Individual eigenvalue distributions for chGSE→chGUE crossover & determination of low-energy constants in two-color QCD + QED

Shinsuke M. Nishigaki Shimane Univ

LATTICE2014 Columbia Univ. June 23, 2014

Two-color QCD

$\mathrm{SU}(N_C{=}2)$: ps-real rep, Pauli-Gursey enhanced flavor symmetry

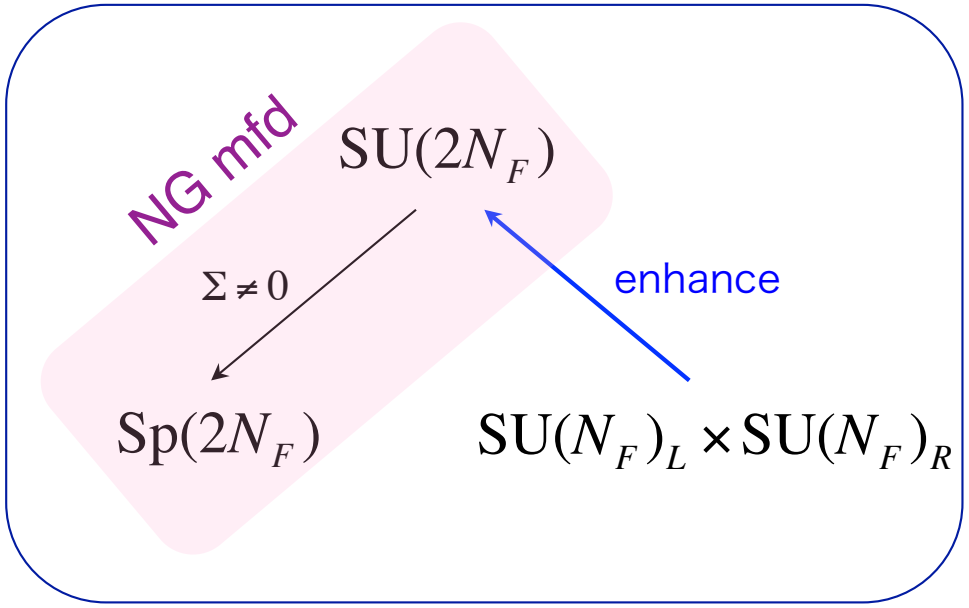
$$(\psi_L)_{i,\alpha}$$

$$(\psi_R)_i^{\dot\alpha} \xrightarrow[\sigma_2(\ )^*]{} (\psi_R^*)_\alpha^i \xrightarrow[\tau_2]{} (\psi_R^*)_{i,\alpha}$$

$$\Longrightarrow \quad \Psi = \left.\begin{pmatrix} \psi_L \\ \sigma_2\tau_2\psi_R^* \end{pmatrix}\right\} 2N_F$$

$$L = i\,\Psi^+ \not{D}\Psi$$

$$-\tfrac{1}{2} m_q \Psi^T \sigma_2\tau_2 \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \Psi \quad \Longrightarrow$$



$$\mathrm{SU/Sp} \cong \{\text{antisym. unitary}\}$$

$$V \;\mapsto\; U = V \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} V^T$$

Classification of RM

σ : **C**-herm. $\rightarrow$ **C**-herm., $\quad \sigma^2 = 1$

Involutive symmetry $\quad H = \sigma(H)$

none	**C**-herm.	GUE
$\sigma(H) = H^*$	**R**-sym.	GOE
$\sigma(H) = \tau_2 H^* \tau_2^{-1}$	**H**-selfdual	GSE
$\sigma(H) = -\gamma_5 H \gamma_5^{-1}$		chG*E
$\cdots$	$\cdots$	

classify by corresponding NLσM [Zirnbauer '96]

Dirac op as Random Matrix

σ : **C**-herm. $\rightarrow$ **C**-herm., $\quad \sigma^2 = 1$

Involutive symmetry $\quad H = \sigma(H)$

$$D_{\mathrm{U(1)}} = \gamma_\mu \left(i\partial_\mu + A_\mu \right) = -\gamma_5 D_{\mathrm{U(1)}} \gamma_5^{-1} \qquad \text{same symm. as chGUE}$$

$$D_{\mathrm{SU(2)\ fnd}} = \gamma_\mu \left(i\partial_\mu + A_\mu^a \tau_a \right) = (\tau_2 C)\, D^*_{\mathrm{SU(2)\ fnd}} (\tau_2 C)^{-1} \qquad \text{chGOE}$$

$$D_{\mathrm{SU(N)\ adj}} = \gamma_\mu \left(i\partial_\mu + A_\mu^a f_a \right) = C\, D^*_{\mathrm{SU(N)\ adj}}\, C^{-1} \qquad \text{chGSE}$$

small Dirac EVs described by chG*E [Verbaarschot '94]

note: chGOE$\leftrightarrows$chGSE for D_{stag}

Dirac Spectrum and LEETs

LECs determined by QCD dynamics

QCD

chPT

$$\left\langle \det\left(\lambda - D(A)\right)\cdots\right\rangle_{A_\mu} \xrightarrow{L^{-1} << \Lambda_{\mathrm{QCD}}} \int_{\mathrm{SU/SO}} DU \ \exp\left\{-\int dx\left(\frac{F^2}{2}\ \mathrm{tr}\left|\partial_\mu U\right|^2 - i\,\Sigma\,\mathrm{Re\ tr}\,\Lambda U\right)\right\}$$

$$\Big\downarrow F^2 L^{-2} >> \Sigma\lambda$$

chRMT

0D NLσM

$$\left\langle \det\left(\lambda - H\right)\cdots\right\rangle_{\mathrm{GAUSSIAN}\ H} \xrightarrow{\text{HS transf}} \int_{\mathrm{SU/SO}} dU \ \exp\left\{i\ V\Sigma\,\mathrm{Re\ tr}\,\Lambda\ U\right\}$$

EV density / indiv. EV distrib., fit to lattice D spectra

measure Σ with λ's as probes

II. Crossover in Spectral Statistics

Symmetry crossover of RM

$$H = H_0 + \alpha H_1 \qquad \sigma(H_0) = H_0 \qquad \sigma(H_1) \neq H_1 \qquad \upsilon = \sqrt{\mathrm{var}(H_{0,1})} \quad , \quad \alpha << 1$$

$$\alpha = O(\Delta/\upsilon)^{<1} \Rightarrow H_0 \text{ dominates}$$

$$\alpha = O(\Delta/\upsilon)^{>1} \Rightarrow H_1 \text{ dominates}$$

$$\alpha = O(\Delta/\upsilon)^{1} \Rightarrow \text{crossover, parametrized by } \rho \equiv \frac{\upsilon}{\Delta}\alpha$$

Δ : mean level spacing of EV window in concern

Dyson's Brownian motion of EVs .... Pfaff process

N particles at finite time $t(\rho)$, initial distribution ◂ H_0 , FP eq ◂ H_1

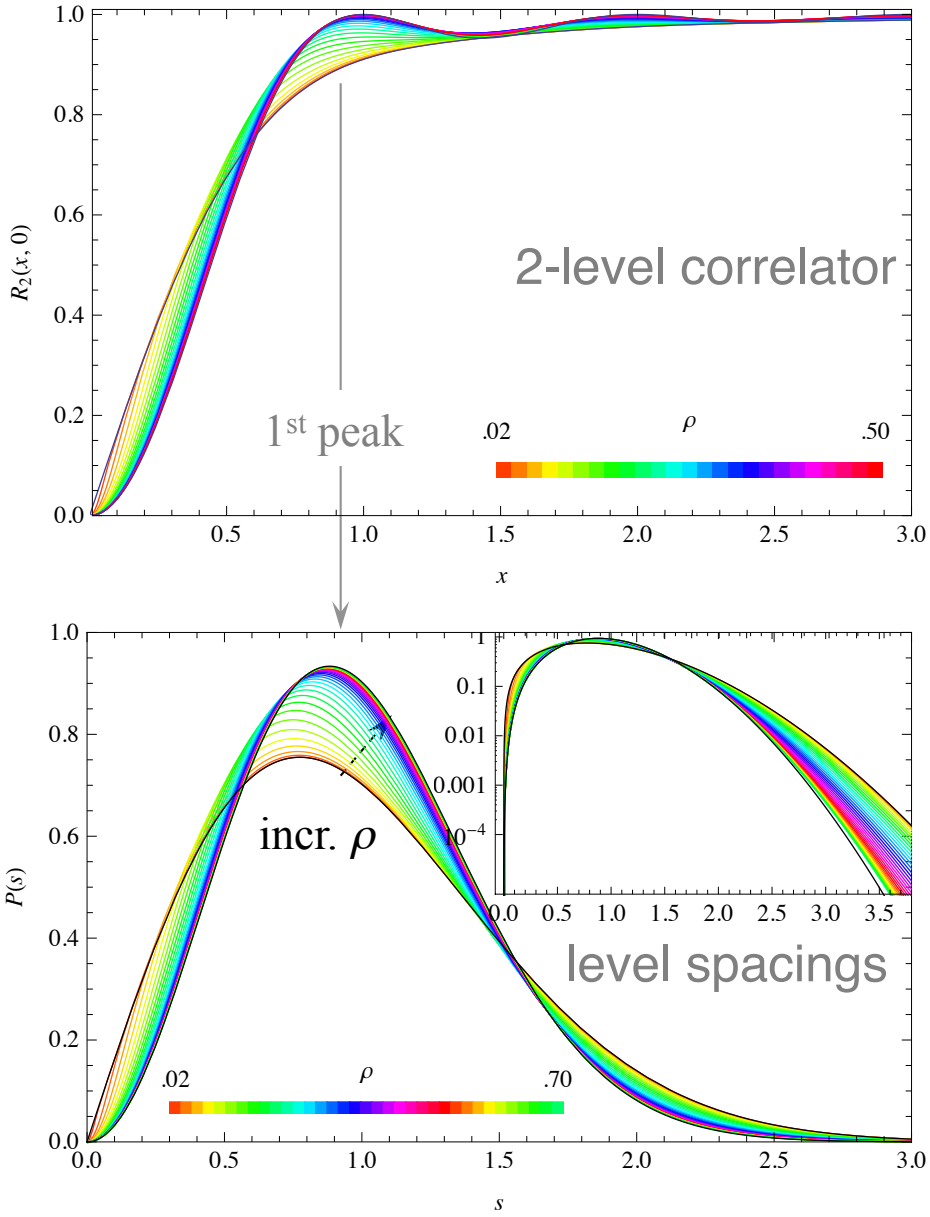
available techniques:

