Crank-Nicolson discretization scheme for lattice fermions

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Outline of the talk

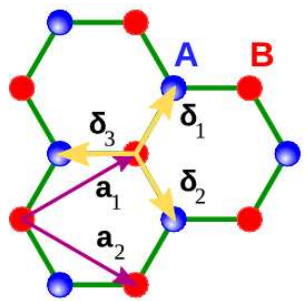
- Motivation
- Crank-Nicolson discretization scheme
- Applications

Motivation

Are we paying a fair price for chiral fermion simulations?

Why minimally Doubled Fermions?

Utility: Overlap/Domain wall are expensive.
Nature tells: 2d graphene electrons are chirally (sublattice) symmetric.



An incomplete list:

- Karsten-Wilczek fermion;
- 4d graphene/Creutz action, orthogonal axis action (A.B.), Bedaque et al., and many more.

Problem: violation of cubic/hypercubic symmetries.

- Add counterterms to restore symmetry, Capitani *et al*:

$\Rightarrow$ needs fine tuning like Wilson fermion!

Minimally doubled fermions violate unitarity

Example: Karsten-Wilczek fermion.

$$D_{KW}^{(r)} = i\gamma_4 \left[\sin p_4 + r \sum_k (1 - \cos p_k) \right] + i\vec{\gamma} \sin \vec{p} \,, \qquad |r| > 1/2$$

$\Rightarrow$ Complex energies:

$$\sinh E = \sigma \sin \vec{p} + ir \sum_k (1 - \cos p_k)$$

Fine tuning may be enough on an Euclidean lattice:
'time' could be along any direction!

Numerical investigations:

- Karsten-Wilczek fermions: Weber (this conference);
- Borici-Creutz fermions: Zeqirllari, in progress.

Fermions on the lattice

Something is always broken!

- Lorentz invariance broken to hypercubic symmetry:

  $\Rightarrow$ A lattice fermion with *all* continuum symmetries is *impossible!*

- Nielsen-Ninomiya theorem:

  Doublers are unaviodable on the lattice.

- Fermion actions:
  - Single fermion with broken chiral symmetry (Wilson,...);
  - Staggered fermions with remnant chiral symmetry;
  - Ginsparg-Wilson fermions with exact chiral symmetry.

How does it work?

- Continuum limit naive fermion spectrum:

  $E_{\vec{p}\,*}(\vec{p}\,) = \cos p_1^* \ \sigma_1 p_1 + \cos p_2^* \ \sigma_2 p_2 + \cos p_3^* \ \sigma_3 p_3$

  $\vec{p}\,^* \in \{(0,0,0),(0,0,\pi),(0,\pi,0),(\pi,0,0),(0,\pi,\pi),(\pi,0,\pi),(\pi,\pi,0),(\pi,\pi,\pi)\}$

- Chirality $\chi(\vec{p}\,^*) = \cos p_1^* \cos p_2^* \cos p_3^*$

$\vec{p}\,^*$	$\chi(\vec{p}\,^*)$
$(0,0,0)$	1
$(0,0,\pi),(0,\pi,0),(\pi,0,0)$	-1
$(0,\pi,\pi),(\pi,0,\pi),(\pi,\pi,0)$	1
(π,π,π)	-1

- Wilson fermions:

  Doublers pushed at the cutoff $\Rightarrow$ Broken chiral symmetry.

- Staggered fermions:

  Spinors distributed around plaquette $\Rightarrow$ Broken 'taste' symmetry.

$\Rightarrow$ Something is always broken!

Overlap/Domain Wall Fermions

On the-mass-shell chirality
Ginsparg-Wilson relation $\{\gamma_5, D\} = aD\gamma_5 D$

- Overlap fermions: zero mode protection;
- Domain Wall fermions: walls isolate and separate chiral modes.

Three great things about GW fermions

1. Great theory: symmetry exact on the lattice;
2. Great complexity: nested/5d inversions;
3. Great difficulty: Hamiltonian almost no spectral gap.

$\Rightarrow$ Full 5D Domain wall: remnant residual mass, however small.

