New fermion discretizations and their applications

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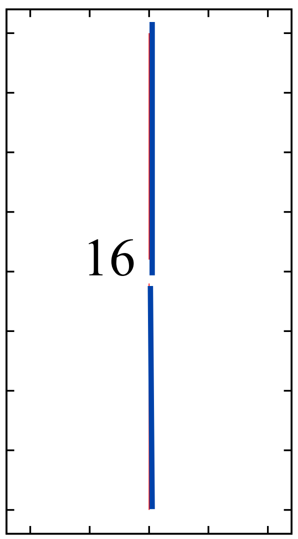
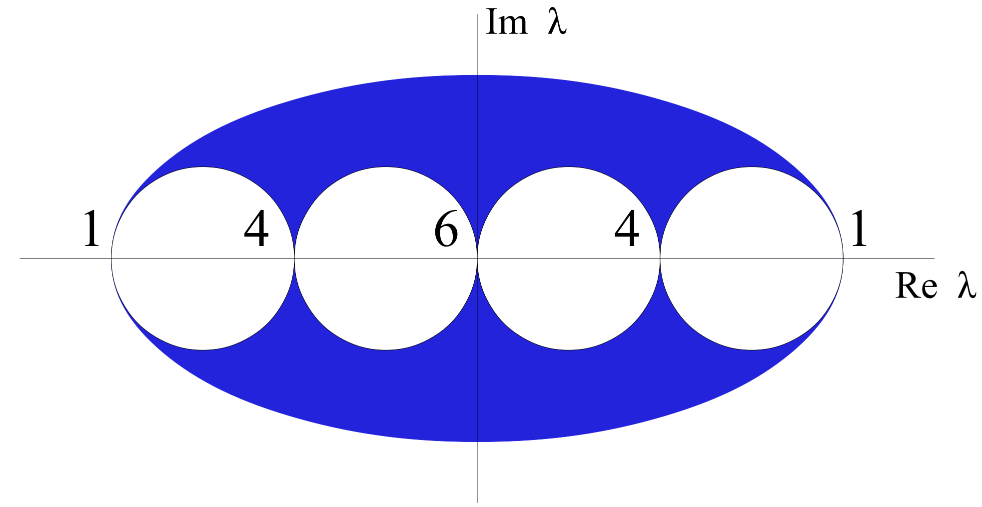
05/14/2012 NHCF workshop@BNL

Table of contents

Naive



Wilson



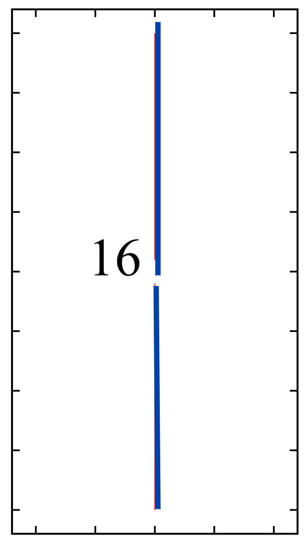
$$\sum_{\mu} C_{\mu}$$

$$C_\mu = (T_{+\mu} + T_{-\mu})/2$$

$$T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu}$$

U(4)×U(4)

Naive



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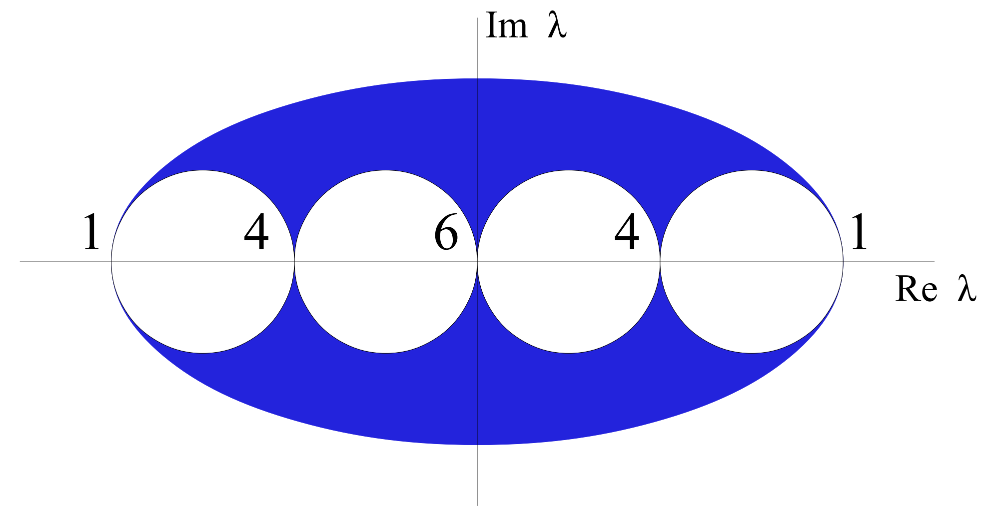
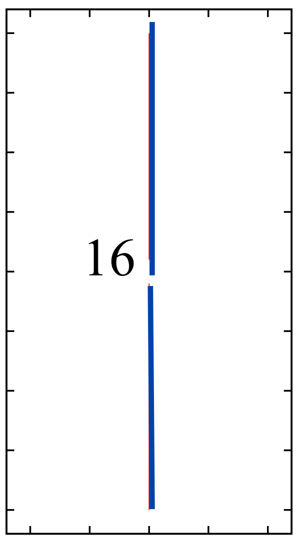


U(4)×U(4)

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4\,,\ (-1)^{n_1+\ldots+n_4}\gamma_5\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\,,\ (-1)^{n_\mu} i\gamma_\mu\gamma_5\,,\ (-1)^{n_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2} \right\}$$
$$\Gamma_X^{(-)} \in \left\{ (-1)^{n_1+\ldots+n_4}\mathbf{1}_4\,,\ \gamma_5\,,\ (-1)^{n_\mu}\gamma_\mu\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\gamma_5\,,\ (-1)^{\check{n}_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2} \right\}$$

$$\psi_n \to \psi'_n = \exp\left[\mathrm{i}\sum_X\left(\theta_X^{(+)}\Gamma_X^{(+)} + \theta_X^{(-)}\Gamma_X^{(-)}\right)\right]\psi_n\,, \qquad \bar{\psi}_n \to \bar{\psi}'_n = \bar{\psi}_n \exp\left[\mathrm{i}\sum_X\left(-\theta_X^{(+)}\Gamma_X^{(+)} + \theta_X^{(-)}\Gamma_X^{(-)}\right)\right]$$

Naive

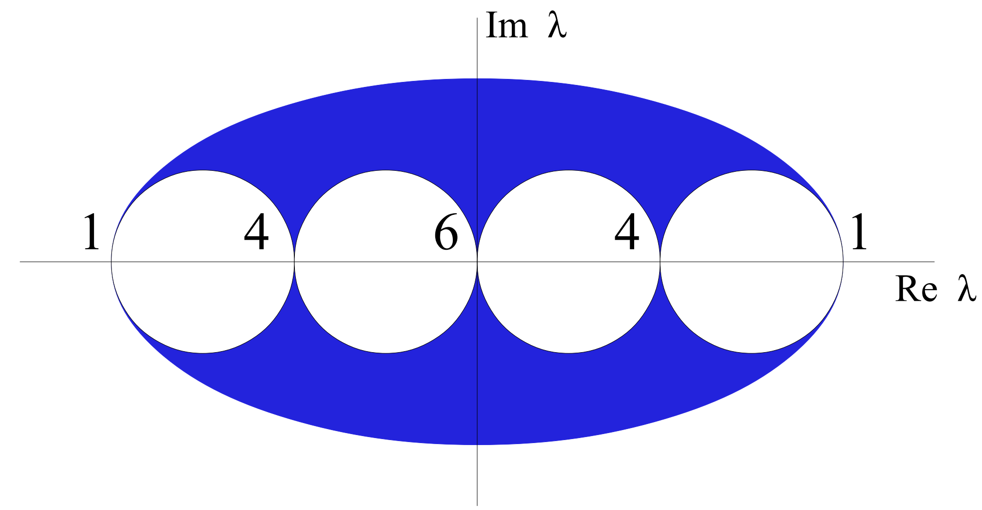


$$\sum_\mu C_\mu$$

$C_\mu = (T_{+\mu} + T_{-\mu})/2$

$T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu}$

Wilson



U(4)×U(4)

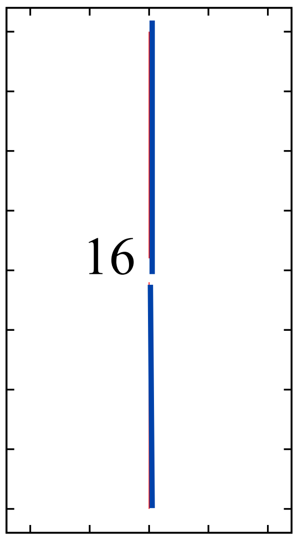
U(1)

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4 \, , \right\}$$

$$\Gamma_X^{(-)} \in \left\{ \right\}$$

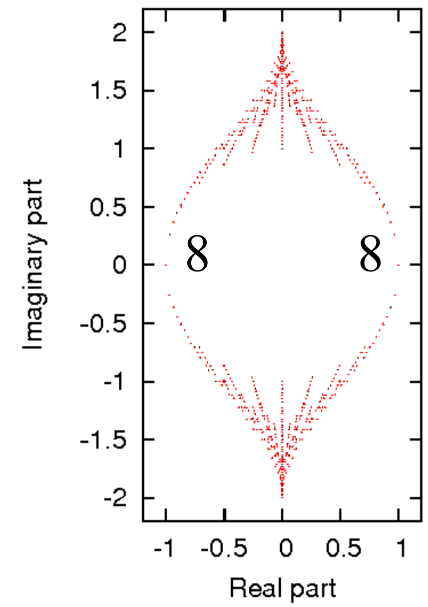
Naive

Wilson'



$$\sum_{sym.} C_1 C_2 C_3 C_4$$

$$C_\mu = (T_{+\mu} + T_{-\mu})/2$$
$$T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu}$$