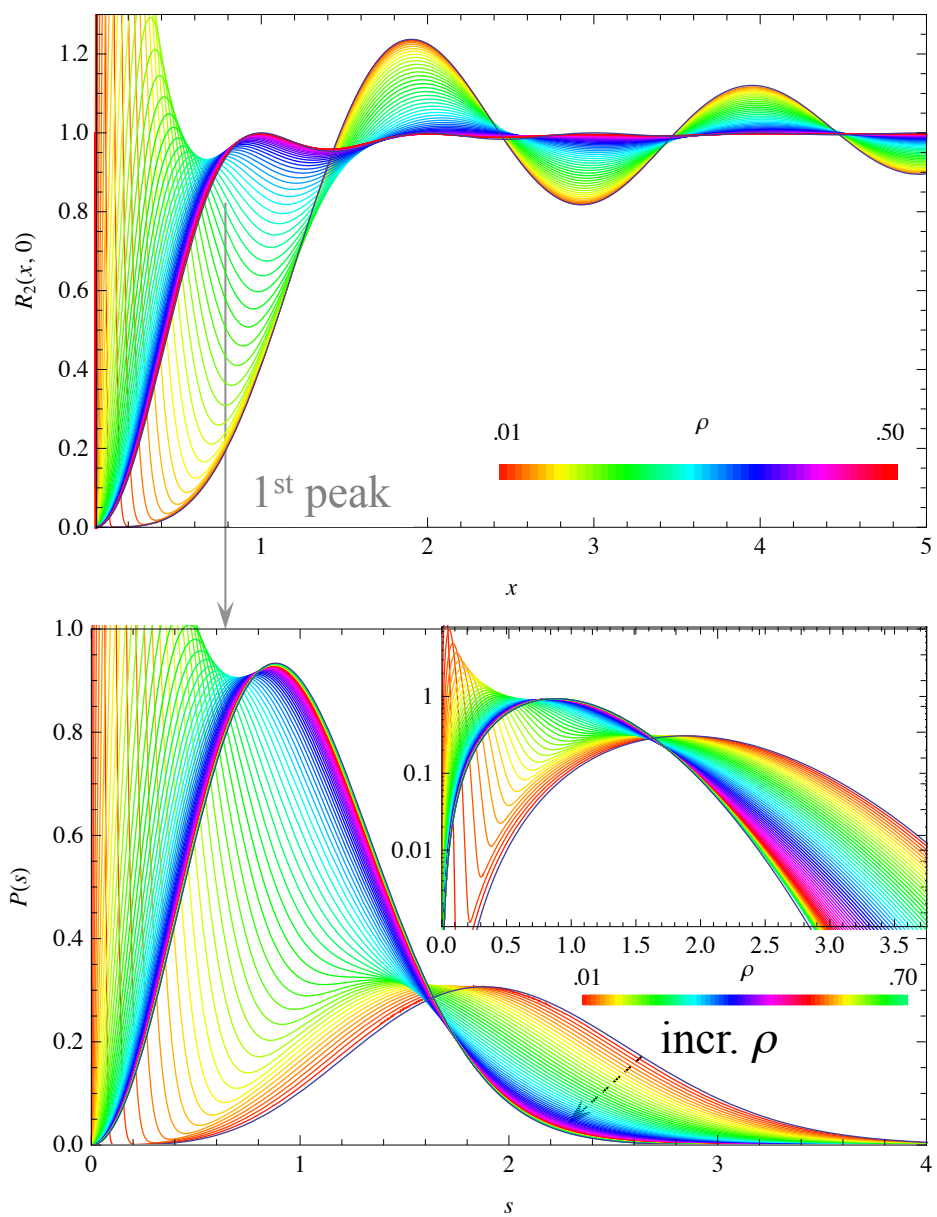
skew-orthogonal polynomials, SUSY NLσM

GOE-GUE (bulk)

GSE-GUE (bulk)

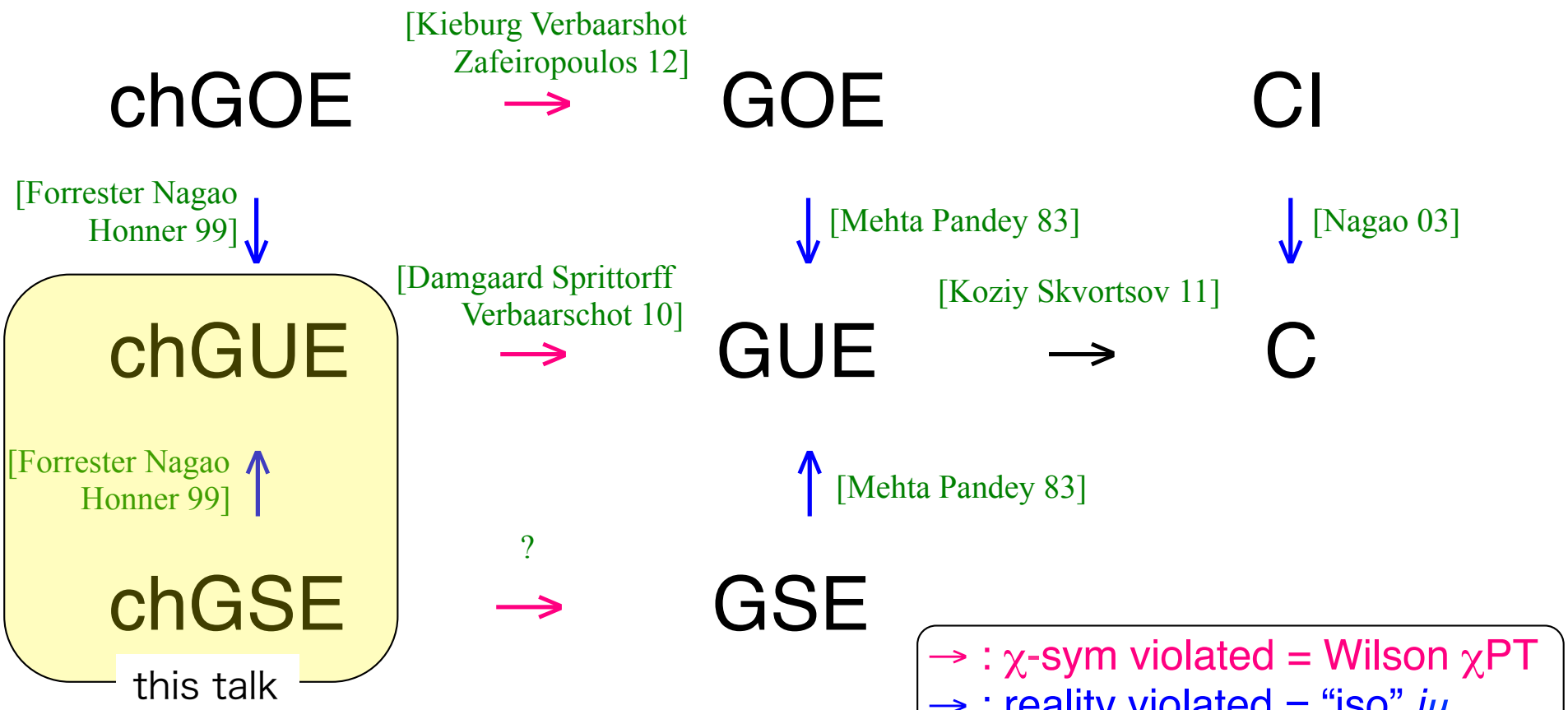
[Mehta Pandey 83]





Symmetry crossover : RM

$$H = H_0 + \alpha H_1 \qquad \sigma(H_0) = H_0 \qquad \sigma(H_1) \neq H_1$$



→ : χ-sym violated = Wilson χPT
→ : reality violated = "iso" $i\mu$

Symmetry crossover : RM

$$H = H_{\text{chGSE}} + \alpha\, H_{\text{chGUE}}$$

$$= \left(\begin{array}{c|c} & \mathbf{H} \\ \hline \mathbf{H}^{q-\text{dual}\cdot T} & \end{array}\right) + \alpha \left(\begin{array}{c|c} & \mathbf{C} \\ \hline \mathbf{C}^{+} & \end{array}\right)$$

$$H_0 = -\gamma_5 H_0 \gamma_5 = H_0^{\text{q-dual}\cdot T}$$

$$H_1 = -\gamma_5 H_1 \gamma_5 \neq H_1^{\text{q-dual}\cdot T}$$

$\mathbf{H}, \mathbf{C}$: Gaussian RM

p-EV correlator in Pfaff process

$$R_p(x_1,\ldots,x_p) = \mathrm{Pf}\left[K(x_i,x_j)\right]_{i,j=1}^{p} \quad , \quad K(x,y) = \begin{pmatrix} D(x,y) & -S(y,x) \\ S(x,y) & I(x,y) \end{pmatrix}$$

[chGSE-chGUE]

$$S(x,y) = \pi\sqrt{xy}\left\{\frac{xJ_1(\pi x)J_0(\pi y) - J_0(\pi x)yJ_1(\pi y)}{x^2-y^2} - \frac{J_0(\pi x)}{2}\int_0^{\pi} dv\, \mathrm{e}^{\rho^2(v^2-\pi^2)} J_0(vy)\right\}$$

$$D(x,y) = \frac{\sqrt{xy}}{2}\int_0^{\pi} dv\, v \int_0^1 du\, \mathrm{e}^{\rho^2 v^2(1+u^2)} \left\{J_0(vux)J_0(vy) - J_0(vx)J_0(vuy)\right\}$$

$$I(x,y) = \frac{\sqrt{xy}}{2}\int_{\pi}^{\infty} dv\, v^2\, \mathrm{e}^{-2\rho^2 v^2} \left\{xJ_1(vx)J_0(vy) - J_0(vx)yJ_1(vy)\right\}$$

[chGOE-chGUE] : compact-noncompact flip of the above

[Forrester Nagao Honner 99]

Nystrom-type approx. for Fredholm det

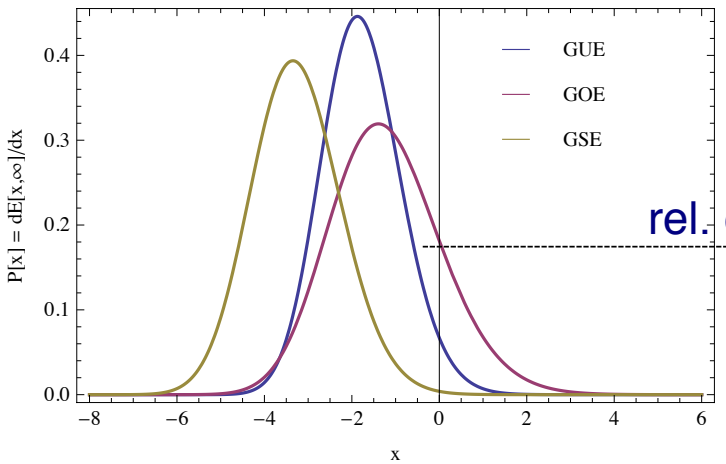
Gauss-Legendre Quadrature : $\{x_1,\ldots,x_M\}\in I,\quad \{w_1,\ldots,w_M\}>0$

$$\int_I f(x)\,dx \;\cong\; \sum_{i=1}^{M} f(x_i)w_i \;,\text{exact for } f(x)=x^M+\text{lower}$$

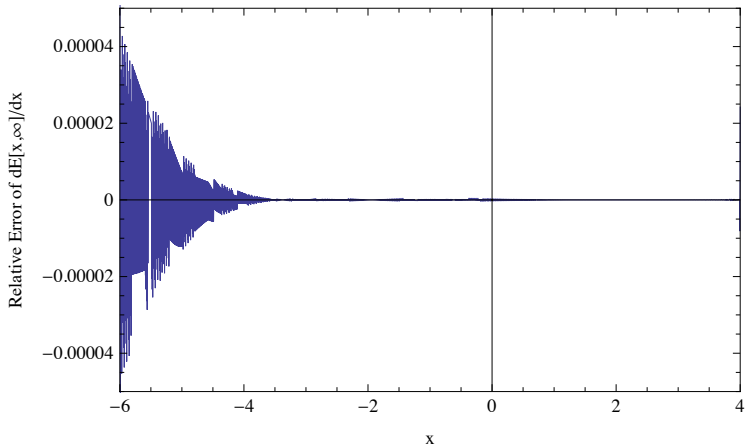
$$\mathrm{Det}\left(I-K_I\right) \;\cong\; \det\left[\delta_{ij}-K(x_i,x_j)\sqrt{w_i w_j}\right]_{i,j=1}^{M} \;+\text{relative error } O(\mathrm{e}^{-\mathrm{const}.M})$$

[Bornemann '10]

example: Nystrom approx (M=30) for K_Airy



rel. error



Symmetry crossover : Dirac op in QCD-like theory

$$H = \left(\begin{array}{c|c} & \mathbf{H} \\ \hline \mathbf{H}^{\text{q-dual } T} & \end{array}\right) + \alpha \left(\begin{array}{c|c} & \mathbf{C} \\ \hline \mathbf{C}^{+} & \end{array}\right), \qquad \left(\begin{array}{c|c} & \mathbf{R} \\ \hline \mathbf{R}^{T} & \end{array}\right) + \alpha \left(\begin{array}{c|c} & \mathbf{C} \\ \hline \mathbf{C}^{+} & \end{array}\right)$$

