Chiral Fermions via Energy Level Splitting

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Outline of the talk

- Overlap/Domain Wall and Minimally Doubled fermions
- Crank-Nicolson discretisation scheme
- Applications: I. Crank-Nicolson-Wilson operator
- Applications: II. Crank-Nicolson chiral operator

Chiral Fermions and the Lattice

- Lorentz invariance broken to hypercubic symmetry:
  - there is no such a thing like spin on the lattice;
    - at least in the sense of SU(2) representations;
    - Dirac field does not fit the lattice.
- $\Rightarrow$ Requiring emerging spin on the lattice brings undesirable results;
- **Theorem** (Nielsen-Ninomiya). *There are no local chiral fermions on the lattice.*
- Broken Lorentz invariance + emerging spin makes a "choose-only-one-item" menu:
  - Many chirality pairs of flavours (naive, minimally doubled,...);
  - One flavour of explicitely broken chirality (Wilson,...).

Overlap/Domain Wall Fermions

Adding a new item in the menu:

On shell chirality but an extra dimension.

- Satisfy Ginsparg-Wilson relation: on shell chiral symmetry;
- Expensive to compute: involves nested Krylov subspaces;
- Large density of near zero modes of the kernel;
- Topology stalling simulation algorithms; Solution:
  - Fixed topology overlap: unitarity violations
  - Fat links: lattice spacing of smoothed configuration?

Minimally Doubled Fermions

- Reduce the number of doubler to two: 25 years old idea revived 5 years ago with new actions;
- That is nice but in expense of hypercubic symmetry:
  - Symmetry restoration requires fine tuning i.e. non-perturbative renormalization;
  - Simulation of one flavour needs rooting.

Lesson: lose symmetry in favour of less doublers.

Question: How far can one push it?

One can isolate a single flavour compensated by ghost particles, i.e. no Dirac/Weyl content:

Weyl and ghost fermions on the lattice, A.B. ArXiv:1010.5156

Nielsen-Ninomiya theorem can be genralized, but the result is essentially the same: the ghosts remain in the continuum limit.

Wilson strategy: distribute doublers in the real axis, revived recently by Creutz and collaborators.

Crank-Nicolson discretization

- Example: Schrödinger equation in Euclidean space:

$$\partial_t \psi(t,x) = H\psi(t,x) \ , \qquad \psi(0,x) = \psi_o(x) \ .$$

- Solution $\psi(t,x) = e^{tH}\psi(0,x)$.
- Numerical problem: exp approximation.
  - Problem: non-local grid in t;
  - Requirement: stay with nearest neighbours, i.e. Euler scheme: $\Rightarrow$ order $O(a)$ errors.
- Crank-Nicholson scheme:

$$\partial\psi(t,x) \rightarrow \frac{1}{a}\left[\psi(t+a,x) - \psi(t,x)\right] \ , \quad H\psi(t,x) \rightarrow \frac{H}{2}\left[\psi(t,x) + \psi(t+a,x)\right] \ .$$

$$\Rightarrow \ \psi(t+a,x) = \frac{\mathbb{1}+\frac{a}{2}H}{\mathbb{1}-\frac{a}{2}H}\psi(t,x) = \left(\mathbb{1} + aH + \frac{a^2}{2}H^2 + O(a^3)\right)\psi(t,x) \ ;$$

$\Rightarrow$ order $O(a^2)$ errors in expense of solving linear systems.

Crank-Nicolson and Dirac operator

- Momentum space spin-1/2 Hamiltonian on the lattice: $H = \vec{\sigma} \sin \vec{p}$;
- $\Rightarrow$ Crank-Nicolson time discretised operator:

$$D(p) = e^{ip_4} - 1 + \frac{1}{2} \vec{\sigma} \sin \vec{p} \, (e^{ip_4} + 1) \; .$$

- Particle content:

$$D(p) = 0 \;\Leftrightarrow\; 4 \sin^2 \frac{p_4}{2} + \sin^2 \vec{p} \, \cos^2 \frac{p_4}{2} = 0 \; .$$

- $\Rightarrow$ 8 particles located at the edges of the 3d Brillouin zone.
- Result:

The number of doublers is reduced by a factor of two!

Crank-Nicolson-Wilson operator

- Reduce the number of doublers using the Wilson approach:

$$D_{CNW}(p) = \gamma_4(e^{ip_4} - 1) + \frac{1}{2}\left[i\vec{\gamma}\sin\vec{p} + \sum_k (1-\cos p_k)\right](e^{ip_4}+1) \ .$$

- $D_{CNW}(p) = 0 \ \Leftrightarrow$

$$4\sin^2\frac{p_4}{2} + \left\{\sin^2\vec{p} + \left[\sum_k (1-\cos p_k)\right]^2\right\}\cos^2\frac{p_4}{2} = 0 \ .$$

- Result:
  - Second order accurate in time;
  - Smaller additive mass renormalisation;
  - Hypercubic symmetry broken to cubic symmetry.