$\Rightarrow$ Overlap operator/DW with Pauli-Villars: potentially nonlocal.

A lot of progress: a question of computer time!

Explore minimally doubled fermions

Fine tuning could be a solution, but it is not yet attractive!

Restoring hypercubic symmetry without fine tuning is more attractive.

At least restore unitarity and be on the safe side.

Crank-Nicolson discretization scheme

- Example: Schrödinger equation in Euclidean space:

$$\partial_t \psi(t,x) = H\psi(t,x) \ , \qquad \psi(0,x) = \psi_o(x) \ .$$

- Forward differences:

$$\partial\psi(t,x) \to \frac{1}{a_t}\left[\psi(t+a_t,x) - \psi(t,x)\right]$$

- $\Rightarrow$ order $O(a_t)$ errors $\Rightarrow$ Euler scheme.
- Crank-Nicholson scheme:

$$H\psi(t,x) \to \frac{H}{2}\left[\psi(t,x) + \psi(t+a_t,x)\right] \ .$$

$$\Rightarrow \ \psi(t+a_t,x) = \frac{\mathbb{1}+\frac{a_t}{2}H}{\mathbb{1}-\frac{a_t}{2}H}\psi(t,x) = \left(\mathbb{1} + a_t H + \frac{a_t^2}{2}H^2 + O(a_t^3)\right)\psi(t,x) \ ;$$

$\Rightarrow$ $O(a_t^2)$ errors!

Immediate implication: halving the speicies of the naive action!

- Momentum space spin-1/2 Hamiltonian on the lattice: $H = \vec{\sigma} \sin \vec{p}$;
- $\Rightarrow$ Crank-Nicolson time discretised operator:

$$d'(p) = e^{ip_4} - 1 + \frac{1}{2}\vec{\sigma} \sin \vec{p} \, (e^{ip_4} + 1) \ .$$

- Solutions:

$$d'(p) = 0 \ \Leftrightarrow \ 2i \sin \frac{p_4}{2} + \vec{\sigma} \sin \vec{p} \ \cos \frac{p_4}{2} = 0 \ .$$

- $\Rightarrow$ 8 zeros located at the edges of the 3d Brillouin zone.

Dirac fermions

Add opposite chirality partner:

$$D^{+}(p) = \gamma_4 \, (e^{ip_4} - 1) + \frac{i}{2}\vec{\gamma} \sin \vec{p} \, (e^{ip_4} + 1)$$

Result:

- 8 degenerate flavors of Dirac fermions;
- Second order accuracy in time;
- Broken cubic symmetry.

Restore cubic symmetry

1. Add isospin partner;
2. Use backward differences for the partner.

$$D^+(p) = \gamma_4 \, (e^{ip_4} - 1) + \frac{i}{2}\vec{\gamma}\sin\vec{p}\,(e^{ip_4} + 1)$$

$$D^-(p) = \gamma_4 \, (1 - e^{-ip_4}) + (e^{-ip_4} + 1) \, \frac{i}{2}\vec{\gamma}\sin\vec{p}$$

$$\mathcal{D}(p) = \begin{bmatrix} 0 & D^+(p) \\ D^-(p) & 0 \end{bmatrix}$$

Result:

- Cubic symmetry restoration using flavored matrix $\gamma_4 \otimes \tau_1$;
- γ_5 symmetric operator;
- Doublet of 8 flavors: 16 degenerate flavors;
- Second order accuracy in time.

Reduction to 2 flavors: Wilson fermions in 3-space

Wilson approach in 3-space: only one fermion in continuum limit

$$D_W^+(p) = \gamma_4 \ (e^{ip_4} - 1) + \frac{1}{2}\left[i\vec{\gamma}\sin\vec{p} + \sum_k (1 - \cos p_k)\right] \ (e^{ip_4} + 1)$$

$$D_W^-(p) = \gamma_4 \ (1 - e^{-ip_4}) + (e^{-ip_4} + 1) \ \frac{1}{2}\left[i\vec{\gamma}\sin\vec{p} + \sum_k (1 - \cos p_k)\right]$$

$$D_W(p) = \begin{bmatrix} 0 & D_W^+(p) \\ D_W^-(p) & 0 \end{bmatrix}$$

Result:

- 2 flavors;
- γ_5 symmetric operator;
- Second order accuracy in time;
- Reduced chiral symmetry breaking w.r.t. Wilson fermion.