$\mathbf{H},\ \mathbf{C},\ \mathbf{R}$: Gaussian RM

symmetry & its breaking

$$D = \gamma_\mu \left(i\partial_\mu + g\, A_\mu^{\mathrm{SU}(2)} + e\, A_\mu^{\mathrm{U}(1)} \right), \qquad \gamma_\mu \left(i\partial_\mu + g\, A_\mu^{\mathrm{SU}(N)\ \mathrm{Adj}} + e\, A_\mu^{\mathrm{U}(1)} \right)$$

dynamical (QED) or constant b.g. $= i\mu$

Symmetry crossover : Dirac op in QCD-like theory

$\mathrm{SU}(N_C{=}2)$: Pauli-Gursey enhanced flavor symmetry

$$(\psi_L)_{i,\alpha}$$

$$(\psi_R)_i^{\dot\alpha} \xrightarrow[\sigma_2(\ )^*]{} (\psi_R^*)_\alpha^i \xrightarrow[\tau_2]{} (\psi_R^*)_{i,\alpha}$$

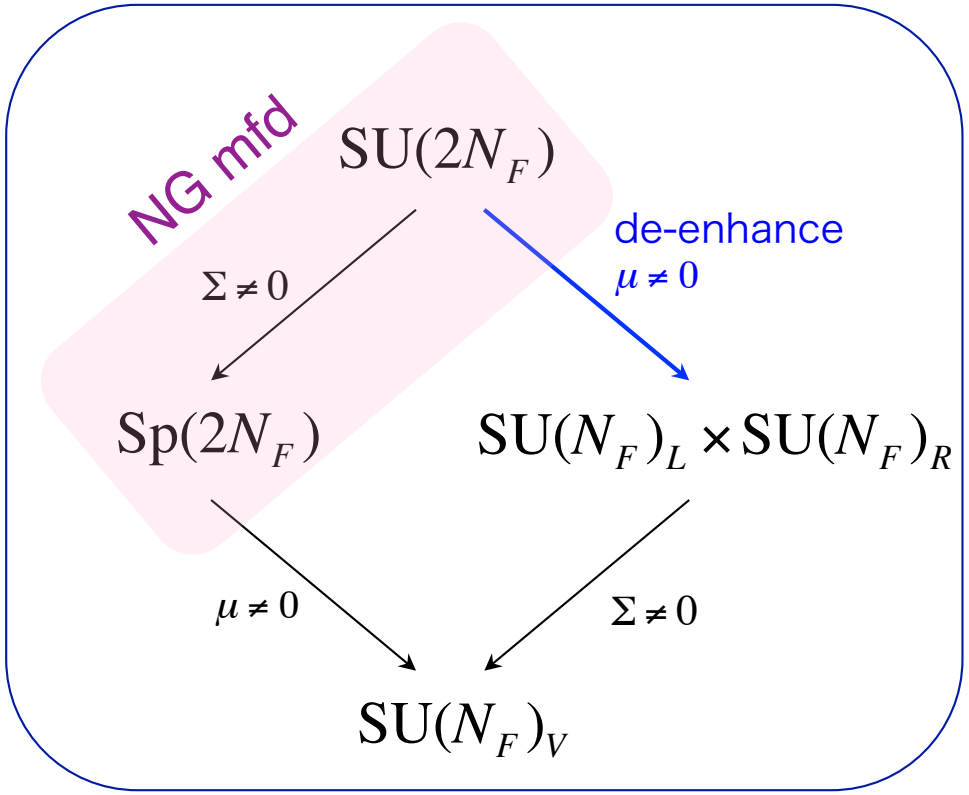
$$\Longrightarrow \quad \Psi = \left.\begin{pmatrix} \psi_L \\ \sigma_2\tau_2\psi_R^* \end{pmatrix}\right\} 2N_F$$

$$L = i\,\Psi^+ \not{D}\Psi \quad + \text{mass term}$$

$$+ i\,\mu\,\Psi^+\gamma_0 \underbrace{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}_{\hat{B}} \Psi$$

$$= i\,\Psi^+ \underbrace{\left(\not{D} + \mu\hat{B}\gamma_0\right)}_{\not{\nabla}} \Psi \quad + \text{mass}$$

$$\Longrightarrow$$



"Flavor-covariantize"

Dirac Spectrum and LEETs

[Kogut Toublan Stephanov 00]

QCD

chPT

LECs

$$\left\langle \det\left(\lambda - D(A)^{(\mu)}\right)\cdots\right\rangle_{A_\mu} \xrightarrow{L^{-1} << \Lambda_{QCD}} \int_{SU/SO} DU \; \exp\left\{-\int dx \left(\frac{F^2}{2}\,\mathrm{tr}\left|\nabla_\mu U\right|^2 - i\,\Sigma\,\mathrm{Re\,tr}\,\Lambda U\right)\right\}$$

$$\nabla_\mu = \partial_\mu + \mu[\hat{B},\ ]\delta_{\mu,0}$$

$$F^2 L^{-2} >> \Sigma\lambda$$

chRMT

0D NLσM

$$\left\langle \det\left(\lambda - (H_0 + \alpha H_1)\right)\cdots\right\rangle_{\text{GAUSSIAN } H} \xrightarrow{\text{HS transf}} \int_{SU/SO} dU \; \exp\left\{\begin{array}{l} VF^2\mu^2\,\mathrm{tr}\,(\hat{B}U^+\hat{B}U + \hat{B}\hat{B}) \\ +\ iV\Sigma\,\mathrm{Re\,tr}\,\Lambda\,U \end{array}\right\}$$

fit individual EV distrib. to LGT

measure Σ, F with λ's , μ as probes

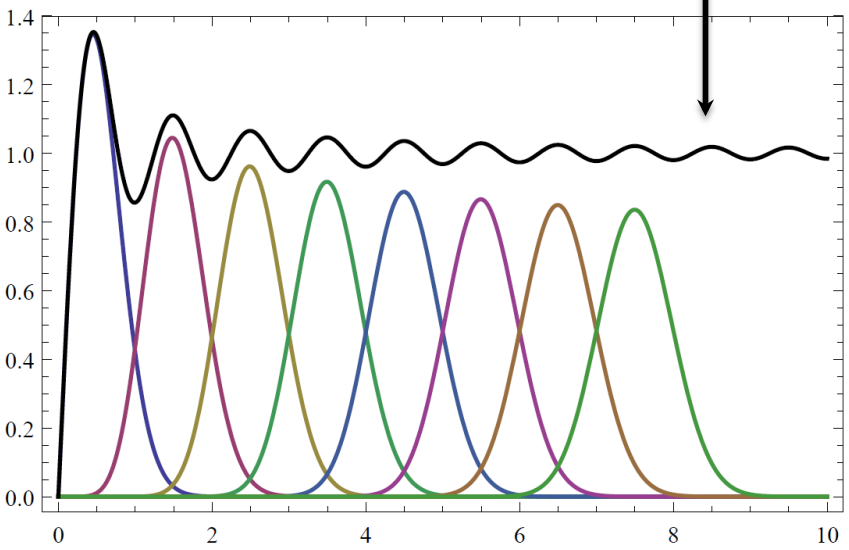
III. Individual EV distribution

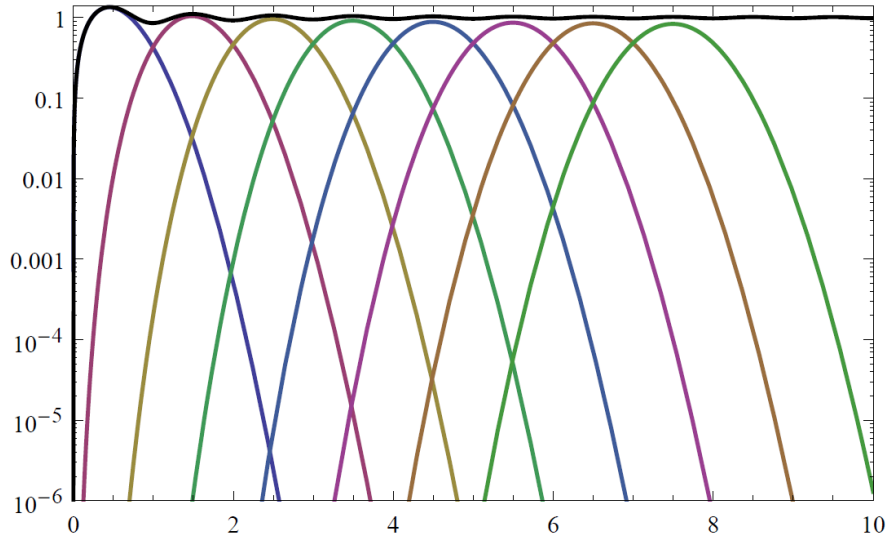
EV density vs Individual distributions

exercise: EV density & $1^{st} \sim 8^{th}$ EV distributions of chGUE

[Damgaard-SMN '01]

flattening oscillation ⇒ larger error for fitting



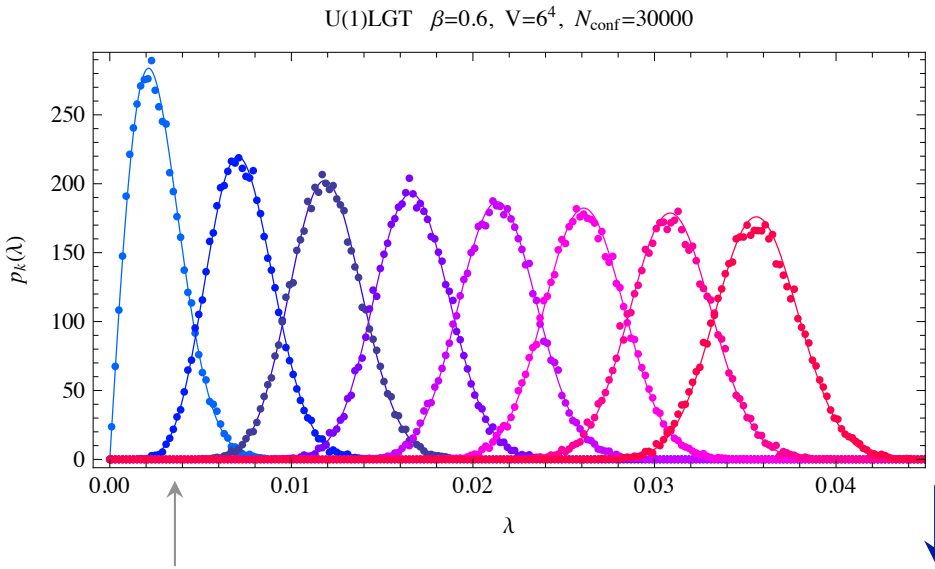


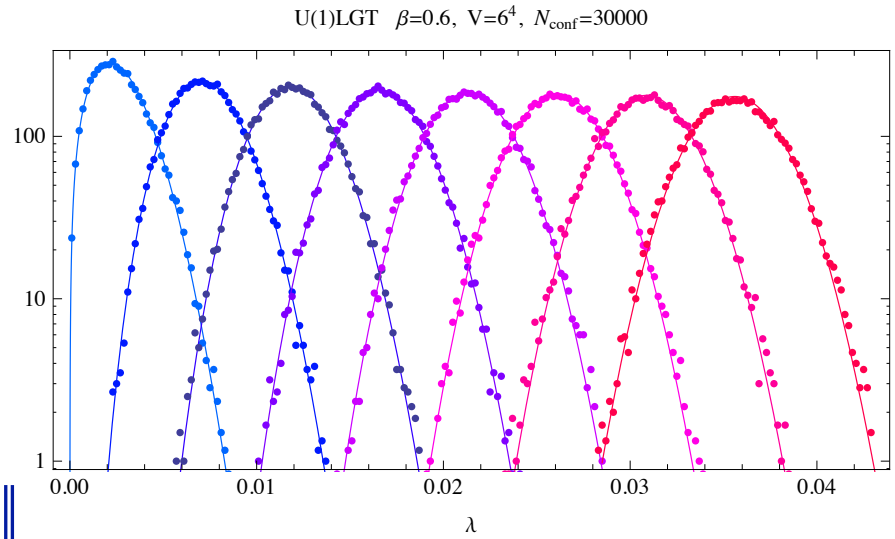
almost Gaussian ⇒ precise fitting possible