Creutz, Kimura, Misumi(2010)

U(4)×U(4) → U(2)×U(2)

Lehner, Misumi(2012)

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4\,,\ (-1)^{n_1+\ldots+n_4}\gamma_5\,, \qquad (-1)^{n_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2} \right\}$$

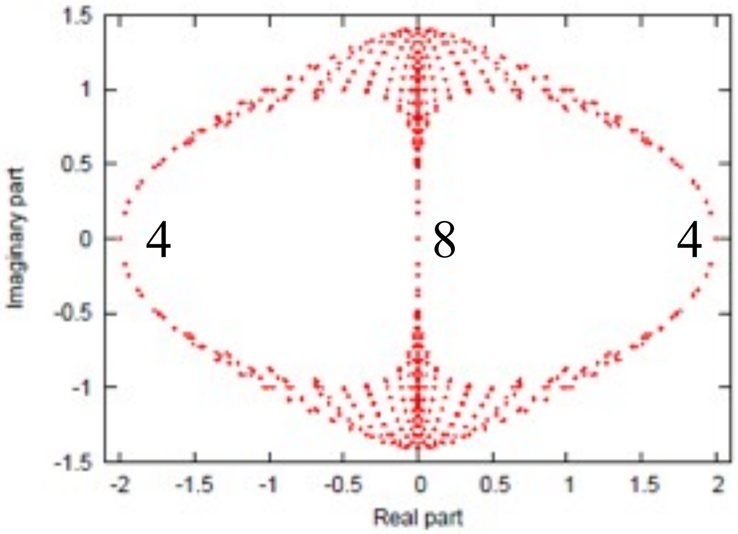
$$\Gamma_X^{(-)} \in \left\{ \qquad \right\}$$

1. Flavored mass

Creutz, Kimura, Misumi(2010)

$$M_{\rm V} = \sum_{\mu} C_{\mu},$$ *Vector (1-link)*

$$M_{\rm T} = \sum_{perm.} \sum_{sym.} C_{\mu} C_{\nu},$$ *Tensor (2-link)*



$$M_{\rm A} = \sum_{perm.} \sum_{sym.} \prod_{\nu} C_{\nu},$$ *Axial-V (3-link)*

$$M_{\rm P} = \sum_{sym.} \prod_{\mu=1}^{4} C_{\mu},$$ *Pseudo-S (4-link)*

- *gamma-5 hermiticity*
- *2nd derivative terms* $\sum_{n} \bar{\psi}_n (M_P - 1) \psi_n \ \rightarrow \ -a \int d^4 x \bar{\psi}(x) D_{\mu}^2 \psi(x) + O(a^2)$
- *Cousins of Wilson fermion*

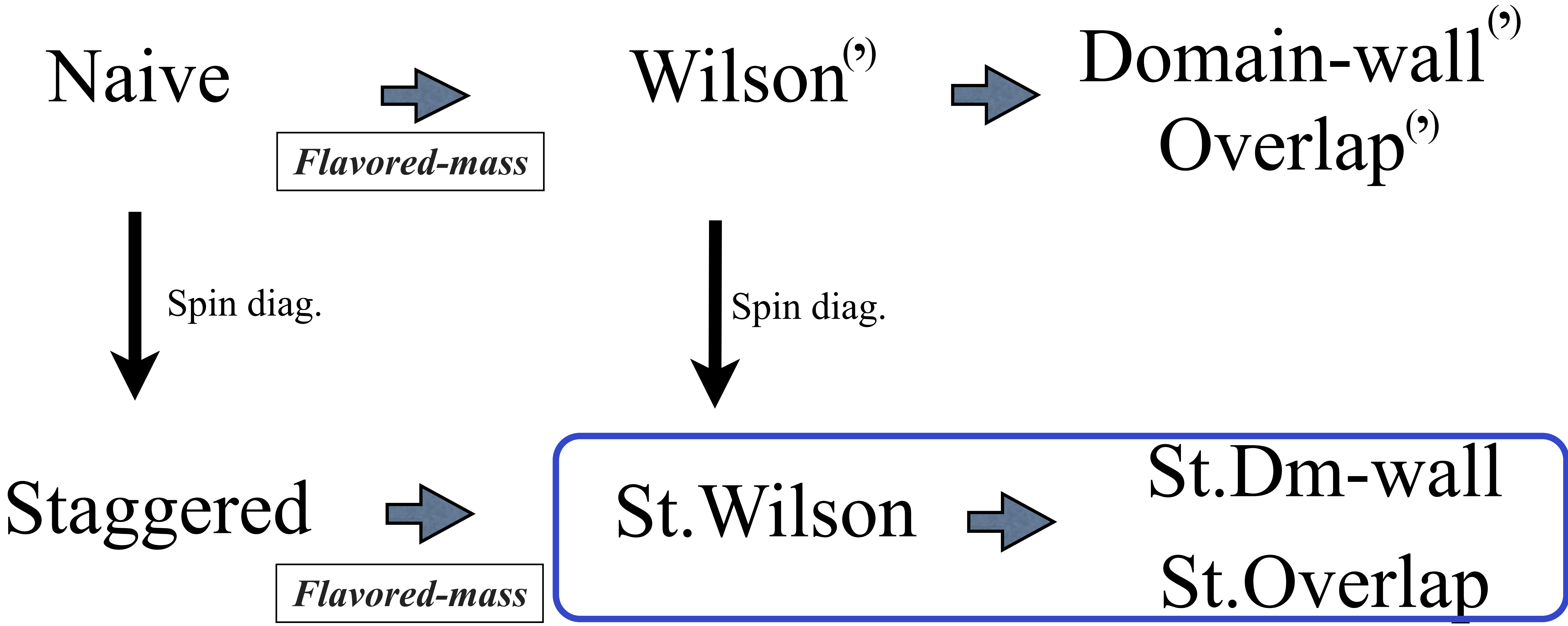
Naive $\rightarrow$ Wilson$^{(')}$ $\rightarrow$ Domain-wall$^{(')}$

Overlap$^{(')}$

Flavored-mass

Spin diag.

Staggered


Naive
Flavored-mass
Wilson(')
Domain-wall(')
Overlap(')
Spin diag.
Spin diag.
Staggered
Flavored-mass
St.Wilson
St.Dm-wall
St.Overlap

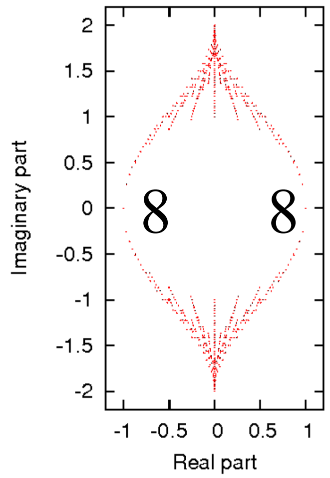
◆Staggered-Wilson

$$\bar{\psi}_x C_1 C_2 C_3 C_4 \psi_x \quad \underset{spin\ diag.}{\longrightarrow} \quad \pm\bar{\chi}_x(\epsilon\eta_1\eta_2\eta_3\eta_4 C_1 C_2 C_3 C_4)\chi_x$$

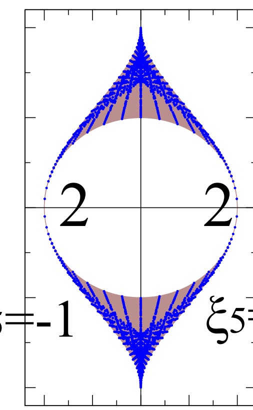
Staggered flavored-mass

Golterman, Smit (1984) Adams(2009)

M_{P}



$\longleftrightarrow$



$M_{\mathcal{A}} \sim (\mathbf{1} \otimes \xi_5)$

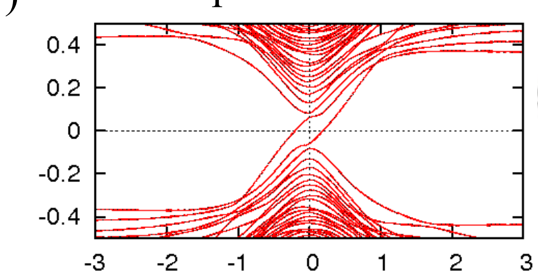
- With mass shift

  ξ_5=-1 → physical sector : ℓ

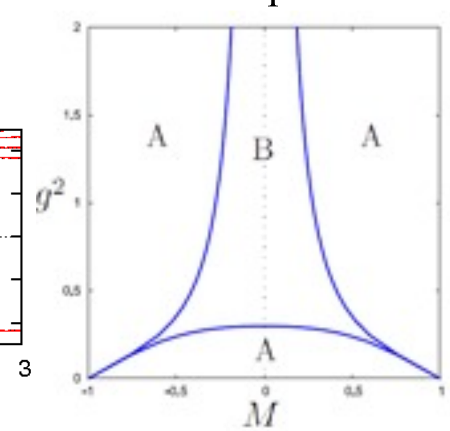
  ξ_5=+1 → decoupled sector : h

- Index theorem Adams (09), Creutz,Kimura,Misumi(10), Follana, et.al.(11)
- Aoki phase Creutz, Kimura Misumi (11)

Spectral flow



Aoki phase



§ Potential advantages of $\eta_\mu D_\mu + r(1 + M_{\mathcal{A}})$

- could reduce numerical costs in 2-flavor overlap de Forcrand, Kurkela, Panero(2012)
- could reduce influence of taste-breaking for 2-flavor Sharpe (2012)

→ Look into symmetries and pion spectrum !

§ Staggered sym. $\{C_0,\, \Xi_\mu,\, I_s,\, R_{\mu\nu}\} \;\times\; \{U^\epsilon(1)\}_{m=0} \Rightarrow \{C_0, \Xi_j, I_s, R_{ij}\}$

Transfer-matrix sym.