Minimally doubled fermions

Start with the theory of 8 flavors:

$$D_N^+(p) = \gamma_4 \, (e^{ip_4} - 1) + \frac{i}{2} \vec{\gamma} \sin \vec{p} \, (e^{ip_4} + 1)$$

Use Borici-Creutz construction in 3-space:

$$D(p) = \gamma_4 \, (e^{ip_4} - 1) + \frac{i}{2} \sum_k \left[\gamma_k \sin p_k + \gamma_k' (\cos p_k - 1) \right] \, (e^{ip_4} + 1)$$

$$\sum_k \gamma_k = \sum_k \gamma_k'$$

$\Rightarrow$ 2 flavors with broken cubic symmetry!

Weyl fermions

Nielsen-Ninomiya theorem: Doublers unavoidable

It is possible, however:

- A theory of many particles with different speed of light:

$$E_n(\vec{p}) = c_n \vec{\sigma}_n \vec{p} \ , \quad n = 1, 2, \ldots, 2m \ .$$

- $\Rightarrow$ Non-degenerate spectrum;
- The ground state is our theory;
- Chiral anomaly is not neccessarily canceled!

Lattice implementation

- Start from Crank-Nicolson discretisation in time and naive discretisation in 3-space:

$$d'(p) = e^{ip_4} - 1 + \frac{1}{2}\vec{\sigma}\sin\vec{p}\,(e^{ip_4} + 1)\ ;$$

- Add a pure imaginary operator of the Wilson type:

$$d(p) = e^{ip_4} - 1 + \frac{1}{2}\left[\vec{\sigma}\sin\vec{p} + ir\sum_k(1-\cos p_k)\right](e^{ip_4} + 1)\ .$$

- $d(p) = 0 \ \Leftrightarrow$

$$\left\{2\sin\frac{p_4}{2} + \left[r\sum_k(1-\cos p_k)\right]\cos\frac{p_4}{2}\right\}^2 + \sin^2\vec{p}\,\cos^2\frac{p_4}{2} = 0\ .$$

Particle spectra

- 8 zeros in the edges of 3d Brillouin zone:

  $\vec{p}^{\,*} \in \{(0,0,0),(0,0,\pi),(0,\pi,0),(\pi,0,0),(0,\pi,\pi),(\pi,0,\pi),(\pi,\pi,0),(\pi,\pi,\pi)\}$

- Define $n(\vec{p}^{\,*}) = \frac{1}{2}\sum_k (1-\cos p_k^*)$

- $\Rightarrow\ D(p) = 0 \ \Leftrightarrow \tan\frac{p_4^*}{2} = -rn(\vec{p}^{\,*})$.

- Define chirality $\chi(\vec{p}^{\,*}) = \cos p_1^* \cos p_2^* \cos p_3^* \quad \Rightarrow$

$\vec{p}^{\,*}$	$n(\vec{p}^{\,*})$	$\chi(\vec{p}^{\,*})$	Degeneracy
$(0,0,0)$	0	1	1
$(0,0,\pi),(0,\pi,0),(\pi,0,0)$	1	-1	3
$(0,\pi,\pi),(\pi,0,\pi),(\pi,\pi,0)$	2	1	3
(π,π,π)	3	-1	1

- Continuum limit dispersion relation: $E_n(\vec{p}) = \frac{\vec{\sigma}_n \vec{p}}{1+r^2n^2}\ , \quad n = 0,1,2,3.$

- Ground state with definite chirality.

Conclusions

CN discretisation offers a new dimension for exploring lattice fermions:

- Explicit unitarity;
- 2 flavors of Wilson type fermions in 3-space and second order accyracy in time;
- 2 flavors of minimally doubled fermions with broken cucic symmetry;
- A theory of 8 non-degenerate Weyl fermions.