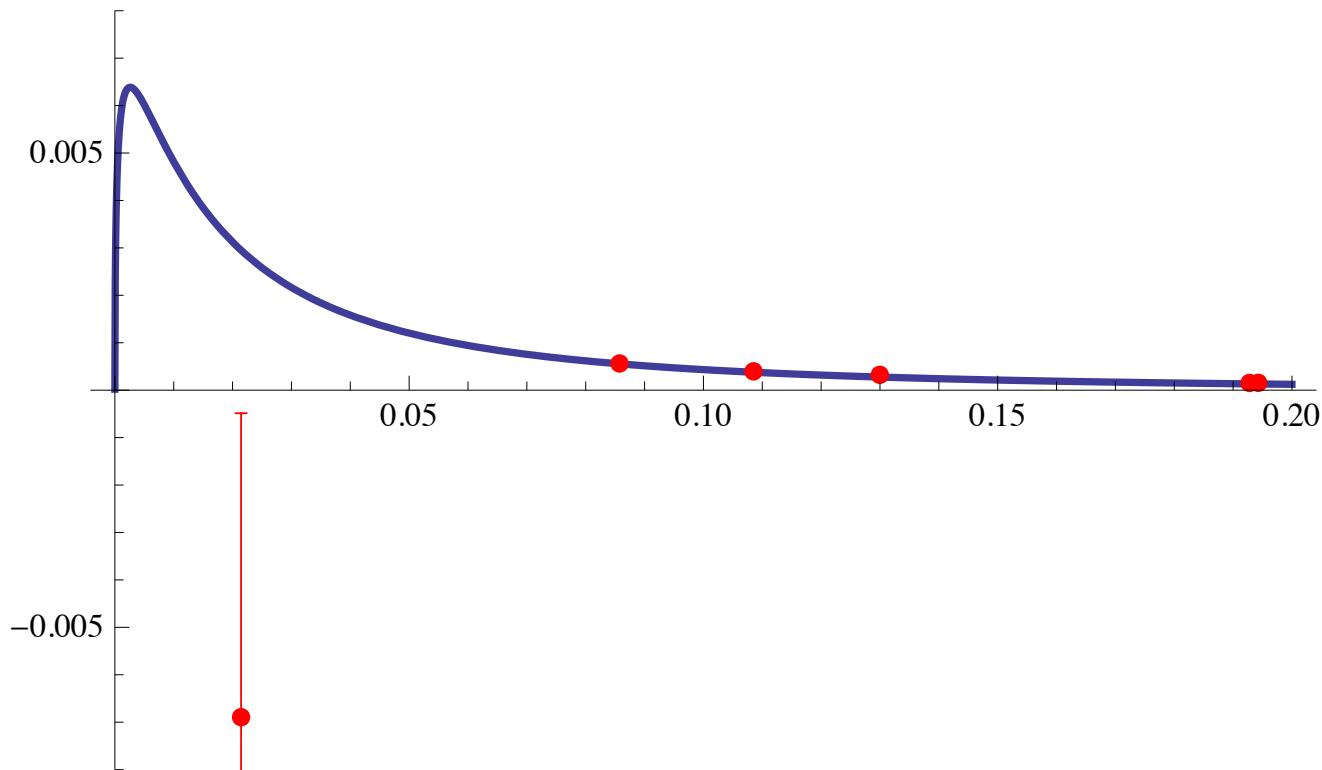


[1,1] corr. (solid) and VMD uncorr. (dashed)

- uncorrelated fits look better at small Q^2
- also considered MILC lattices with $a = 0.06$ fm , $m_\pi = 220$ MeV – similar

$$a_\mu^{\text{HLO}, Q^2 \leq 1} = 572(41) \times 10^{10} \text{ [1,1] corr. , } \quad a_\mu^{\text{HLO}, Q^2 \leq 1} = 646(8) \times 10^{10} \text{ VMD uncorr.}$$

- not possible to decide which fit is best, based on current data



Integrand of $a_\mu^{\text{HLO}}/(4\alpha^2)$ compared with data
(MILC, $a = 0.06$ fm , $m_\pi = 220$ MeV)

⇒ need more data at low Q^2 with smaller errors! In progress...

Conclusions

- New method to parametrize hadronic vacuum polarization;
avoid model dependence of vector meson dominance.
Based on representation of vacuum polarization in terms of Stieltjes function.
- Tested on two examples of numerical data for vacuum polarization.
Padé approximant fits can lead to larger statistical errors, but avoid
unknown systematic error associated with VMD.
- Method looks promising, but data at lower momenta and smaller errors
are indispensable (difference between $a_\mu(VMD)$ and $a_\mu([1, 1])$ is about 17%).
- Note: long chiral extrapolation – also need data with small pion mass!

Backup slide 1: comparison with polynomial fits

	Poly 3		Poly 4		PA [1,1]		PA [1,2]	
# points	χ^2/dof	$a_\mu^{(1)}$	χ^2/dof	$a_\mu^{(1)}$	χ^2/dof	$a_\mu^{(1)}$	χ^2/dof	$a_\mu^{(1)}$
16	9.6/12	543	9.5/11	483	9.7/12	564	9.7/11	565
18	11.4/14	526	10.5/13	596	11.2/14	541	11.5/13	561
20	13.1/16	536	13.1/15	535	13.9/16	572	13.9/15	572
22	16.5/18	541	15.9/17	513	18.5/18	566	18.5/17	566
24	16.6/20	537	16.4/19	521	19.4/20	583	19.4/19	583
26	30.7/22	505	23.6/21	580	26.8/22	557	26.7/21	560

- Poly 3, PA [1,1] and PA [1,2] correlated fits all good, not so Poly 4.
- Stability from PA [1,1] to PA [1,2], not from Poly 3 to Poly 4.

Backup slide 2: chiral extrapolation

Assume VMD, and approximate $\Pi(Q^2)_{subtr} = g_\rho^2 Q^2 / m_\rho^2$ (continuum)

$$\Pi(Q^2)_{subtr} = g_V^2 Q^2 / m_V^2 \text{ (lattice)}$$

Define $I_H = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w \left((Q^2/m_\mu^2)(H_{phys}^2/H_{latt}^2) \right) \Pi(Q^2)_{subtr}$
(Feng et al. 2011)

Then $I_H = \left(\frac{H_{latt}}{H_{phys}} \right)^2 \frac{g_V^2}{g_\rho^2} \frac{m_\rho^2}{m_V^2} I_{model}$ hence choose $H_{latt} = m_V/g_V$

Whatever choice: model dependent! 1st PA pole not equal to m_V^2
Cannot avoid small pion masses (much smaller than 300 MeV)