⇒ classify 15 pseudoscalar operators Golterman (1986)

$\mathbf{1} \;:\; \xi_4,\, \xi_{45},\, \xi_5,$

$\mathbf{3} \;:\; \xi_i,\, \xi_{i5},\, \xi_{ij}\; \xi_{i4}$

7 irreps

cf.) ChPT by Lee, Sharpe (1999)

4 irreps of SO(4) upto

$O(a^4),\, O(a^2 m)\;\; O(a^2 p^2)$

$\mathbf{1} \;:\; \xi_5,$

$\mathbf{4} \;:\; \xi_\mu,\, \xi_{\mu 5},$

$\mathbf{6} \;:\; \xi_{\mu\nu}.$

§ Staggered-Wilson $\boxed{\{C_0, \Xi'_\mu, R_{\mu\nu}\}} \Rightarrow \{C_0, \Xi'_j, R_{ij}\}$ Transfer-matrix sym.

$C \quad P \quad SW_4$

Irreps mix in ξ_5 pairs

Physical sector

⇒

$\mathbf{1} \;\&\; \xi_5 \longrightarrow \bar{\ell}\ell \;,\; \bar{h}h \longrightarrow \boxed{\bar{\ell}(\gamma_5 \otimes \mathbf{1})\ell} \quad \eta'$

$\xi_4 \;\&\; \xi_{45} \longrightarrow \bar{\ell}h \;,\; \bar{h}\ell$

$\xi_{i4} \;\&\; \xi_{i45} \longrightarrow \bar{\ell}\sigma_j\ell \;,\; \bar{h}\sigma_j h \longrightarrow \boxed{\bar{\ell}(\gamma_5 \otimes \sigma_i)\ell} \quad \pi_0,\, \pi_\pm$

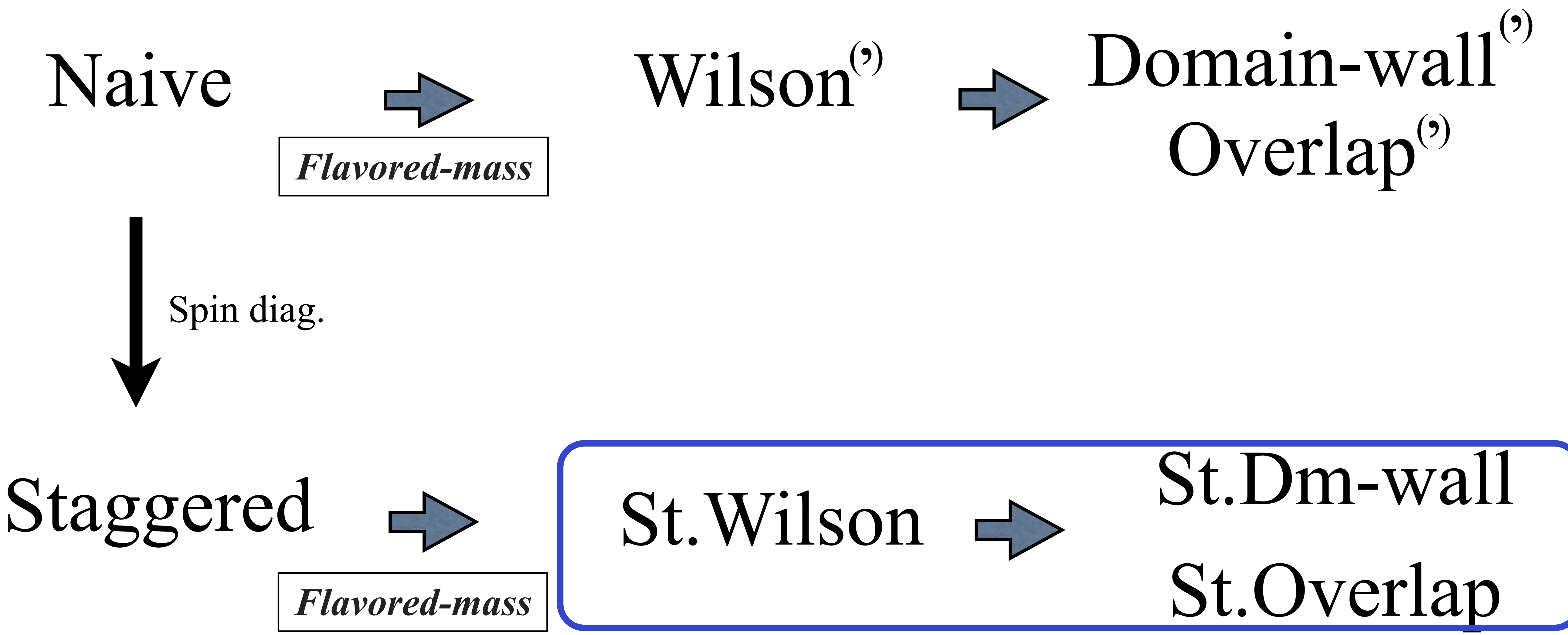
$\xi_i \;\&\; \xi_{i5} \longrightarrow \bar{\ell}\sigma_j h \;,\; \bar{h}\sigma_j \ell$

States in 3d irrep

Sharpe (2012)

Discrete symmetries seem sufficient for degenerate pion triplet!

cf.) ChPT → SU(2) restoration upto $O(a^4),\, O(a^2 m)\; O(a^2 p^2)$


Naive
Wilson(')
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Flavored-mass
Spin diag.
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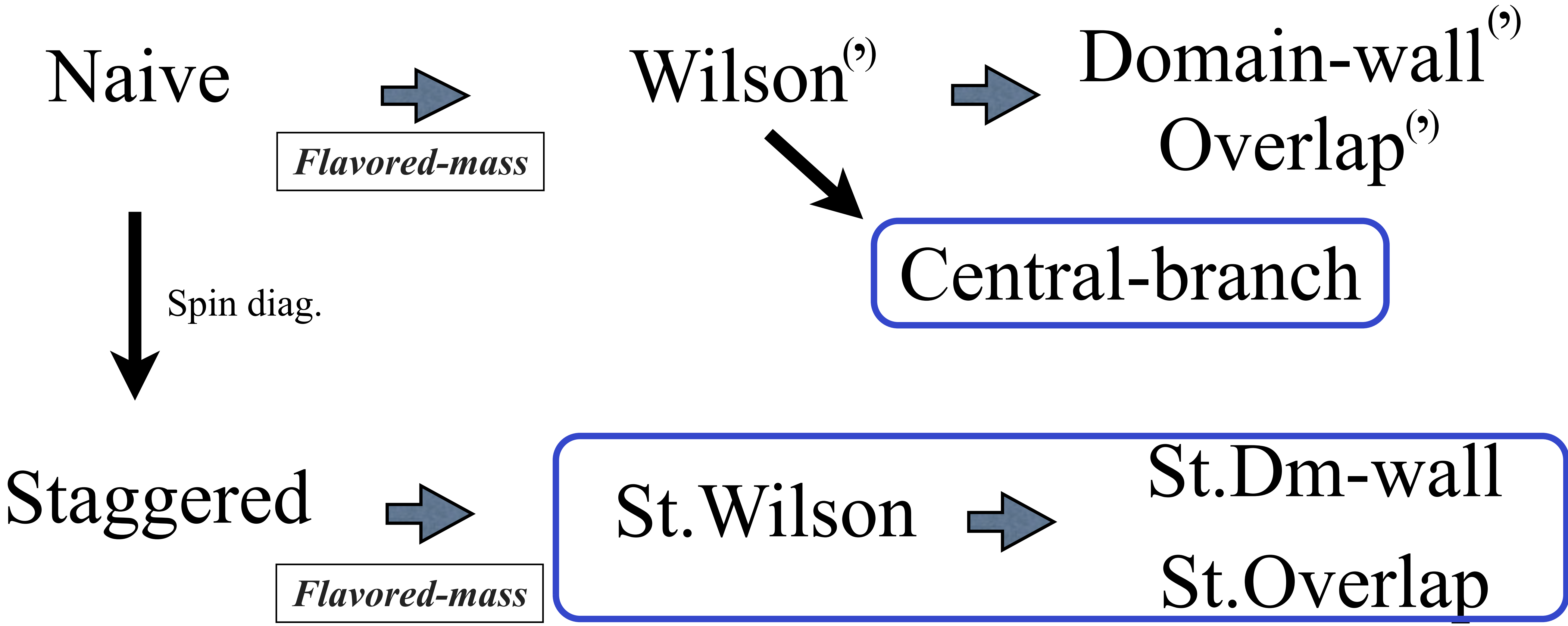
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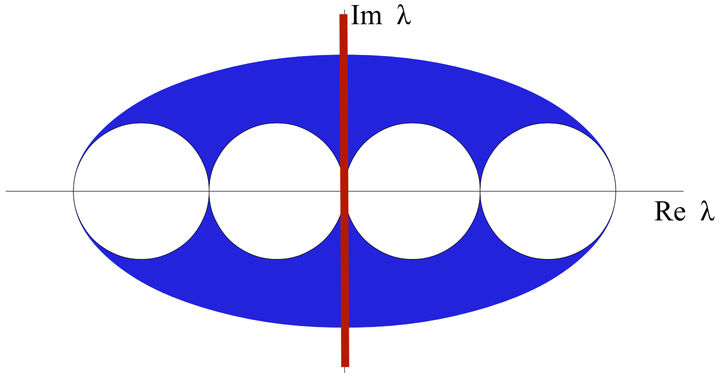
2. Central-branch

Creutz, Kimura, Misumi (2011)
Kimura, Komatsu, Misumi, Noumi, Torii, Aoki (2012)

- Wilson w/o onsite term $M_W \equiv m + 4r = 0$

$$S = \frac{1}{2}\sum_{x,\mu} \bar{\psi}_x[\gamma_\mu(\psi_{x+\mu} - \psi_{x-\mu}) - (\psi_{x+\mu} + \psi_{x-\mu})]$$



Another U(1) !