EV density vs Individual distributions

exercise: U(1) Dirac spectrum vs chGUE

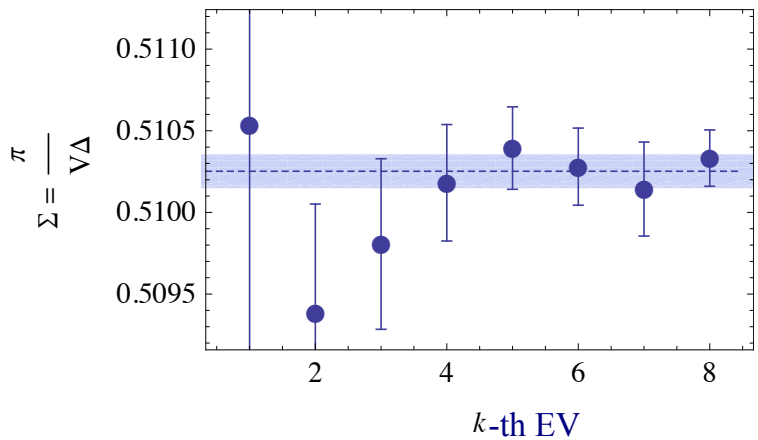
[SMN unpub.]





best-fit curves by adjusting m.l.s. Δ

chiral condensate



$$\Sigma a^3 = 0.51025(10)$$

Gap probabilty

$$\mathrm{Prob}\left[\text{any EV} \notin I\right] = \left(\int_{\mathbf{R}} - \int_I d\lambda_1\right) \cdots \left(\int_{\mathbf{R}} - \int_I d\lambda_N\right) P(\lambda_1, \ldots, \lambda_N)$$

$$= \int_{\mathbf{R}} d\lambda_1 \cdots d\lambda_N P(\lambda_1, \ldots, \lambda_N) - \int_I d\lambda_1 \underbrace{{}_N C_1 \int_{\mathbf{R}} d\lambda_2 \cdots d\lambda_N P(\lambda_1, \ldots, \lambda_N)}_{\text{1-level correlator}} + \int_I d\lambda_1 d\lambda_2 \underbrace{{}_N C_2 \int_{\mathbf{R}} d\lambda_3 \cdots d\lambda_N P(\lambda_1, \ldots, \lambda_N)}_{\text{2-level correlator}} - \cdots$$

$$= 1 - \int_I d\lambda_1 \mathrm{Pf}\left[K(\lambda_1, \lambda_1)\right] + \int_I d\lambda_1 d\lambda_2 \ \mathrm{Pf}\begin{bmatrix} K(\lambda_1, \lambda_1) & K(\lambda_1, \lambda_2) \\ K(\lambda_2, \lambda_1) & K(\lambda_2, \lambda_2) \end{bmatrix} - \cdots$$

$$= \sqrt{\mathrm{Det}\left(J - K_I\right)} \qquad (K_I \circ \mathbf{f})(\lambda) := \int_I d\lambda' \ K(\lambda, \lambda') \cdot \mathbf{f}(\lambda')$$

$$\mathrm{Prob}\left[k \text{ EVs} \in I\right] = \frac{1}{k!}\left(-\partial_\xi\right)^k \sqrt{\mathrm{Det}\left(J - \xi K_I\right)}\Big|_{\xi=1} = \sqrt{\mathrm{Det}\left(J - K_I\right)} \times \left(\text{resolvents of } K_I\right)$$

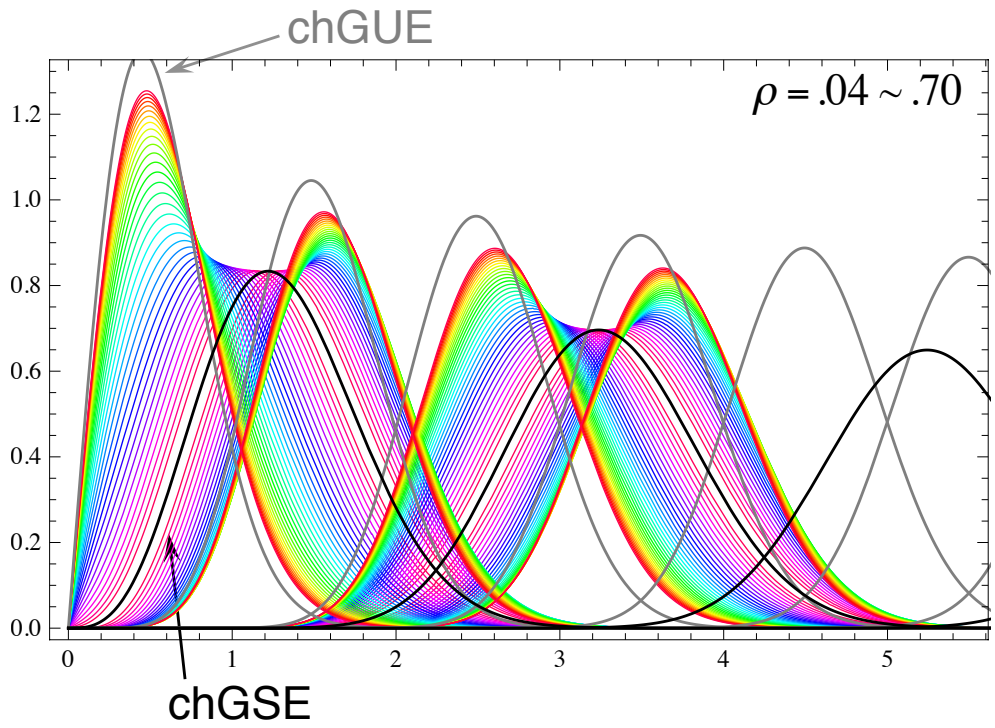
$$\mathrm{tr}\left[K_I \left(J - K_I\right)^{-1}\right]^k \text{'s}$$

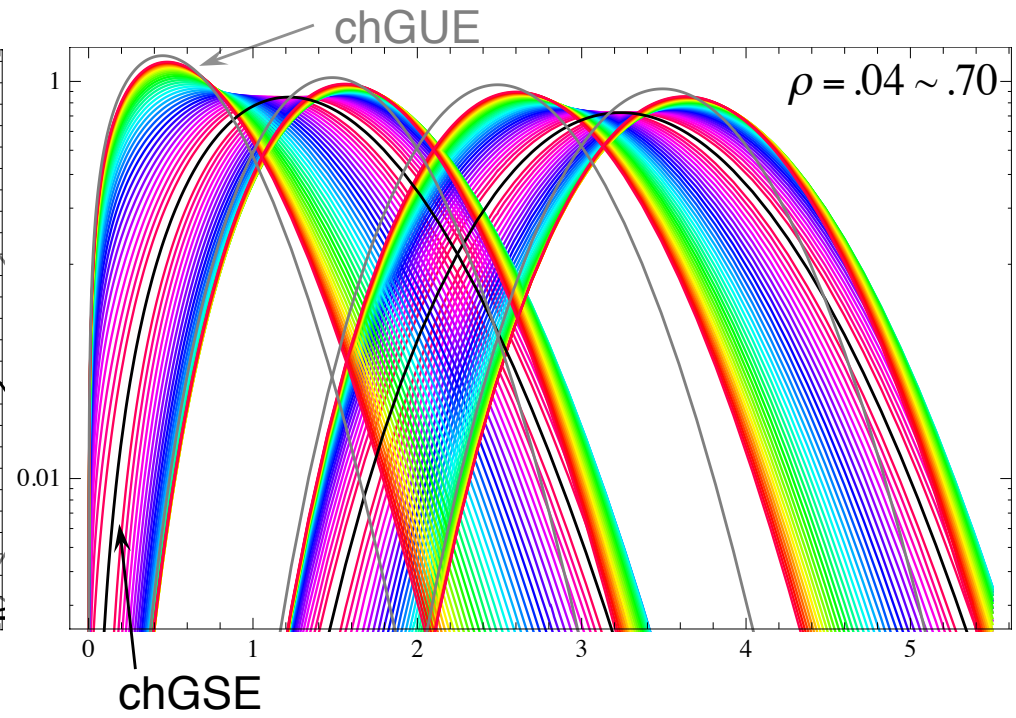
$$\Rightarrow \quad k\text{th EV distribution}: \ p_k(s) = \frac{d}{ds} \sum_{j=0}^{k-1} \mathrm{Prob}\left[j \text{ EVs} \in [0, s]\right]$$

Fredholm det & resolvents $\Leftarrow$ Quadrature method applicable [Bornemann '10]

Individual Dirac EV distribution

1st ~4th EV distributions for chGSE-chGUE crossover



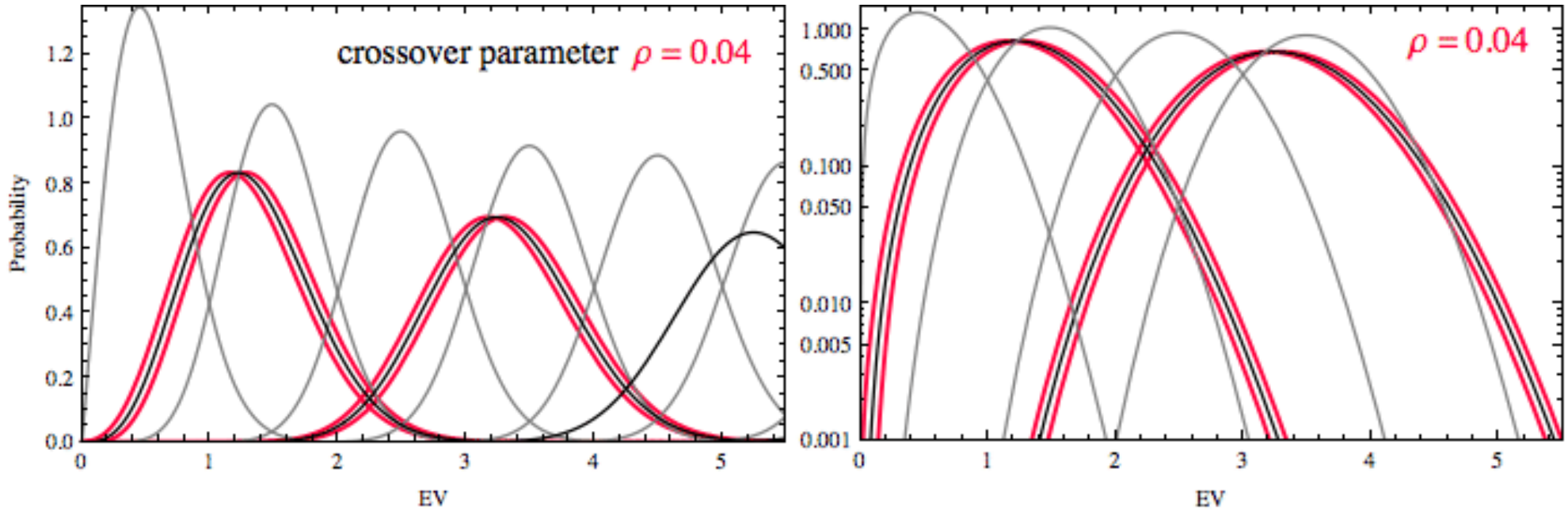


$$H = \left(\begin{array}{c|c} & \mathbf{H} \\ \hline \mathbf{H}^D & \end{array}\right) + \alpha \left(\begin{array}{c|c} & \mathbf{C} \\ \hline \mathbf{C}^\dagger & \end{array}\right), \qquad \rho \equiv \frac{v}{\Delta}\alpha$$