Degenerate doublet of Crank-Nicolson-Wilson fermions

Restore hybercubic symmetry defining a doublet of Crank-Nicolson-Wilson fermions:

$$\mathcal{D}_{CNW} = \begin{pmatrix} D^{(+)}_{CNW}(m) & 0 \\ 0 & D^{(-)}_{CNW}(m) \end{pmatrix}$$

where:

$$\begin{aligned} D^{(+)}_{CNW}(m) &= m\mathbb{1} + \gamma_4(e^{ip_4} - 1) \; + \; \frac{1}{2}\left[i\vec{\gamma}\sin\vec{p} + \sum_k (1 - \cos p_k) \right] (e^{ip_4} + 1) \\ D^{(-)}_{CNW}(m) &= m\mathbb{1} + \gamma_4(1 - e^{-ip_4}) + \frac{1}{2}\left[i\vec{\gamma}\sin\vec{p} + \sum_k (1 - \cos p_k) \right] (e^{-ip_4} + 1) \end{aligned}$$

- Hypercubic symmetry restored under flavour exchanging operation;
- Result: $O(a_t^2)$ errors theory without fine tuning + 3-space Wilson.

A chiral theory with doublers

- Start with a general theory of many chirality pairs flavours:

$$E_n(\vec{p}) = \alpha_n \vec{\sigma}\vec{p} \; , \quad n = 1, 2, \ldots, m \; ,$$

- For example naive fermions have this continuum limit with $\alpha_n = \pm 1$;
- We are seeking a theory with a singlet chirality ground state:
  - the opposite chirality counterpart should occupy a different level;
  - the theory has a energy gap $\Delta(\vec{p}) = E_1(\vec{p}) - E_0(\vec{p})$.
- Energy gap increases with momenta:
  - A non-uniform gap;
  - Maximal gap at cuttoff.
- $\Rightarrow$ A chiral theory of free fermions with doublers.

Lattice implementation

- Start from Crank-Nicolson discretisation in time and naive discretisation in 3-space:

$$D(p)' = e^{ip_4} - 1 + \frac{1}{2}\vec{\sigma}\sin\vec{p}\,(e^{ip_4}+1)\ ;$$

- Add a pure imaginary operator of the Wilson type:

$$D(p) = e^{ip_4} - 1 + \frac{1}{2}\left[\vec{\sigma}\sin\vec{p} + ir\sum_k (1-\cos p_k)\right](e^{ip_4}+1)\ .$$

- $D(p) = 0 \;\Leftrightarrow$

$$\left\{2\sin\frac{p_4}{2} + \left[r\sum_k(1-\cos p_k)\right]\cos\frac{p_4}{2}\right\}^2 + \sin^2\vec{p}\,\cos^2\frac{p_4}{2} = 0\ .$$

Particle spectra

- 8 zeros in the edges of 3d Brillouin zone:

  $\vec{p}^{\,*} \in \{(0,0,0),(0,0,\pi),(0,\pi,0),(\pi,0,0),(0,\pi,\pi),(\pi,0,\pi),(\pi,\pi,0),(\pi,\pi,\pi)\}$

- Define $n(\vec{p}^{\,*}) = \frac{1}{2}\sum_k (1-\cos p_k^*)$
- $\Rightarrow \; D(p) = 0 \;\Leftrightarrow \tan\frac{p_4^*}{2} = -rn(\vec{p}^{\,*})$.
- Define chirality $\chi(\vec{p}^{\,*}) = \cos p_1^* \cos p_2^* \cos p_3^* \quad \Rightarrow$

$\vec{p}^{\,*}$	$n(\vec{p}^{\,*})$	$\chi(\vec{p}^{\,*})$	Degeneracy
$(0,0,0)$	0	1	1
$(0,0,\pi),(0,\pi,0),(\pi,0,0)$	1	-1	3
$(0,\pi,\pi),(\pi,0,\pi),(\pi,\pi,0)$	2	1	3
(π,π,π)	3	-1	1

- Continuum limit dispersion relation: $E_n(\vec{p}) \to \frac{\vec{\sigma}\vec{p}}{1+r^2n^2}\;, \quad n = 0,1,2,3.$

Broken hypercubic symmetry

- Restore hypercubic symmetry defining a quartet of such fermions:

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 0 & D_r^{(+)} \\ 0 & 0 & D_{-r}^{(+)} & 0 \\ 0 & D_r^{(-)} & 0 & 0 \\ D_{-r}^{(-)} & 0 & 0 & 0 \end{pmatrix}$$

- $\Rightarrow$ symmetry is restored under $(+)$ and $(-)$ flavour exchange as well as under r and $-r$ flovour exchange.
- Result: a free theory of four flavours with definity chirality.
- Interacting theory: close to continumm theory should be Ok;
- Strong coupling: doubler mixing.

Conclusions

- Crank-Nicolson discretisation scheme offers new definitions of fermions on the lattice:

  I. Crank-Nicolson-Wilson operator is a one flavour theory with broken chiral symmetry and second order discretisation errors in a_t.

  II. A free fermion chiral theory via doubler level splitting.