$$\psi_x \;\to\; e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}, \quad \bar{\psi}_x \;\to\; \bar{\psi}_x e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}$$

$$\Gamma_X^{(+)} \;\in\; \left\{\mathbf{1}_4\,,\; (-1)^{n_1+\ldots+n_4}\gamma_5\,,\; (-1)^{\check{n}_\mu}\gamma_\mu\,,\; (-1)^{n_\mu} i\gamma_\mu\gamma_5\,,\; (-1)^{n_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2}\right\}$$

$$\Gamma_X^{(-)} \;\in\; \left\{(-1)^{n_1+\ldots+n_4}\mathbf{1}_4\,,\; \gamma_5\,,\; (-1)^{n_\mu}\gamma_\mu\,,\; (-1)^{\check{n}_\mu}\gamma_\mu\gamma_5\,,\; (-1)^{\check{n}_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2}\right\}$$

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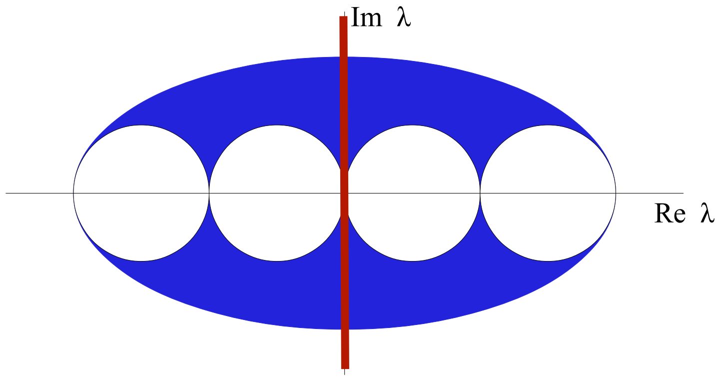


⇨ Another U(1) !

$$\psi_x \;\to\; e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}, \quad \bar{\psi}_x \;\to\; \bar{\psi}_x e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}$$

$$\Gamma_X^{(+)} \;\in\; \left\{ \mathbf{1}_4 \,, \qquad\qquad \right\}$$

$$\Gamma_X^{(-)} \;\in\; \left\{ \underline{(-1)^{n_1+\ldots+n_4}\mathbf{1}_4} \,, \qquad\qquad \right\}$$

Prohibits additive mass renormalization !
SSB gives NG boson !

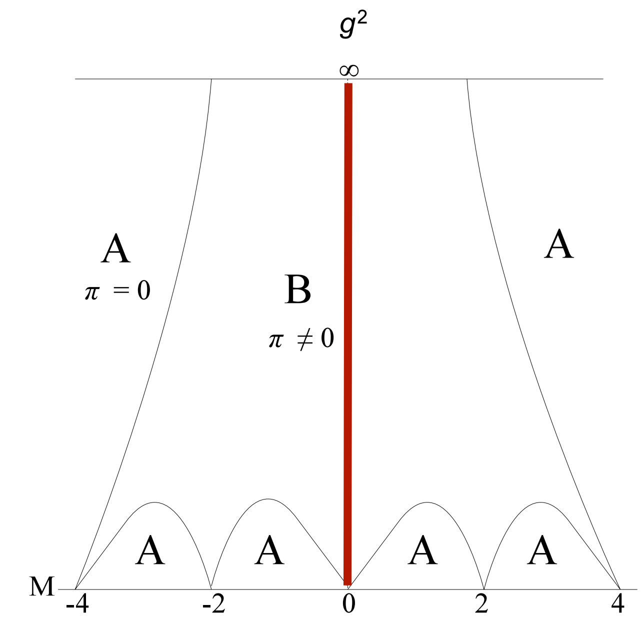
◆ Strong-coupling QCD

$$\cosh(m_{SPA}) = 1 + \frac{2M_W^2(16 + M_W^2)}{16 - 15M_W^2}$$

NG boson associated with SSB of U(1)

- Pion (eta) condensate $\langle \bar{\psi}\gamma_5\psi \rangle \neq 0$
- No chiral condensate $\langle \bar{\psi}\psi \rangle = 0$

$$\bar{\psi}\psi \overset{?}{\leftrightarrow} \bar{\psi}\gamma_5\psi$$



Aoki phase

§ Advantages

- No additive mass renormalization (no fine-tuning)
- SSB of U(1) and massless NG boson

$\bar{\psi}\psi \leftrightarrow \bar{\psi}\gamma_5\psi$ *change of mass base*

$\boxed{\textit{6-flavor massless QCD}}$

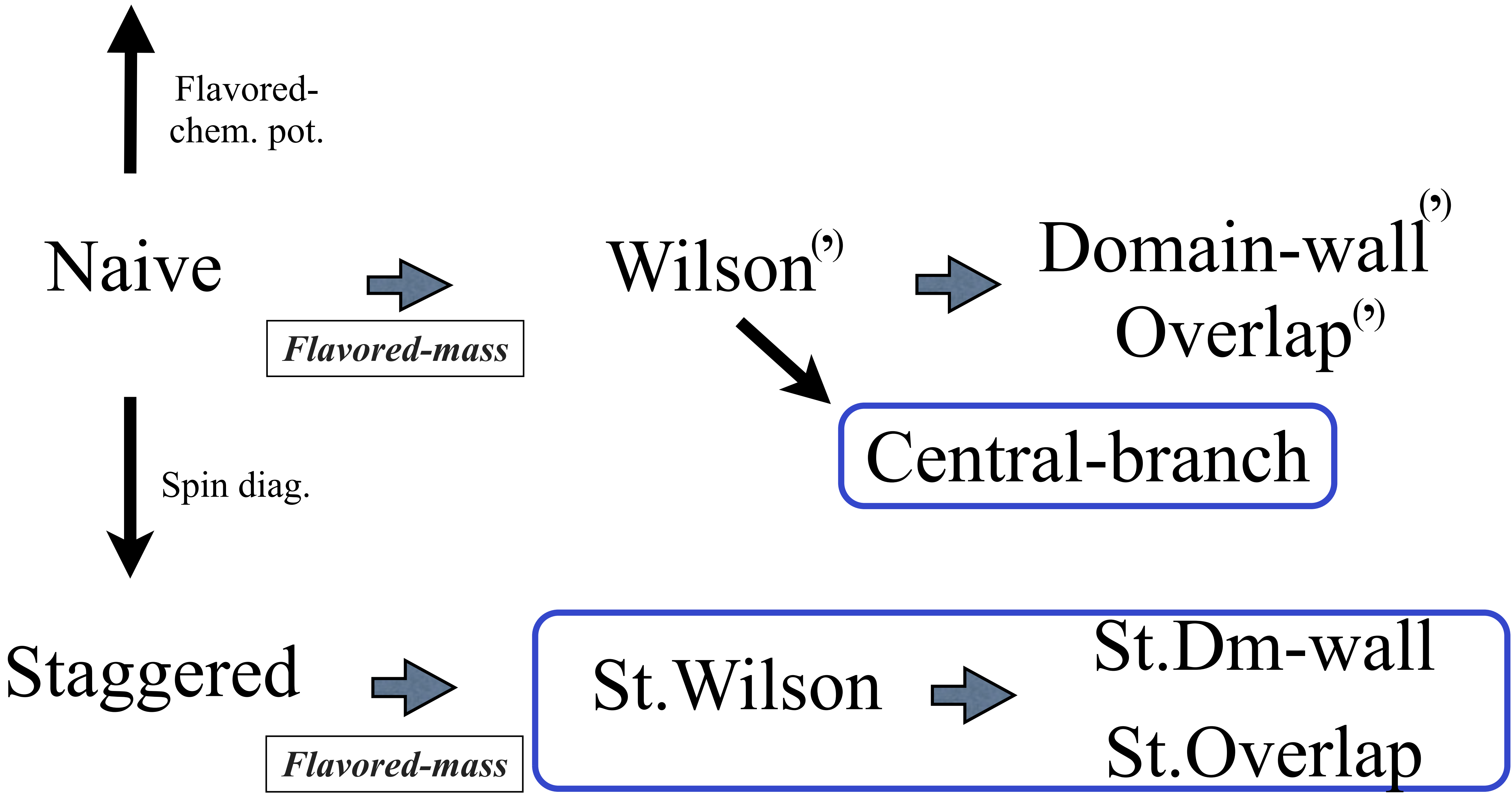
§ Potential drawbacks

- sign problem
- U(1) problem
- Quark mass

Twisted-mass works ?

$\boxed{\rightarrow \textit{12-flavor massless QCD}}$

New possibility of many-flavor lattice QCD !