[SMN '12/14]

Individual Dirac EV distribution

1st ~4th EV distributions for chGSE-chGUE crossover



breaking Kramers degeneracy

$$H = \left(\begin{array}{c|c} & \mathbf{H} \\ \hline \mathbf{H}^D & \end{array}\right) + \alpha \left(\begin{array}{c|c} & \mathbf{C} \\ \hline \mathbf{C}^\dagger & \end{array}\right), \qquad \rho \equiv \frac{v}{\Delta}\alpha$$

[SMN '12/14]

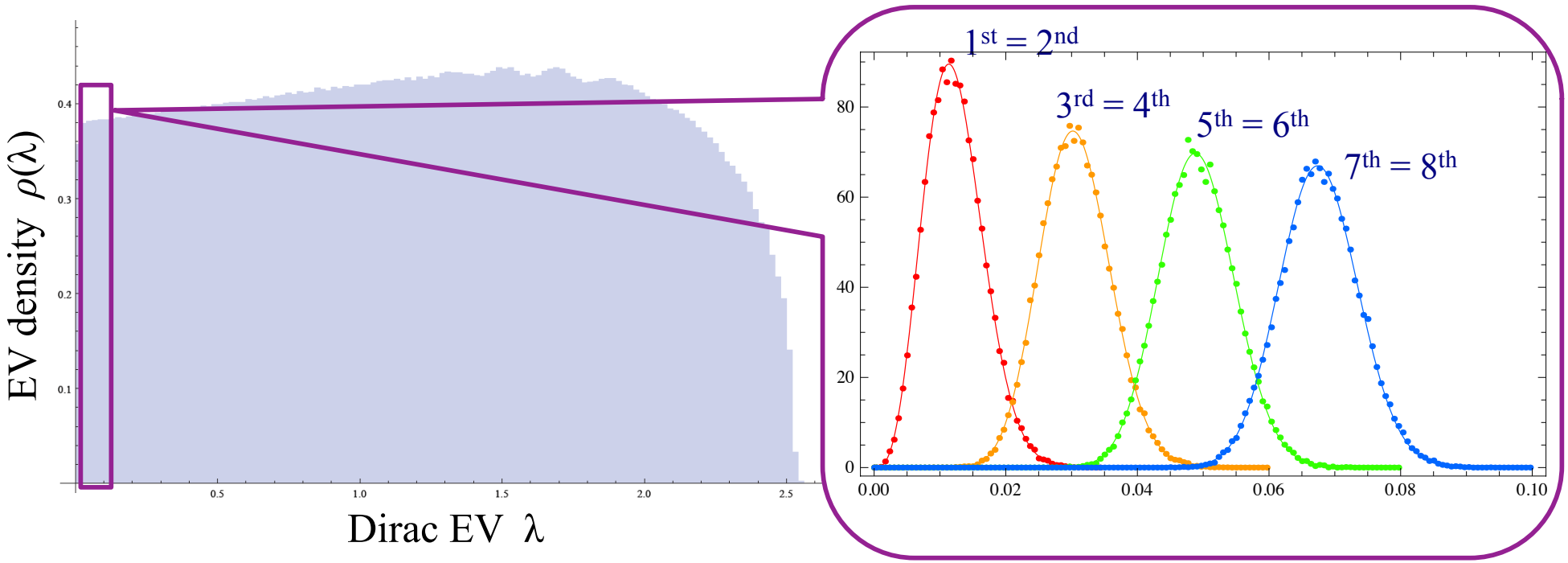
IV. Lattice Dirac Spectrum

Measurement

Spec(D_{stag}) of SU(N_C=2) ×

U(1)	V	parameter range	$\beta_{SU(2)}$	N_{conf}
noncompact QED	6^4	$e_{U(1)} = 0.0004\sim.0024$	0~1.75	$1\cdot10^4$
	4^4	$e_{U(1)} = 0.002\sim.012$	0~1.5	$4\cdot10^4$
const background	6^4	$i\mu = 0.005\sim.042$	0~1.75	$1\cdot10^4$
	4^4	$i\mu = 0.016\sim.094$	0~1.5	$4\cdot10^4$

step 1: pure SU(2) → fit $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$ to chGSE → mean spacing Δ

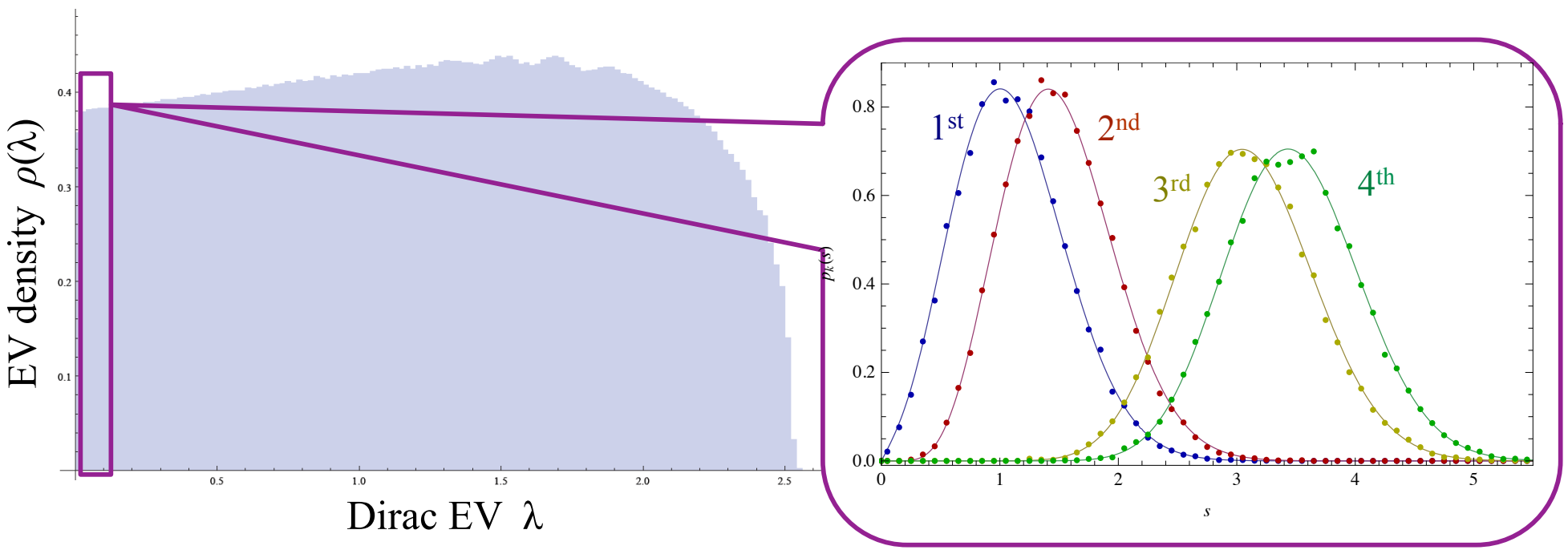


Measurement

Spec(D_{stag}) of SU(N_C=2) ×

U(1)	V	parameter range	$\beta_{SU(2)}$	N_{conf}
noncompact QED	6^4	$e_{U(1)} = 0.0004 \sim .0024$	0~1.75	$1 \cdot 10^4$
	4^4	$e_{U(1)} = 0.002 \sim .012$	0~1.5	$4 \cdot 10^4$
const background	6^4	$i\mu = 0.005 \sim .042$	0~1.75	$1 \cdot 10^4$
	4^4	$i\mu = 0.016 \sim .094$	0~1.5	$4 \cdot 10^4$

step 2: include $i\mu$ or U(1) → fit λ_1/Δ, ... λ_4/Δ to chGSE-chGUE crossover → parameter ρ



Measurement

Spec(D_{stag}) of SU(N_C=2) ×

U(1)	V	parameter range	$\beta_{SU(2)}$	N_{conf}
noncompact QED	6^4	$e_{U(1)} = 0.0004\sim.0024$	0~1.75	$1\cdot10^4$
	4^4	$e_{U(1)} = 0.002\sim.012$	0~1.5	$4\cdot10^4$
const background	6^4	$i\mu = 0.005\sim.042$	0~1.75	$1\cdot10^4$
	4^4	$i\mu = 0.016\sim.094$	0~1.5	$4\cdot10^4$

step 3: convert Δ, ρ to LECs Σ, F by chRMT⇔NLσM correspondence

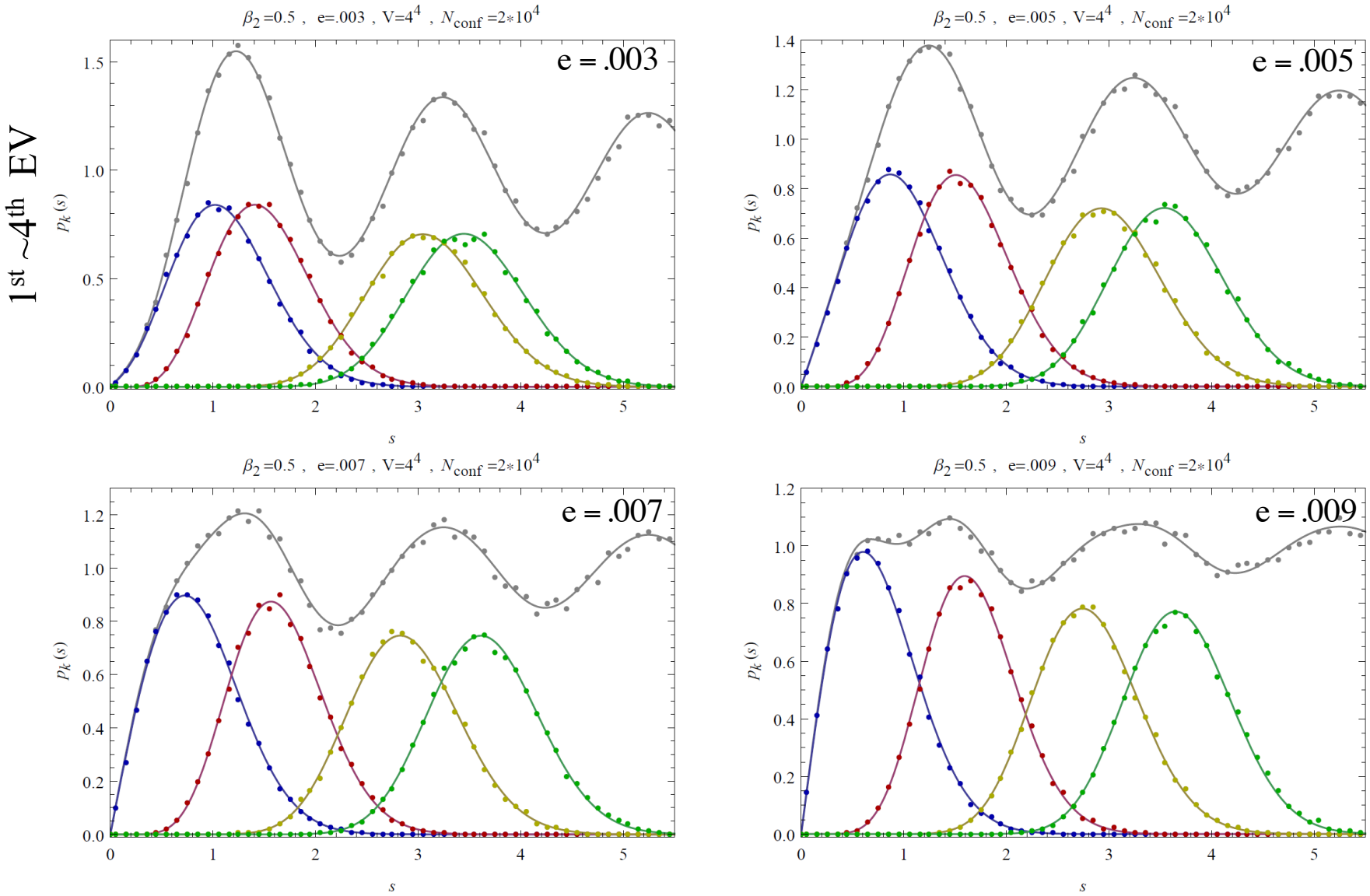
chRMT NLσM

$$\left\langle \det\left(\lambda - (H + \alpha H')\right)\cdots\right\rangle_{\text{GAUSSIAN}} \xrightarrow{\text{HS transf}} \int_{\text{SU/SO}} dU \, \exp\left\{ \begin{array}{l} VF^2\mu^2 \, \text{tr}\,(\hat{B}U^+\hat{B}U + \hat{B}\hat{B}) \\ + \; iV\Sigma \, \text{Re tr}\, \Lambda U \end{array} \right\}$$

$$\Sigma = \frac{\pi}{V\Delta}, \quad F = \sqrt{\frac{\pi}{2V}}\frac{\rho}{\mu}$$