Flavored-
chem. pot.
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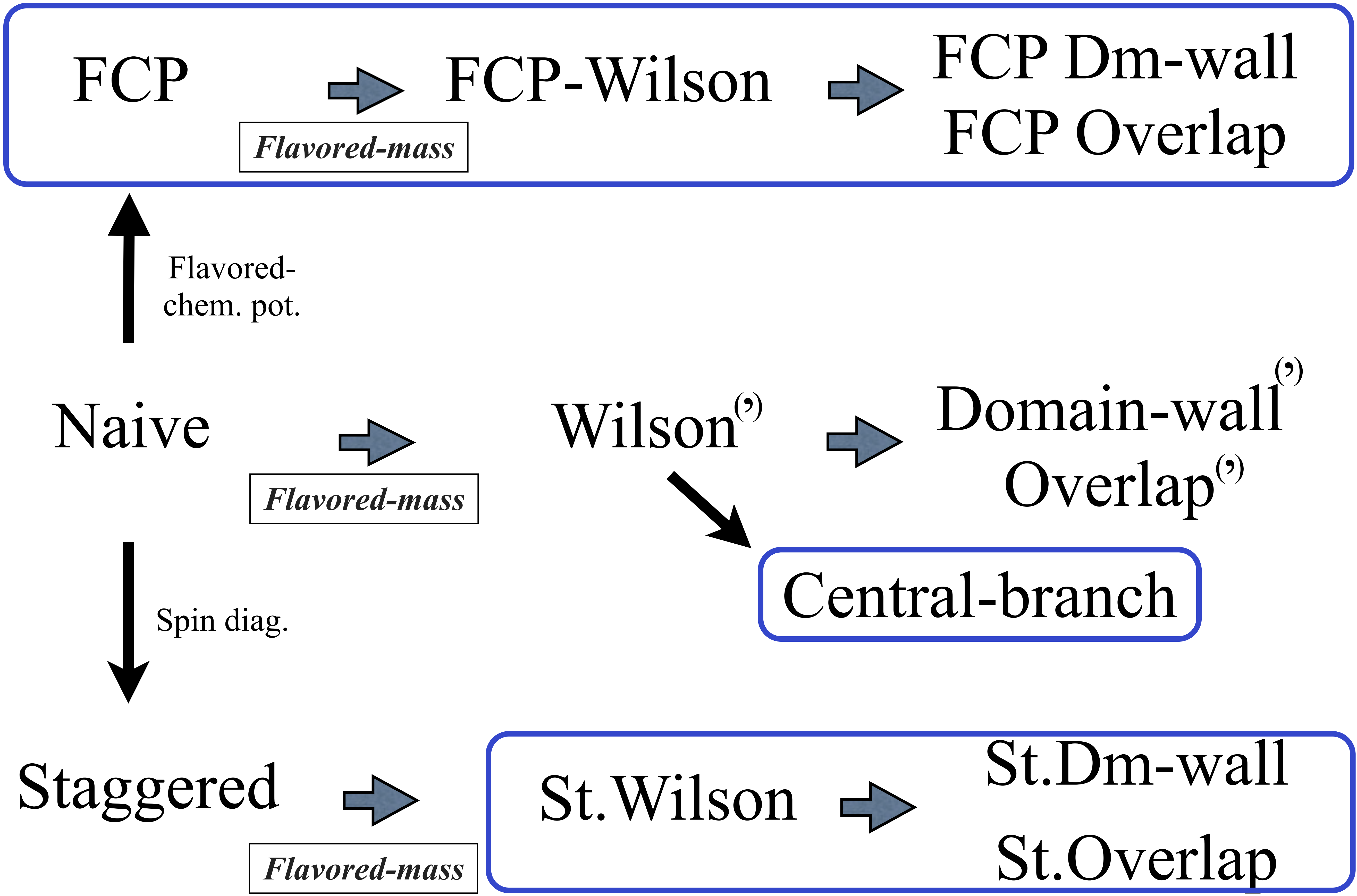
FCP
FCP-Wilson
FCP Dm-wall
FCP Overlap
Flavored-mass
Flavored-
chem. pot.
Naive
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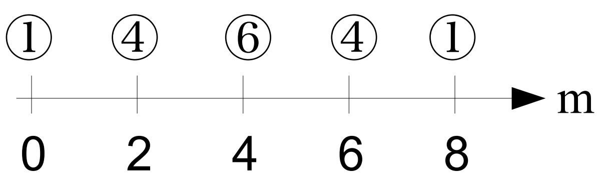
St.Overlap
Flavored-mass

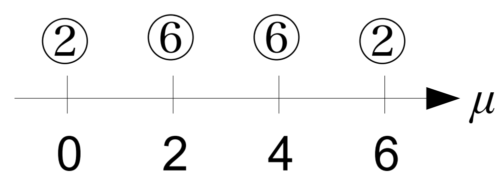
3. Flavored chemical potential
Misumi (2012)

Wilson



$$\sum_{\mu}(1-\cos p_\mu)$$

Flavored chemical-pot.



$$\text{(i)}\ \gamma_4 \sum_{j=1}^{3}(1-\cos p_j)$$

- Real type $\rightarrow$ Sign problem
- Imaginary type $\rightarrow$ No sign problem

"*Minimal-doubling*" Karsten(81)Wilczek(87)Creutz, Borici(07)Creutz&Misumi(10)

cf.)Bedaque, Buchoff, Tiburzi, Walker-Loud(08)

Finite-mass system(Wil) $\leftrightarrows$ Finite-density system(FCP)

◆ Advantage

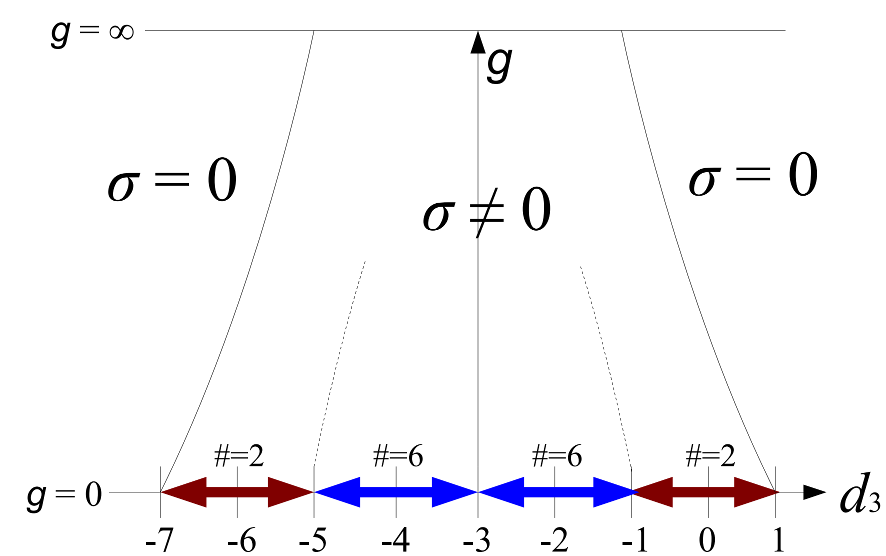
- U(1) chiral symmetry
- 2 flavor possible

(1) $U(1)_V \times \underline{U(1)_A}$
(2) P
(3) CT
(4) Cubic symmetry

◆ Drawbacks

- O($1/a$) chemical potential renormalization
- Tuning a parameter even for finite-μ QCD

Aoki phase-like structure

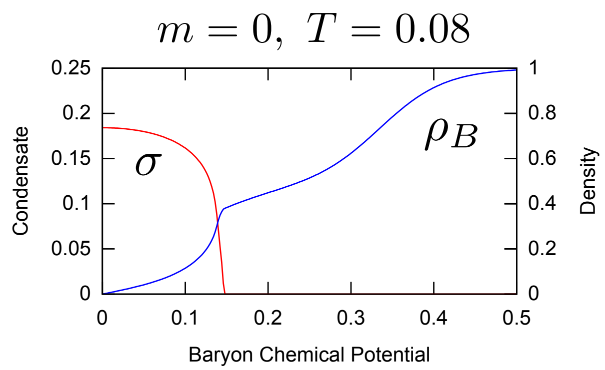


Creutz & Misumi(2012)

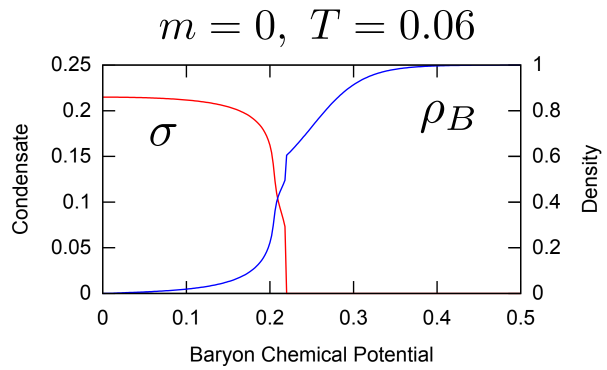
◆ Finite (T, μ) QCD with FCP

Misumi, Kimura, Ohnishi (2012)

$$S_{\rm md}=\sum_x\left[\frac{1}{2}\sum_{j=1}^{3}\bar\psi_x\gamma_j\left(U_{x,x+j}\psi_{x+j}-U_{x,x-j}\psi_{x-j}\right)+\frac{1}{2}\bar\psi_x\gamma_4\left(e^{\mu}U_{x,x+4}\psi_{x+4}-e^{-\mu}U_{x,x-4}\psi_{x-4}\right)\right.$$
$$\left.+\frac{i}{2}\sum_{j=1}^{3}\underline{\bar\psi_x\gamma_4\left(2\psi_x-U_{x,x+j}\psi_{x+j}-U_{x,x-j}\psi_{x-j}\right)}+id_3\bar\psi_x\gamma_4\psi_x\right]$$



§ Strong-coupling study

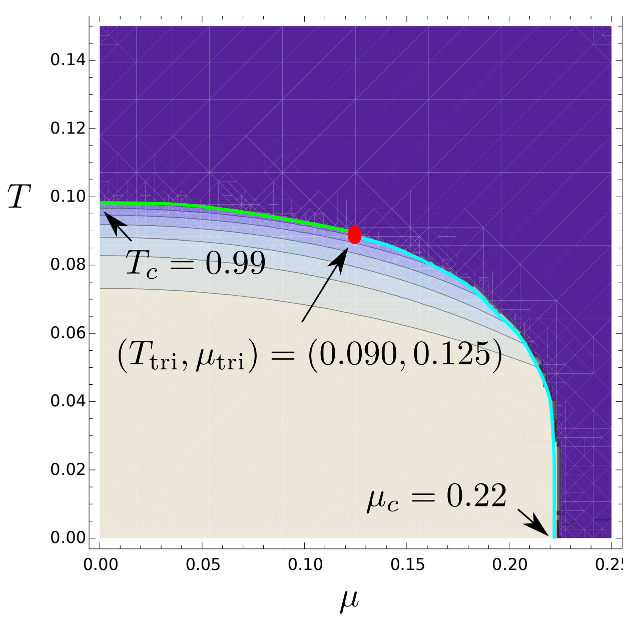


Effective potential of σ as a function of T, μ and d_3

$$\mathcal{F}_{\rm eff}(\sigma;m,T,\mu,d_3)=\frac{9}{2}\sigma^2-\frac{3}{2}\log\left(1+(d_3+3)^2\right)$$
$$-\max\left\{3\ \mathrm{arcsinh}\left(\frac{3\sqrt{2}\sigma}{2\sqrt{1+(d_3+3)^2}}\right),\mu_B\right\}.$$

➡ Chiral phase structure

- 1st and 2nd phase transition (m=0)
- 1st, critical point and crossover (m≠0)



New possibility of (T,μ) lattice QCD !