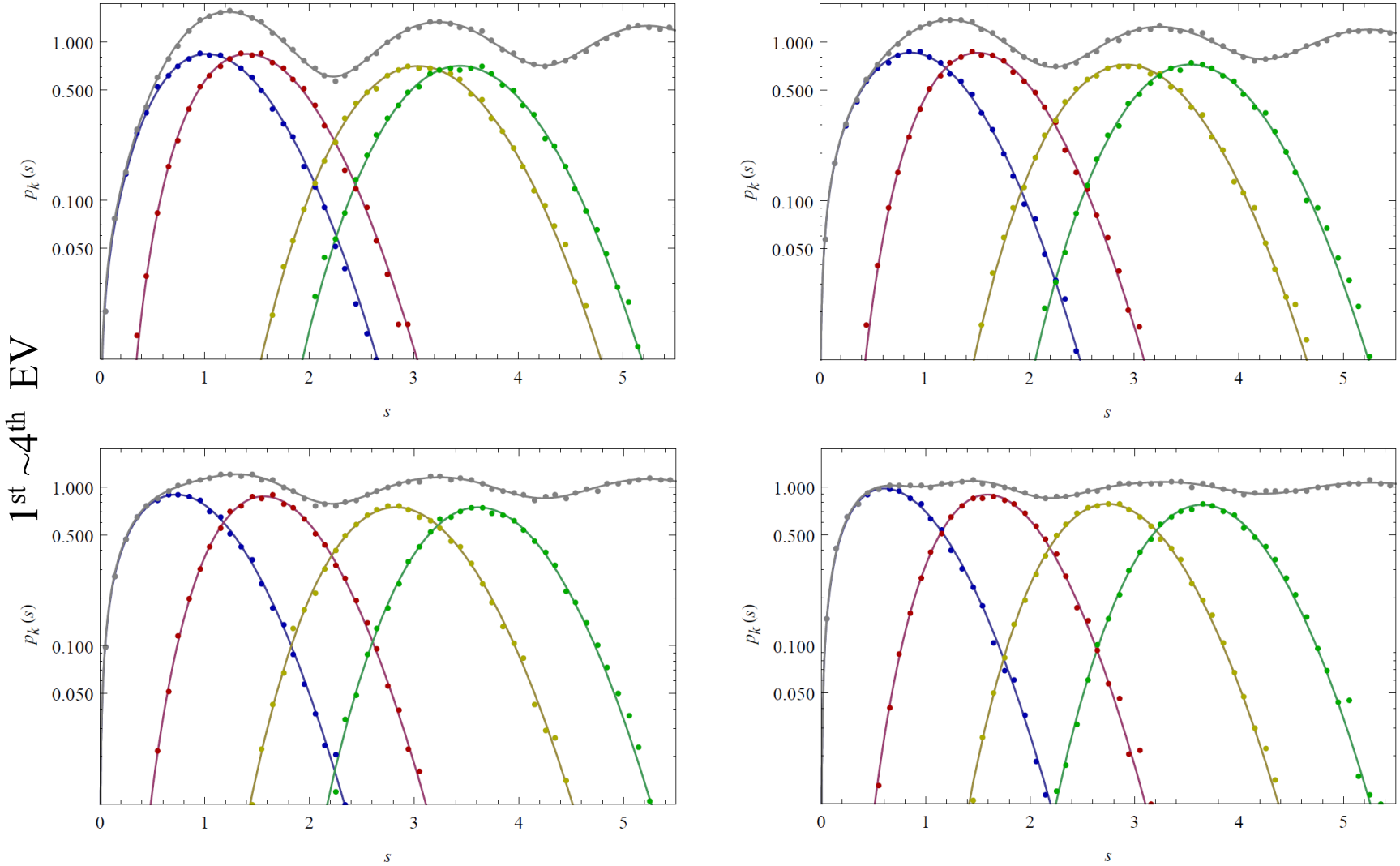
SU(2) × U(1)

example: $\beta_{SU(2)} = 0.5,\ V = 4^4$



SU(2) × U(1)

example: $\beta_{SU(2)} = 0.5,\ V = 4^4$

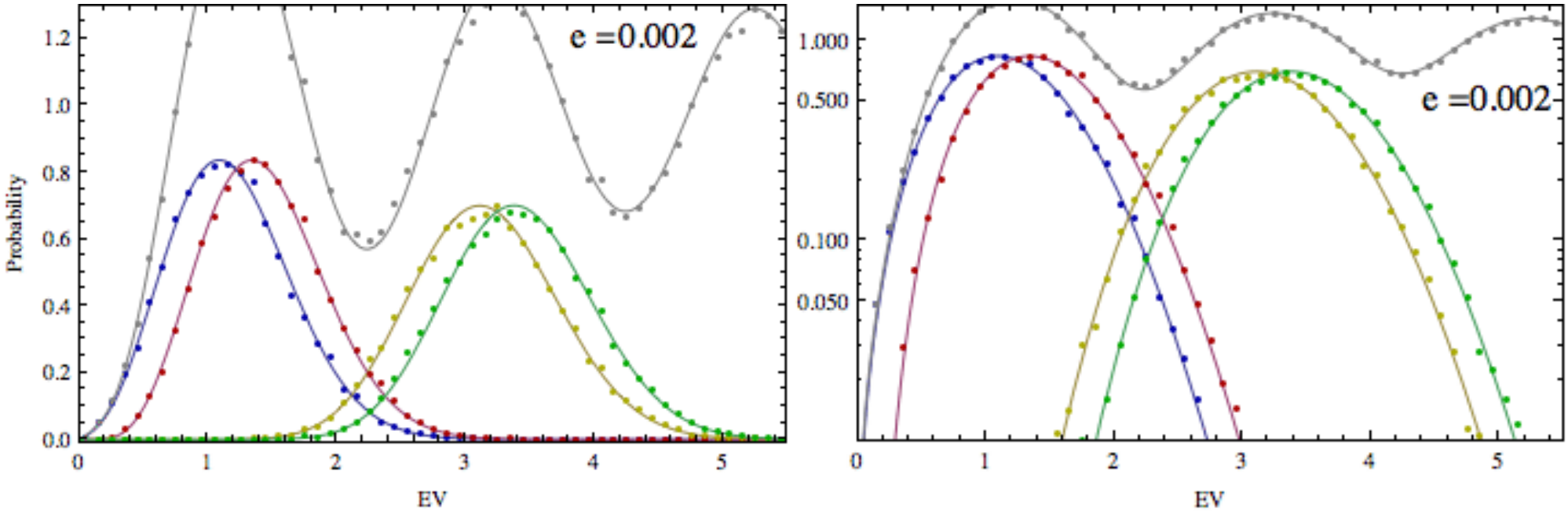


$1^{st} \sim 4^{th}$: $\chi^2/\text{d.o.f} = 0.6 \sim 1.5$

SU(2) × U(1)

example: $\beta_{SU(2)} = 0.5,\ V = 4^4$

$1^{st} \sim 4^{th}$ EV

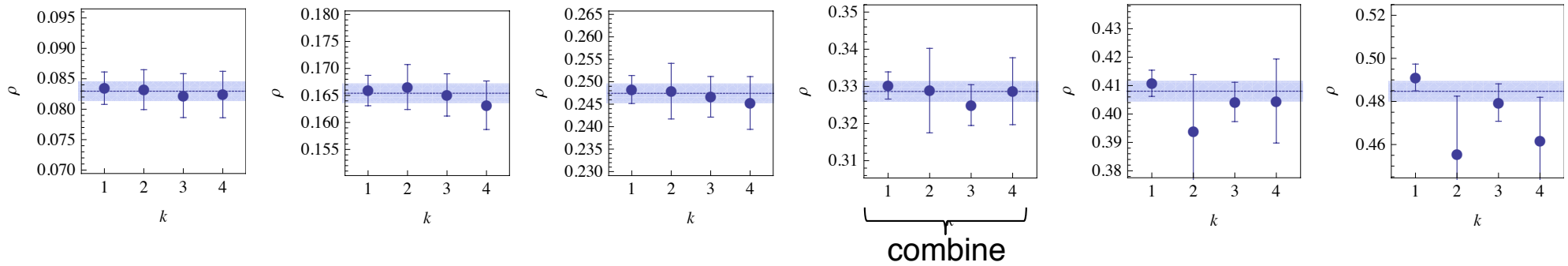


$1^{st} \sim 4^{th}$: $\chi^2/\text{d.o.f} = 0.5 \sim 1.5$

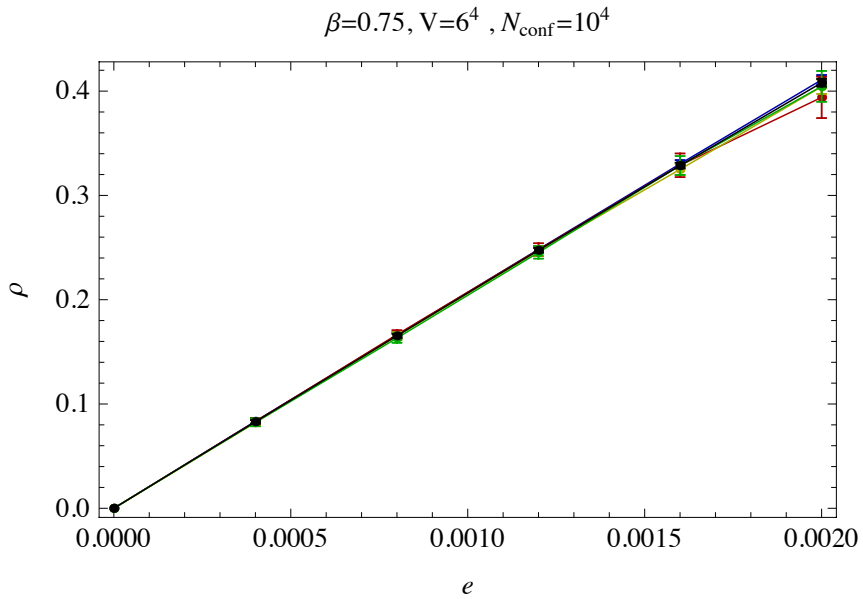
SU(2) × U(1)

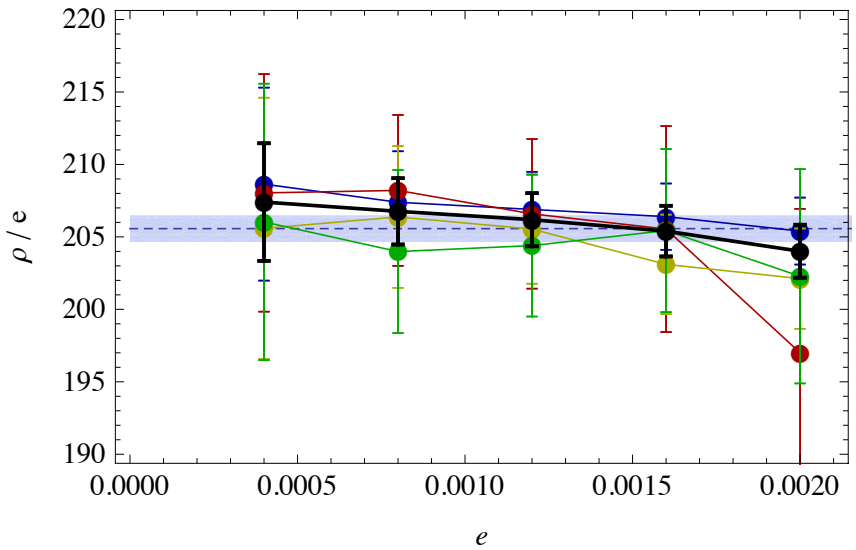
example: $\beta_{\mathrm{SU(2)}} = 0.75,\ \ e_{\mathrm{U(1)}} = 0.0004 \sim 0.0024,\ \ \ V = 6^4$

- consistency of ρ parameters from 1st ~4th EV



- linear response of ρ parameter

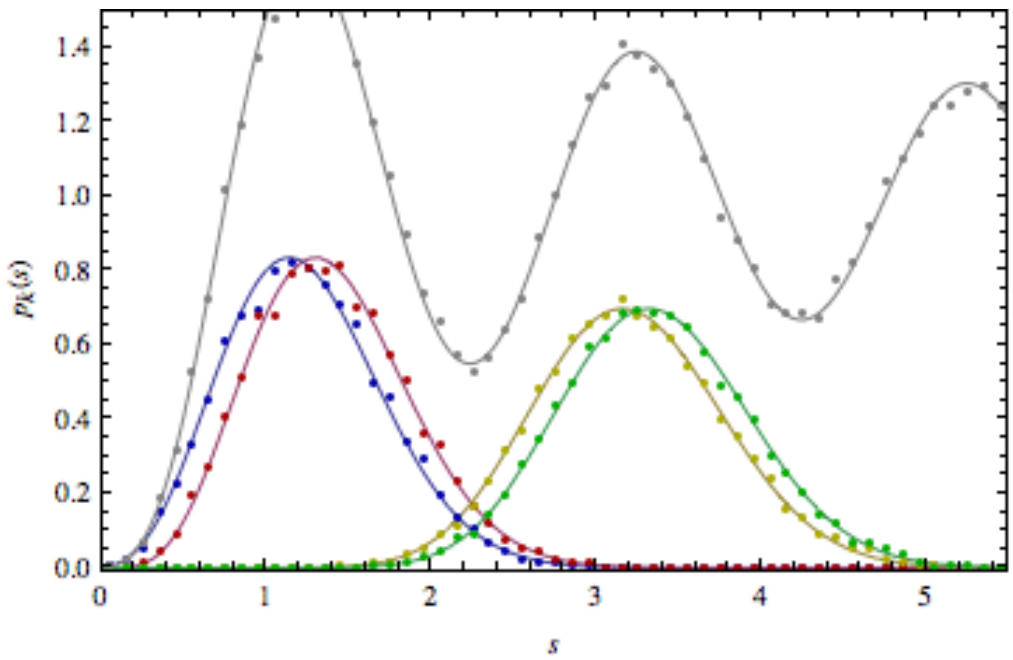


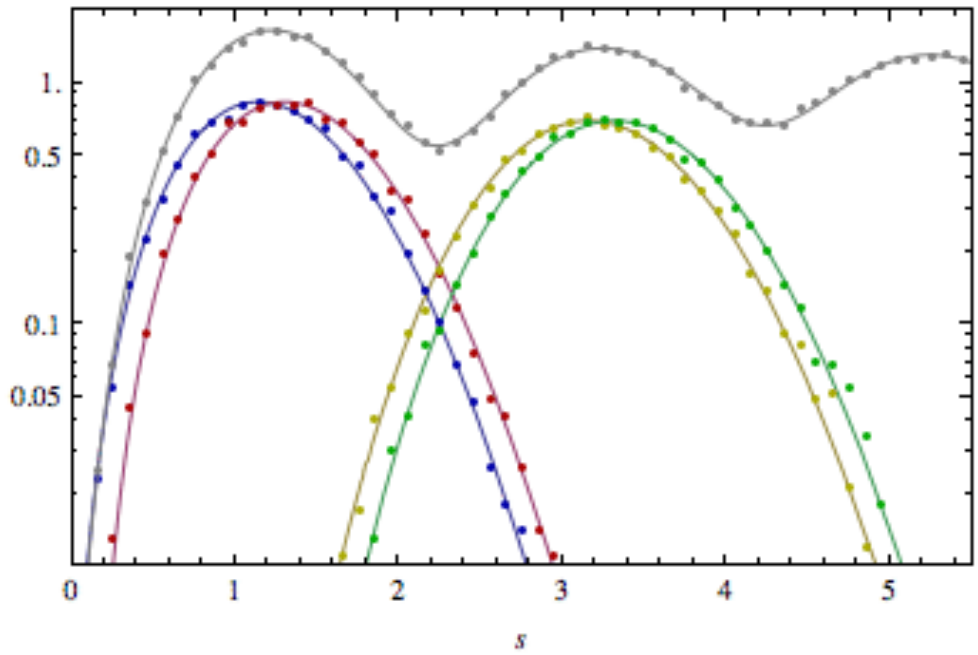


SU(2) + $i\mu$

example: $\beta_{SU(2)} = 0,\ V = 6^4,\ i\mu = 0.005 \sim .042$

$1^{st} \sim 4^{th}$ EV



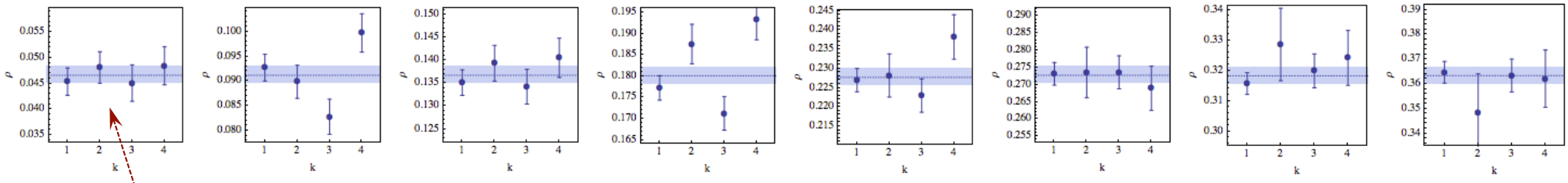


$1^{st} \sim 4^{th}$: $\chi^2/\text{d.o.f} = 0.5 \sim 1.5$

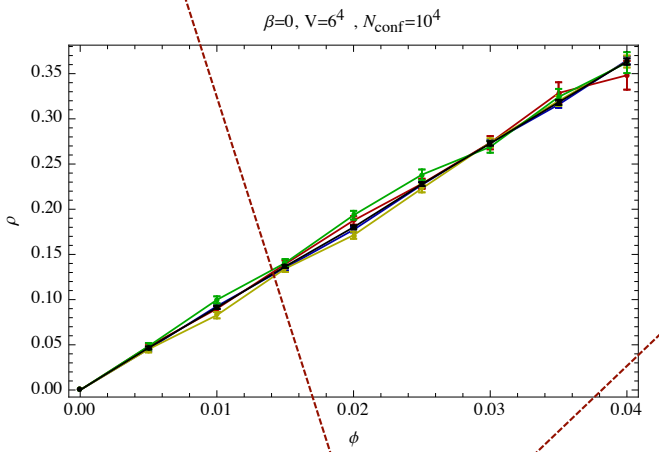
SU(2) + $i\mu$

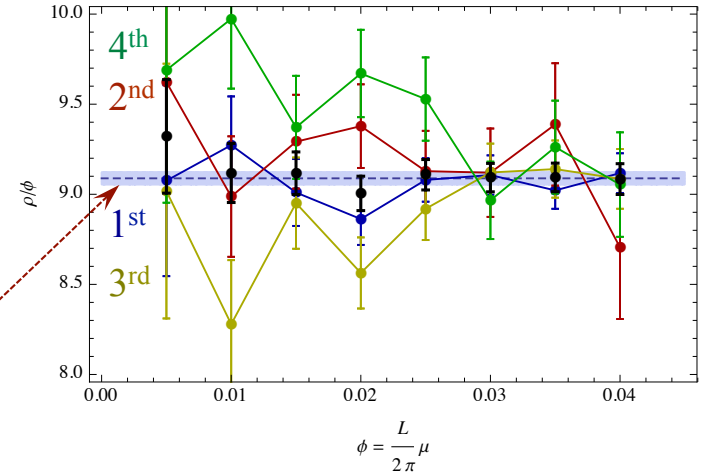
example: $\beta_{\mathrm{SU(2)}} = 0,\ V = 6^4,\ i\mu = 0.005 \sim .042$

- ρ parameters from 1st ~4th EV

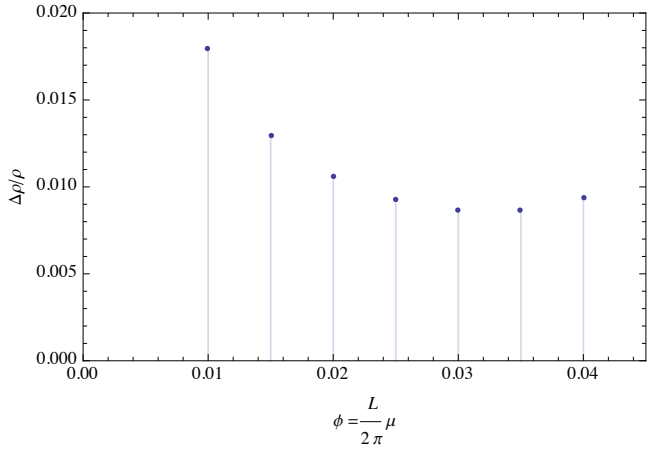


- linear response of ρ parameter





relative stat. error



statistical error in determining the mean EV spacing ($\propto \Sigma$)