4. Summary

1. Flavored-mass terms give us new types of Wilson and overlap fermions.

2. Staggered-Wilson can be an alternative Wilson and overlap for 2-flavor QCD (3 degenerate pion spectrum)

3. Central-branch fermion is a new possibility of use of Wilson for many-flavor QCD without fine-tuning of parameters.

4. Flavored-chemical-potential fermion would be useful for finite-temperature & density lattice QCD.

Back-up slides

Adams-type flavored mass

$$\bar{\psi}_x \psi_{x+\hat{1}+\hat{2}+\hat{3}+\hat{4}} = \bar{\chi}_x \gamma_4^{x_4}\gamma_3^{x_3}\gamma_2^{x_2}\gamma_1^{x_1}\gamma_1^{x_1+1}\gamma_2^{x_2+1}\gamma_3^{x_3+1}\gamma_4^{x_4+1}\chi_{x+\hat{1}+\hat{2}+\hat{3}+\hat{4}}$$
$$= (-1)^{x_2+x_4}\bar{\chi}_x \gamma_5 \chi_{x+\hat{1}+\hat{2}+\hat{3}+\hat{4}} \qquad (\gamma_5 \;\; \text{diagonalized})$$
$$\to \pm \bar{\chi}_x \epsilon \eta_1\eta_2\eta_3\eta_4 \chi_{x+\hat{1}+\hat{2}+\hat{3}+\hat{4}}$$

Hoelbling-type flavored mass

Hoelbling (2010), de Forcrand (2010)

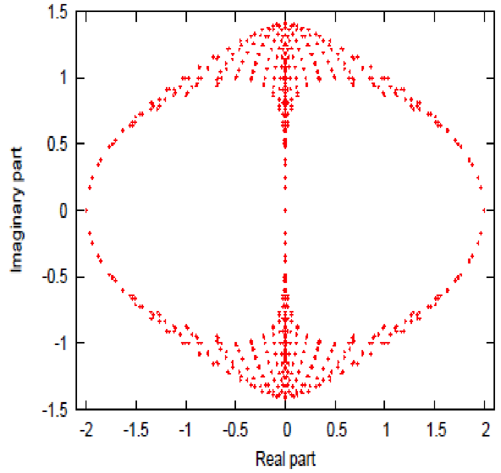
- *spin diagonalization*

$$\bar{\psi}_x \psi_{x+\hat{1}+\hat{2}} + \bar{\psi}_x \psi_{x+\hat{3}+\hat{4}} = (-1)^{x_2}\bar{\chi}_x \gamma_1\gamma_2 \chi_{x+\hat{1}+\hat{2}} + (-1)^{x_4}\bar{\chi}_x \gamma_3\gamma_4 \chi_{x+\hat{3}+\hat{4}}$$
$$\to \pm \bar{\chi}_x i\epsilon_{12}\eta_1\eta_2 \chi_{x+\hat{1}+\hat{2}} \pm \bar{\chi}_x i\epsilon_{34}\eta_3\eta_4 \chi_{x+\hat{3}+\hat{4}}$$

※ two terms simultaneously diagonalizable : $[\sigma_{12}, \sigma_{34}] = 0$

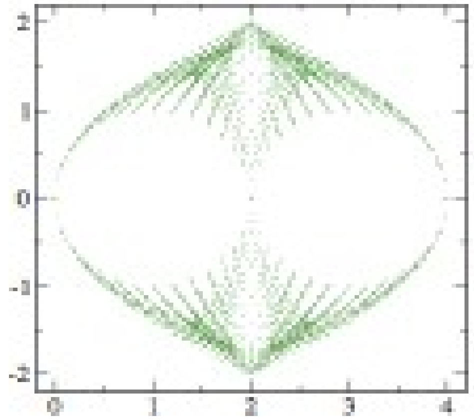
(4,8,4)

$M_{\mathrm{T}}^{(i)}$



$\longrightarrow$

(1,2,1)



M_H

Hoelbling, PLB696, 422(2011) [1009.5362].

- ***Shift symmetry*** $\longrightarrow$ broken to *2-link shift for S_A*
broken to *4-link shift for S_H*

$$\mathcal{S}_\rho: \ \chi_x \to \zeta_\rho(x)\chi_{x+\hat{\rho}}, \quad \bar{\chi}_x \to \zeta_\rho(x)\bar{\chi}_{x+\hat{\rho}}, \quad U_{\mu,x} \to U_{\mu,x+\hat{\rho}}$$

$$\mathcal{S}_\mu: \ \phi(p) \ \to \ \exp(ip_\mu)\Xi_\mu\,\phi(p)$$

- ***Axis reversal*** $\longrightarrow$ broken to *shifted axis reversal*

$$\mathcal{I}_\rho: \ \chi_x \to (-1)^{x_\rho}\chi_{Ix}, \quad \bar{\chi}_x \to (-1)^{x_\rho}\bar{\chi}_{Ix}, \quad U_{\mu,x} \to U_{\mu,Ix}$$

$$\mathcal{I}_\rho: \phi(p) \ \to \ \Gamma_\rho\Gamma_5\Xi_\rho\Xi_5\,\phi(Ip)$$

- ***Rotation*** $\longrightarrow$ remain *in S_A*
broken to *subgroup in S_H*

$$\mathcal{R}_{\rho\sigma}: \ \chi_x \to S_R(R^{-1}x)\chi_{R^{-1}x}, \quad \bar{\chi}_x \to S_R(R^{-1}x)\bar{\chi}_{R^{-1}x}, \quad U_{\mu,x} \to U_{\mu,Rx}$$

$$\mathcal{R}_{\rho\sigma}: \ \phi(p) \ \to \ \exp(\frac{\pi}{4}\Gamma_\rho\Gamma_\sigma)\exp(\frac{\pi}{4}\Xi_\rho\Xi_\sigma)\,\phi(R^{-1}p)$$

- ***Conjugation*** $\longrightarrow$ remain *in S_A*
broken *in S_H*

$$\mathcal{C}: \ \chi_x \to \epsilon_x\bar{\chi}_x^T, \ \bar{\chi}_x \to -\epsilon_x\bar{\chi}_x^T, \ U_{\mu,x} \to U_{\mu,x}^*$$

$$\mathcal{C}: \ \phi(p) \ \to \ \bar{\phi}(-p)^T$$

Axis and Rotation $\rightarrow$ $(\Gamma_4 \rtimes SW_{4,\mathrm{diag}})$

Details of StWil symmetries

$$\{\Xi_\mu, I_s, R_{\mu\nu}\} \quad \longrightarrow \quad \Gamma_4 \rtimes SW_4$$

$$\{\Xi'_\mu, R_{\mu\nu}\} \quad \longrightarrow \quad \Gamma_3 \rtimes SW_4$$

Physical-sector symmetry

$$\begin{aligned} \Xi'_j \Xi'_4 R^2_{j4} &= \Xi_j \Xi_4 \sim (1 \otimes \sigma_j) \\ \Xi'_4 R^2_{34} R^2_{12} &= \Xi_4 I_s \sim (\gamma_4 \otimes \mathbf{1}) \\ C_0 \Xi'_2 \Xi'_4 R^2_{24} &\sim C \end{aligned}$$

Details of timeslice symmetries

Enlarged staggered sym : $\{C_0, \Xi_\mu, I_s, R_{\mu\nu}, T_\mu^{1/2}\}$ $\qquad \Xi_\mu^2 = 1$

$$\longrightarrow T_\mu^{1/2} \rtimes [\{C_0, \Xi_\mu\} \rtimes \{R_{\mu\nu}, I_s\}] = (\otimes_j Z_{N_\mu}) \rtimes [\Gamma_{4,1} \rtimes W_4]$$

Timeslice sym : $T_\mu^{1/2} \rtimes [\{C_0, \Xi_\mu\} \rtimes \{R_{ij}, I_s\}] = (\otimes_j Z_{N_j}) \rtimes \underline{[\Gamma_{4,1} \rtimes W_3]}$

Relevant group at rest

$$\begin{aligned}\Gamma_{4,1} \rtimes W_3 &\sim [\underline{\{R_{ij}, \Xi_{ij}\}} \times \{C_0, \Xi_4, \Xi_{123}, I_s\}]/Z_2 \\ &= [\underline{\{R_{ij}, \tilde{R}_{4i} \equiv \epsilon_{ijk} R_{jk} \Xi_{kj}\}} \times \{C_0, \Xi_4, \Xi_{123}, C_0 \Xi_4 I_s\}]/Z_2 \\ &= [\underline{SW_4} \times \Gamma_{2,2}]/Z_2\end{aligned}$$

Staggered-Wilson

$\{C_0, \Xi'_\mu, R_{\mu\nu}, T'^{1/2}_\mu\}$

$$\begin{aligned}&\sim [\{R_{ij}, \Xi'_{ij}\} \times \{C_0, \Xi'_4, \Xi'_{123}, I_s\}]/Z_2 \\ &= [\{R_{ij}, \tilde{R}_{4i} \equiv \epsilon_{ijk} R_{jk} \Xi'_{kj}\} \times \{C_0, \Xi_4, \Xi_{123}]/Z_2 \\ &= \underline{[SW_4 \times \Gamma_{1,2}]}/Z_2\end{aligned}$$

Dim3, 4 : $\bar{Q}(1\otimes\xi_F)Q \quad \bar{Q}(\gamma_\mu\otimes\xi_F)D_\mu Q$ for $\xi_F = 1$ or ξ_5 $\Rightarrow$ $\bar{\ell}\gamma_\mu D_\mu \ell, \quad \bar{\ell}\ell$

Dim5 O(a): $\bar{Q}(i\sigma_{\mu\nu}F_{\mu\nu}\otimes\xi_F)Q$ for $\xi_F = 1$ or ξ_5 $\Rightarrow$ $\bar{\ell}i\sigma_{\mu\nu}F_{\mu\nu}\ell$

No unphysical term nor taste-breaking term up to O(a)

Dim6 O(a^2) : 2 types of four-fermi operators $\mathcal{L}_6^{FF(A)}$ and $\mathcal{L}_6^{FF(B)}$

In $\mathcal{L}_6^{FF(A)}$ the spin and flavor independently forms scalar

$\Rightarrow$ 25 operators with ξ_5 pair $\longrightarrow$ 50 operators

SA, SV, AS, VS, PV, PA, VP, AP, TV, TA, VT, AT, AA, PP , SP, PS, ST, PT, TS, TP, VV, AA, VA, AV, TT

$\longrightarrow$ No taste-breaking. No derivative terms. Contributes to potential $\mathcal{V}_6^{FF(A)}$

In $\mathcal{L}_6^{FF(B)}$ the spin and flavor are not independent

$\Rightarrow$ 10 operators with ξ_5 pair $\longrightarrow$ 20 operators

TV, TA, VT, AT, VV, AA, VA, AV, TT+, TT-

$\longrightarrow$ Taste-breaking. Derivative terms. No contribution to potential $\mathcal{V}_6^{FF(B)}$

No taste-breaking in ChPT potential upto O(a^2): SU(2)

◆ Central cusps for other flavored masses

progress in NTFL workshop (2012)

- For other naive flavored mass terms

M_{A} : U(1) restored

$M_{\mathrm{T}}^{(\mathrm{i})}$: U(2) restored

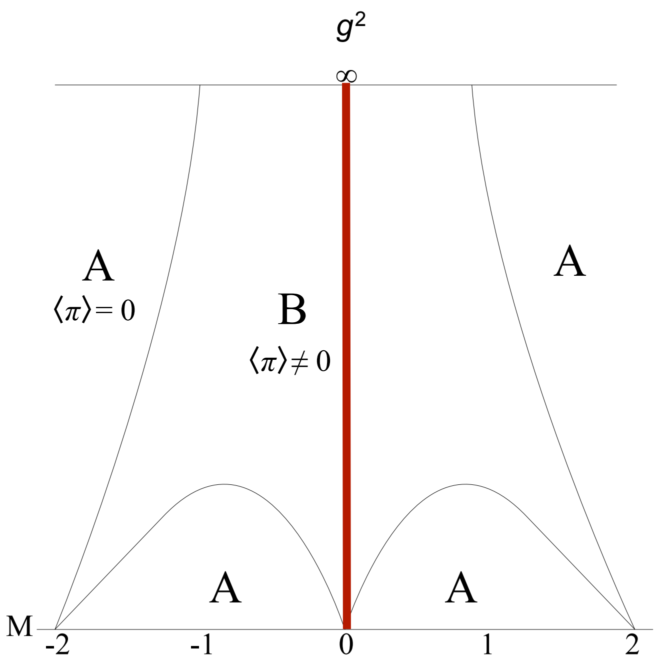
M_{P} : U(2)×U(2) restored

- For staggered flavored mass terms

$M_{\mathcal{A}}$: $C_T\Xi$, $C_T I$ restoed

$M_{\mathcal{H}}$: C_T restored

$$\mathcal{C}: \ \chi_x \to \bar{\chi}_x^T, \quad \bar{\chi}_x \to \chi_x^T, \quad U_{\mu,x} \to U_{\mu,x}^*$$



	C'_T	Ξ_μ	I_μ	$C'_T\Xi_\mu$	$C'_T I_\mu$	$\Xi_\mu I_\mu$
S_{st}	○	○	○	○	○	○
$S_{\mathcal{A}}$	×	×	×	○	○	○
$S_{\mathcal{H}}$	○	×	×	×	×	○
S_m	×	○	○	×	×	○

3.2 Strong coupling analysis

Now we employ the strong coupling analysis to show that there appears an NG boson associated with the $\mathrm{U}(1)_V^-$ symmetry breaking in the presence of the pion condensate. An effective action for mesons in the strong coupling limit [42, 9, 10] can be written in general as

$$S_{\rm eff}(M) = N_c \sum_n \left[\sum_\mu \mathrm{Tr}\, f(\Lambda_{n,\mu}) + \mathrm{tr}\, \hat{M} M(n) - \mathrm{tr}\, \log M(n) \right], \tag{23}$$

$$\Lambda_{n,\mu} = \frac{V_{n,\mu}\bar{V}_{n,\mu}}{N_c^2}, \quad M(n)^{\alpha\beta} = \frac{\sum_a \bar{\psi}_n^{a,\alpha}\psi_n^{a,\beta}}{N_c},$$

where N_c is the number of colors, Tr (tr) means a trace over color(spinor) index, and $M(n)$ is a meson field. The explicit form of the function f is determined by performing a one-link integral of the gauge field. More explicitly we can write

$$V_{n,\mu}^{ab} = \bar{\psi}_n^b P_\mu^- \psi_{n+\hat{\mu}}^a, \quad \bar{V}_{n,\mu}^{ab} = -\bar{\psi}_{n+\hat{\mu}}^b P_\mu^+ \psi_n^a, \tag{24}$$

$$\mathrm{Tr}\, f(\Lambda_{n,\mu}) = -\mathrm{tr}\, f\left(-M(n)(P_\mu^+)^{\rm T} M(n+\hat{\mu})(P_\mu^-)^{\rm T}\right), \tag{25}$$

where 4×4 matrices $P_\mu^\pm$ are specified later. In the large N_c limit, it is known that $f(x)$ can be analytically evaluated as

$$f(x) = \sqrt{1+4x} - 1 - \ln\frac{1+\sqrt{1+4x}}{2} = x + O(x^2). \tag{26}$$

However, in the following part of this paper, we will approximate $f(x)$ as $f(x) = x$ unless otherwise stated because qualitative features such as an appearance of NG bosons remain unchanged by this approximation.

To calculate meson masses we expand the meson field as[5]

$$M(n) = M_0^{\rm T} + \sum_X \pi^X(n)\Gamma_X^{\rm T}, \quad X \in \{S, P, V_\alpha, A_\alpha, T_{\alpha\beta}\}, \tag{27}$$

where M_0 is the vacuum expectation value (VEV) of $M(n)$, and

$$\Gamma_S = \frac{\mathbf{1}_4}{2},\ \Gamma_P = \frac{\gamma_5}{2},\ \Gamma_{V_\alpha} = \frac{\gamma_\alpha}{2},\ \Gamma_{A_\alpha} = \frac{\mathrm{i}\gamma_5\gamma_\alpha}{2},\ \Gamma_{T_{\alpha\beta}} = \frac{\gamma_\alpha\gamma_\beta}{2\mathrm{i}}\ (\alpha<\beta). \tag{28}$$

Then the effective action at the second order of π^X is given by

$$\begin{aligned} S_{\rm eff}^{(2)} &= N_c \sum_n \left[\frac{1}{2}\mathrm{tr}\,(M_0^{-1}\Gamma_X M_0^{-1}\Gamma_Y)\,\pi^X(n)\pi^Y(n) + \sum_\mu \mathrm{tr}\,(\Gamma_X P_\mu^- \Gamma_Y P_\mu^+)\pi^X(n)\pi^Y(n+\hat{\mu}) \right] \\ &= N_c \int \frac{d^4p}{(2\pi)^4} \pi^X(-p) D_{XY}(p)\pi^Y(p), \end{aligned} \tag{29}$$

where

$$D_{XY}(p) = \frac{1}{2}\left(\widetilde{D}_{XY}(p) + \widetilde{D}_{YX}(-p)\right), \tag{30}$$

$$\widetilde{D}_{XY}(p) = \frac{1}{2}\mathrm{tr}\,(M_0^{-1}\Gamma_X M_0^{-1}\Gamma_Y) + \sum_\mu \mathrm{tr}\,(\Gamma_X P_\mu^- \Gamma_Y P_\mu^+) e^{\mathrm{i}p_\mu}. \tag{31}$$

In the case of the Wilson fermion, $\hat{M} = (m+4r)\mathbf{1}_4 \equiv M_W \mathbf{1}_4$ and $P_\mu^\pm = \frac{\gamma_\mu \pm r}{2}$. By taking $M_0 = \sigma\mathbf{1}_4 + \mathrm{i}\pi\gamma_5$, we have

$$\begin{cases} \sigma = \dfrac{-M_W \pm \sqrt{M_W^2 + 8(1-r^2)}}{4(1-r^2)}, & \pi = 0, \qquad\qquad M_W^2 \geq M_c^2 \\ \sigma = \dfrac{M_W}{4r^2}, & \pi^2 = \dfrac{1}{16r^4(1+r^2)}(8r^4 - M_W^2(1+r^2)), \quad M_W^2 < M_c^2 \end{cases} \tag{32}$$

where $M_c^2 = \dfrac{8r^4}{1+r^2}$.

As discussed in the previous subsection, at $M_W = 0$ we have an additional U(1) symmetry, $\mathrm{U}(1)_V^-$. Since this parameter regime resides in the parity broken phase, in which $\pi^2 \neq 0$ and $M_W^2 < M_c^2$, $\mathrm{U}(1)_V^-$ is spontaneously broken by the VEV of π in this case.

To compute the meson mass, we hereafter take $r^2 = 1$ for simplicity. Because $D(p)$ is block-diagonal, we concentrate on its submatrix $D_{XY}(p)$ with $X, Y \in \{S, P, A_\alpha\}$. Then, by setting $p = (\pi, \pi, \pi, \pi + \mathrm{i}m_{SPA})$, we find that the S-P-A_α sector mass m_{SPA} is given by

$$\cosh(m_{SPA}) = 1 + \frac{20M_W^2}{6 - 7M_W^2}. \tag{33}$$

Note that since the transformation (22) involves the site-dependent quantity $(-1)^{n_1+\cdots+n_4}$, it is natural to expand the momentum p around (π, π, π, π). Eq. (33) tells us that the meson becomes a massless NG boson at $M_W = 0$ as expected. If we use the exact form of $f(x)$ in the large N_c limit, we then obtain

$$\cosh(m_{SPA}) = 1 + \frac{2M_W^2(16 + M_W^2)}{16 - 15M_W^2}, \tag{34}$$

which again shows that a massless NG boson appears at $M_W = 0$.