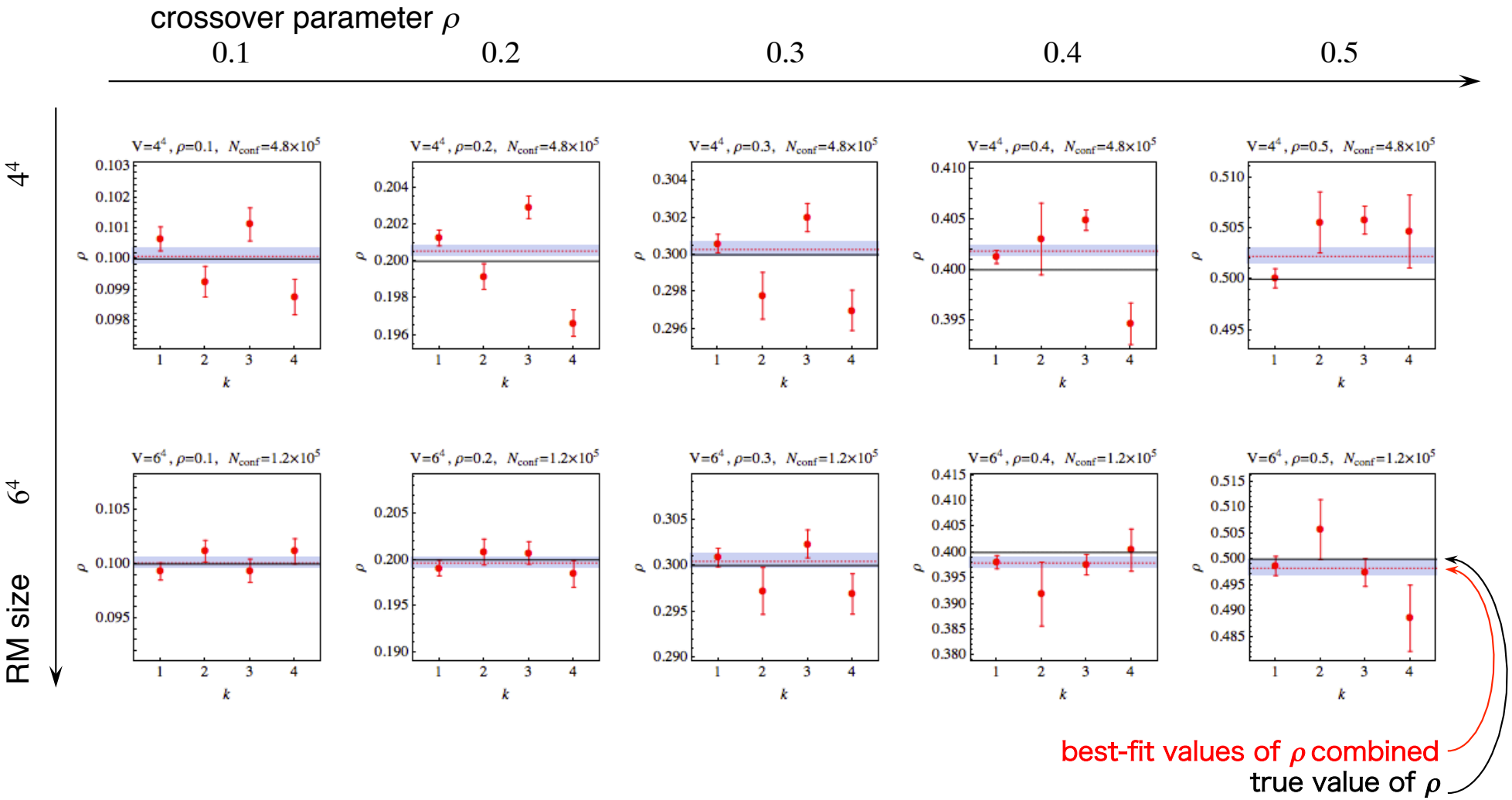
$\Rightarrow$ counter-deviation of crossover parameters ρ ($\propto F$) for once-Kramers-deg. pair of EVs

$\Rightarrow$ almost-cancelled by combining ρ's

combined relative stat. error $.008 \le \left|\frac{\Delta\rho}{\rho}\right| \le .018$

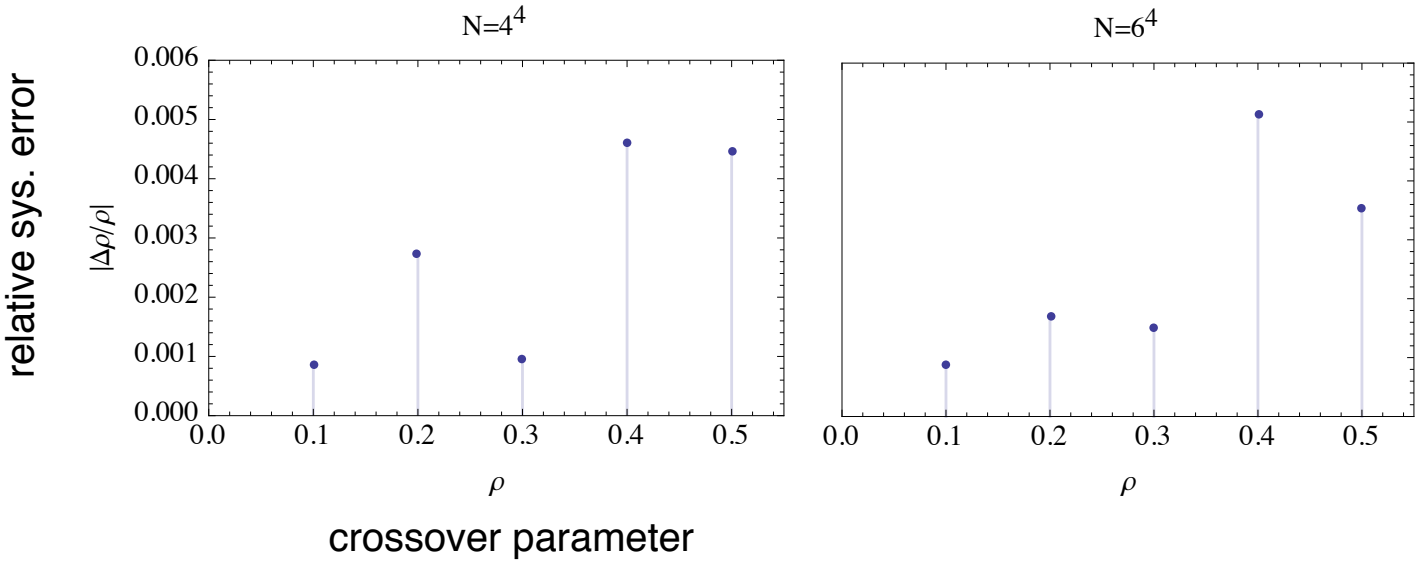
Estimation of systematic error

fit numerically-generated EVs of chGSE-chGOE RMs
to analytically-computed individual EV distributions



Estimation of systematic error

fit numerically-generated EVs of chGSE-chGOE RMs
to analytically-computed individual EV distributions



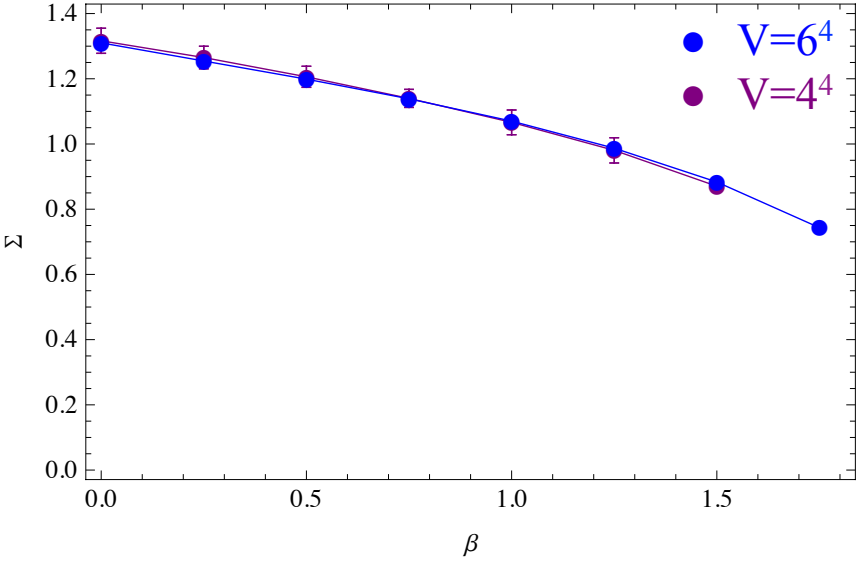
$$.001 \le \left|\frac{\Delta\rho}{\rho}\right| \le .005$$

... sys. error under
very good control

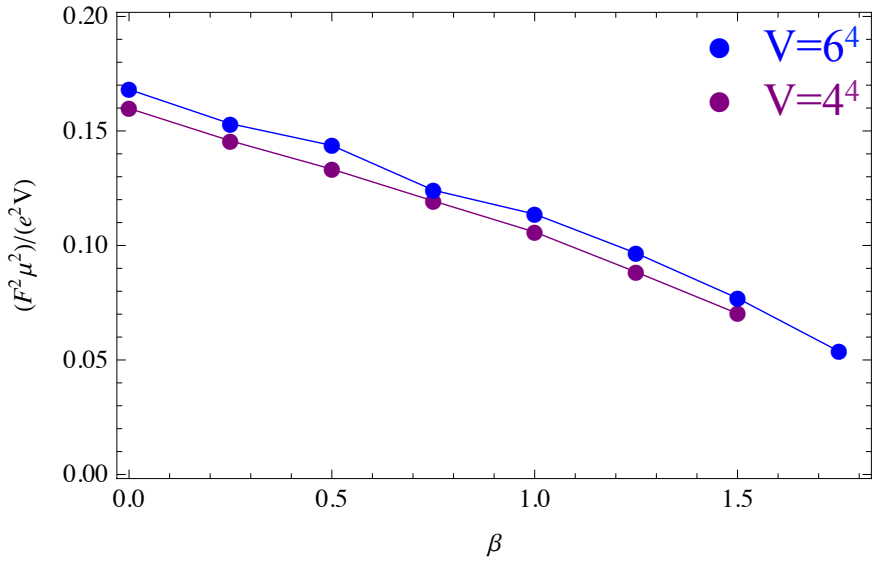
SU(2) × U(1)

- LECs vs bare coupling, system size

chiral condensate



pion decay constant



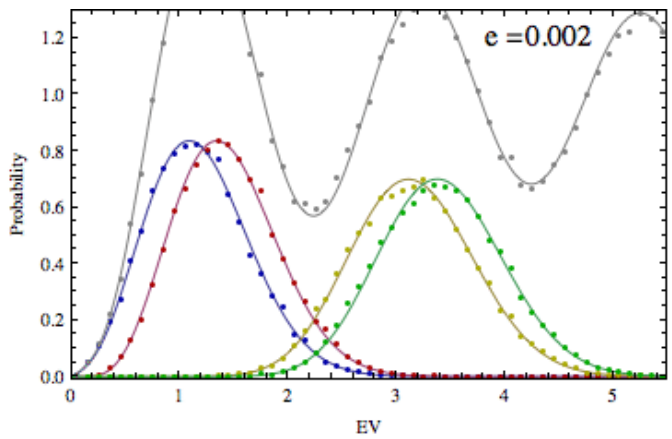
precise measurement possible on small lattices

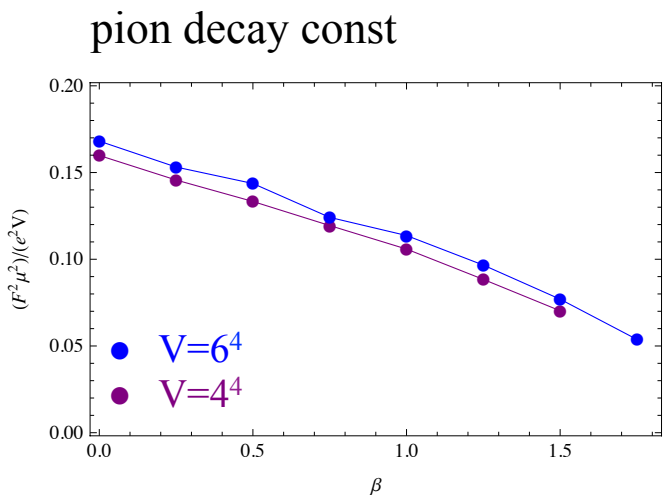
relative error of F : no larger than $\pm .018$ (stat) $\pm .005$ (sys)

V Summary

√ computed k^{th} EV distribtions of chGSE-chGUE crossover

√ fitted to Dirac EVs of SU(2)×U(1) and $i\mu$ → determine parameters in chL





√ good control of errors in F on small lattices

- unquenching feasible
- indiv. D EV of chGUE-GUE crossover
  ⇔ Wilson chPT : underway
- consistency with F_π from conventional $\left\langle J_5^\mu(x)\pi(0)\right\rangle$
- SU(2) D_{DW} on ≤ $16^3\cdot 32$
  : underway (chGSE⇆chGOE for D_{stag})