A. Effective potential

We first need to derive effective potential of meson fields including scalar one σ from (4) in the strong-coupling limit ($g^2 \to \infty$). We here consider general color number as N_c for $SU(N_c)$ gauge group and general space-time dimensions as $D+1$. For the purpose we perform the 1-link integral for the gauge field in the D-dimensional spatial part, and introduce auxiliary fields to eliminate the 4-point interaction as

$$\begin{aligned}
&\int \mathcal{D}U_1 \cdots \mathcal{D}U_D \exp\left[-\sum_x \sum_{j=1}^{D} \left(\bar{\psi}_x P_j^+ U_j(x) \psi_{x+\hat{j}} - \bar{\psi}_{x+\hat{j}} P_j^- U_j^\dagger(x) \psi_x\right)\right] \\
&\qquad = \exp\left[N_c \sum_x \left(\sum_{j=1}^{D} \mathrm{tr}\, \mathcal{M}(\mathrm{x})(P_\mathrm{j}^+)^\mathrm{T} \mathcal{M}(\mathrm{x}+\hat{\mathrm{j}})(P_\mathrm{j}^-)^\mathrm{T}\right) + \mathcal{O}(1/\sqrt{D})\right] \\
&\qquad = \int \mathcal{D}\sigma \mathcal{D}\pi_4 \exp\left[-N_c \sum_x \left(D\left((1+r^2)\sigma^2 + (1-r^2)\pi_4^2\right)\right.\right. \\
&\qquad\qquad \left.\left. - \frac{D}{2} \mathrm{tr}\left(\sqrt{1+\mathrm{r}^2}\sigma - \mathrm{i}\sqrt{1-\mathrm{r}^2}\pi_4\gamma_4\right) \mathcal{M}(\mathrm{x})\right)\right],
\end{aligned} \tag{5}$$

with

$$P_\mu^\pm = \begin{cases} (\gamma_\mu \pm ir\gamma_4)/2 & (\mu \neq 4) \\ \gamma_4/2 & (\mu = 4) \end{cases} \tag{6}$$

where we introduce the mesonic field as

$$\mathcal{M}^{\alpha\beta}(x) = \frac{1}{N_c} \delta_{ab} \bar{\psi}_x^{a,\alpha} \psi_x^{b,\beta}. \tag{7}$$

We note that two auxiliary fields σ and π_4 are required to get rid of four-fermi interactions in this case because of the Karsten-Wilczek term. π_4 condensate can be a signal of nonzero effective imaginary chemical potential, and we will discuss on this later. We also note that we dropped next-leading order of $O(1/\sqrt{D})$ expansions in (5), which corresponds to large D limit. We now have an intermediate form of the effective action from (4),

$$\begin{aligned} S_{\rm eff} &= \sum_x \Bigg[\frac{1}{2}\left(\bar{\psi}_x e^{\mu} U_4(x)\gamma_4 \psi_{x+\hat{4}} - \bar{\psi}_{x+\hat{4}} e^{-\mu} U_4^{\dagger}(x)\gamma_4\psi_x\right) + \bar{\psi}_x \left(m\mathbf{1} + i(d_3 + Dr)\gamma_4\right)\psi_x \\ &\quad + N_c D\left((1+r^2)\sigma^2 + (1-r^2)\pi_4^2\right) + \frac{N_c}{2} D \,\mathrm{tr}\left(\sqrt{1+\mathrm{r}^2}\sigma - \mathrm{i}\sqrt{1-\mathrm{r}^2}\pi_4\gamma_4\right)\mathcal{M}(\mathrm{x})\Bigg]. \end{aligned} \tag{8}$$

We here defined complex chemical potential as $\mu \equiv \mu_{\rm Re} + i\mu_{\rm Im}$. We make fourier transformation of the temporal direction ($\mu = 4$) by introducing Matsubara modes as,

$$\psi_{\tau,\vec{x}} = \frac{1}{\sqrt{N_\tau}}\sum_{n=1}^{N_\tau} e^{ik_n\tau}\tilde{\psi}_{n,\vec{x}}, \qquad \bar{\psi}_{\tau,\vec{x}} = \frac{1}{\sqrt{N_\tau}}\sum_{n=1}^{N_\tau} e^{-ik_n\tau}\tilde{\bar{\psi}}_{n,\vec{x}}, \qquad k_n = \frac{2\pi}{N_\tau}\left(n - \frac{1}{2}\right). \tag{9}$$

We here take Polyakov gauge and the link variable in the temporal direction is given by,

$$U_4(\vec{x}) = \begin{pmatrix} e^{i\phi_1(\vec{x})/N_\tau} & & & \\ & e^{i\phi_2(\vec{x})/N_\tau} & & \\ & & \ddots & \\ & & & e^{i\phi_{N_c}(\vec{x})/N_\tau} \end{pmatrix}, \qquad \sum_{a=1}^{N_c}\phi_a(\vec{x}) = 0. \tag{10}$$

with ϕ_a defined as components of gauge fields. It enables us to calculate fermionic determinant analytically as,

$$\begin{aligned} \det D &= \prod_{\vec{x}}\prod_{a=1}^{N_c}\prod_{n=1}^{N_\tau}\det\left[\left(m + \frac{D}{2}\sqrt{1+r^2}\sigma\right)\mathbf{1} + i\gamma_4\left(\sin\bar{k}_n^{(a)} + d_3 + Dr - \frac{D}{2}\sqrt{1-r^2}\pi_4\right)\right] \\ &\equiv \prod_{\vec{x}}\prod_{a=1}^{N_c}\prod_{n=1}^{N_\tau}\det\left[B + i\gamma_4 A\sin\tilde{k}_n^{(a)}\right] \\ &= \prod_{\vec{x}}\prod_{a=1}^{N_c}\prod_{n=1}^{N_\tau}\left(A^2\sin^2\tilde{k}_n^{(a)} + B^2\right)^2 \\ &= \prod_{\vec{x}} A^{4N_cN_\tau}\prod_{a=1}^{N_c}\left(2\cosh N_\tau E + 2\cos\left(\phi_a - iN_\tau\mu\right)\right)^4, \end{aligned} \tag{11}$$

where we define

$$A^2 = 1 + \left(d_3 + Dr - \frac{D}{2}\sqrt{1-r^2}\pi_4\right)^2, \qquad B = m + \frac{D}{2}\sqrt{1+r^2}\sigma, \tag{12}$$

$$E = \mathrm{arcsinh}\left(\frac{B}{A}\right) = \log\left[\frac{B}{A} + \sqrt{1 + \left(\frac{B}{A}\right)^2}\right], \tag{13}$$

with $\bar{k}_n^{(a)} = k_n + \phi_a/N_\tau - i\mu$. By integrating the temporal gauge field ϕ_a we derive

$$\int \mathcal{D}U_4 \prod_{\vec{x}} A^{4N_cN_\tau}\prod_{a=1}^{N_c}\left(2\cosh N_\tau E + 2\cos\left(\phi_a - iN_\tau\mu\right)\right)^4 = \prod_{\vec{x}}\left[\sum_{n\in Z}\det\left(Q_{n+i-j}\right)_{1\le i,j\le N_c}\right], \tag{14}$$

$$Q_n = \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\left(2\cosh N_\tau E + 2\cos\theta\right)^4 e^{-in\phi}, \qquad \theta = \phi - iN_\tau\mu. \tag{15}$$

For $N_c = 3$ these Q_n are explicitly given as

$$\begin{aligned} Q_0 &= 2(8\cosh^4 N_\tau E + 24\cosh^2 N_\tau E + 3) \\ Q_{\pm 1} &= 8\cosh N_\tau E(4\cosh^2 N_\tau E + 3)e^{\pm N_\tau\mu} \\ Q_{\pm 2} &= 4(6\cosh^2 N_\tau E + 1)e^{\pm 2N_\tau\mu} \\ Q_{\pm 3} &= 8\cosh N_\tau E\, e^{\pm 3N_\tau\mu} \\ Q_{\pm 4} &= e^{\pm 4N_\tau\mu} \\ Q_{|n|\ge 5} &= 0 \end{aligned} \tag{16}$$

As a result, the effective free energy is given by

$$\begin{aligned} \mathcal{F}_{\rm eff}(\sigma,\pi_4; m, T, \mu, d_3) &= \frac{N_c D}{4}\left((1+r^2)\sigma^2 + (1-r^2)\pi_4^2\right) - N_c\log A \\ &\quad - \frac{T}{4}\log\left(\sum_{n\in Z}\det\left(Q_{n+i-j}\right)_{1\le i,j\le N_c}\right). \end{aligned} \tag{17}$$

Here we redefine the free energy $4\mathcal{F}_{\rm eff} \to \mathcal{F}_{\rm eff}$. The calculation of the determinant part for $N_c = 3$ is moved to Appendix A 2. We here show only the result as

$$\begin{aligned} &\sum_{n\in Z}\det\left(Q_{n+i-j}\right)_{1\le i,j\le N_c} \\ &= 8\left(1 + 12\cosh^2\frac{E}{T} + 8\cosh^4\frac{E}{T}\right)\left(15 - 60\cosh^2\frac{E}{T} + 160\cosh^4\frac{E}{T} - 32\cosh^6\frac{E}{T} + 64\cosh^8\frac{E}{T}\right) \\ &\quad + 64\cosh\frac{\mu_B}{T}\cosh\frac{E}{T}\left(-15 + 40\cosh^2\frac{E}{T} + 96\cosh^4\frac{E}{T} + 320\cosh^8\frac{E}{T}\right) \\ &\quad + 80\cosh\frac{2\mu_B}{T}\left(1 + 6\cosh^2\frac{E}{T} + 24\cosh^4\frac{E}{T} + 80\cosh^6\frac{E}{T}\right) \\ &\quad + 80\cosh\frac{3\mu_B}{T}\cosh\frac{E}{T}\left(-1 + \cosh^2\frac{E}{T}\right) + 2\cosh\frac{4\mu_B}{T}. \end{aligned} \tag{18}$$

with

$$\mu_B = 3\mu. \tag{19}$$