Finite volume quantization conditions for multiparticle states

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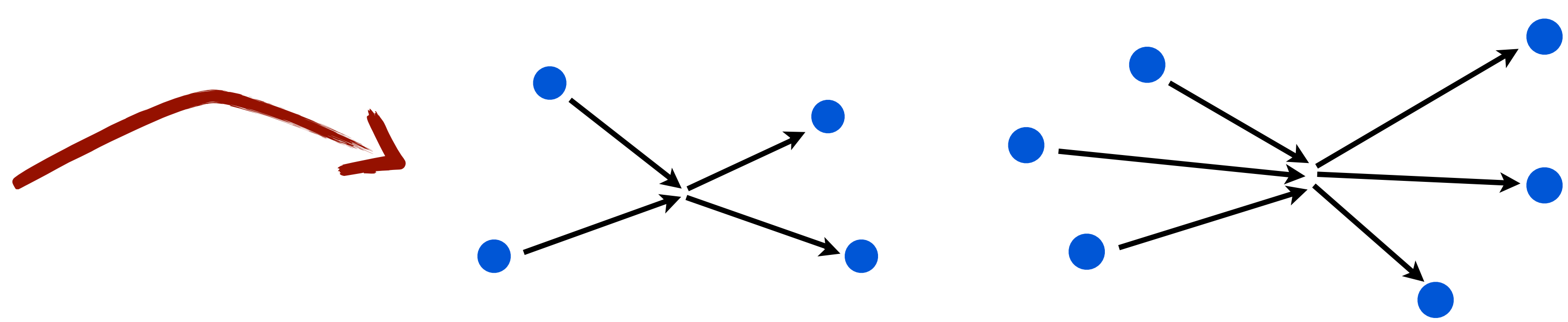
The fundamental issue

- Lattice simulations are done in finite volumes
- Experiments are not



How do we connect these?

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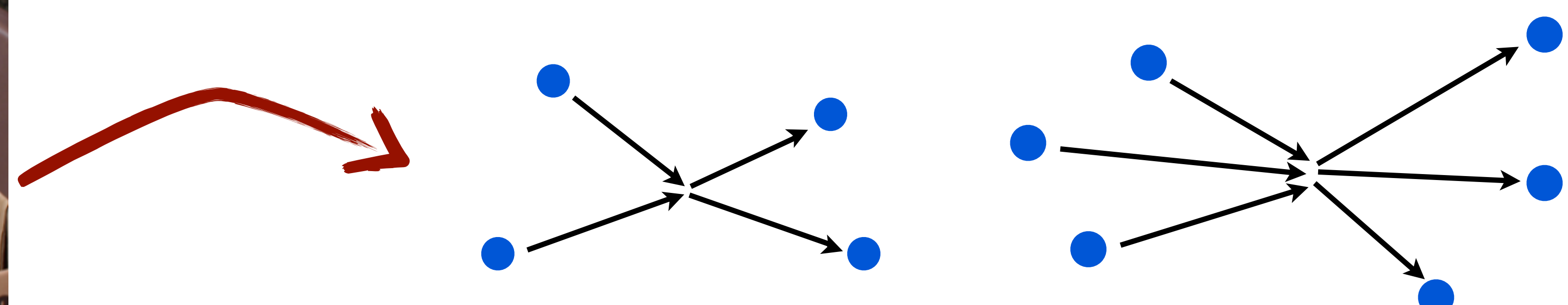

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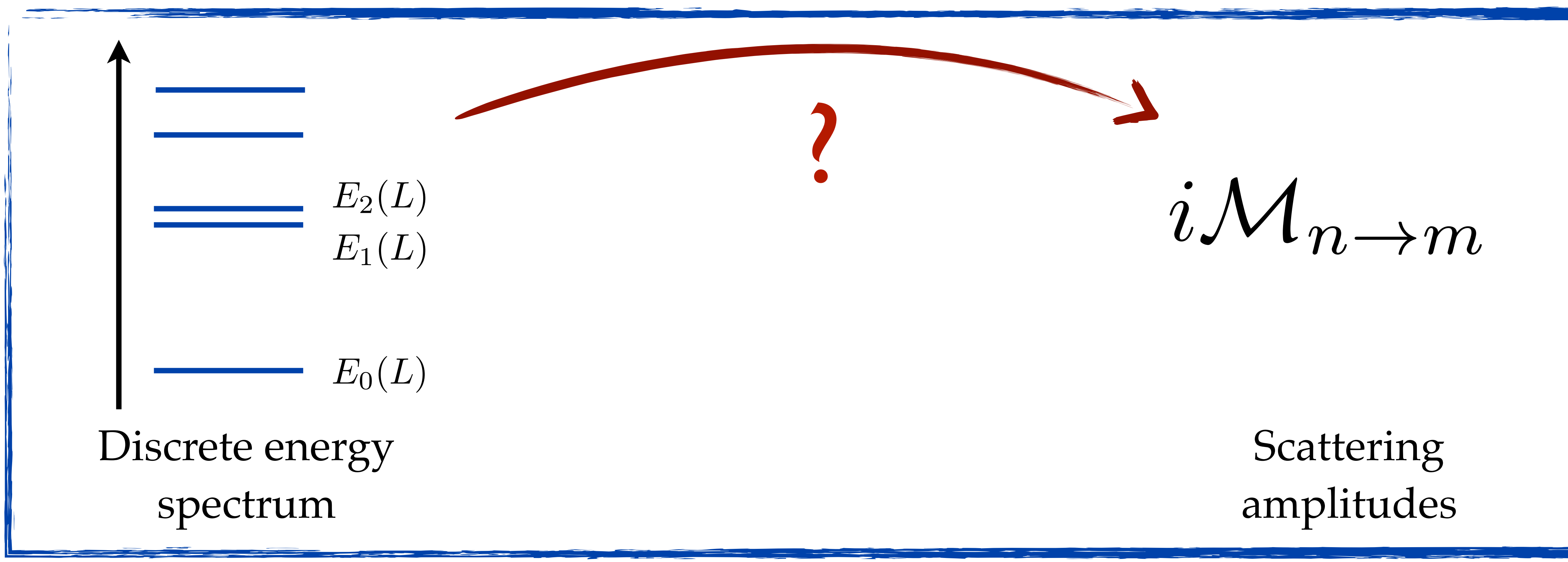
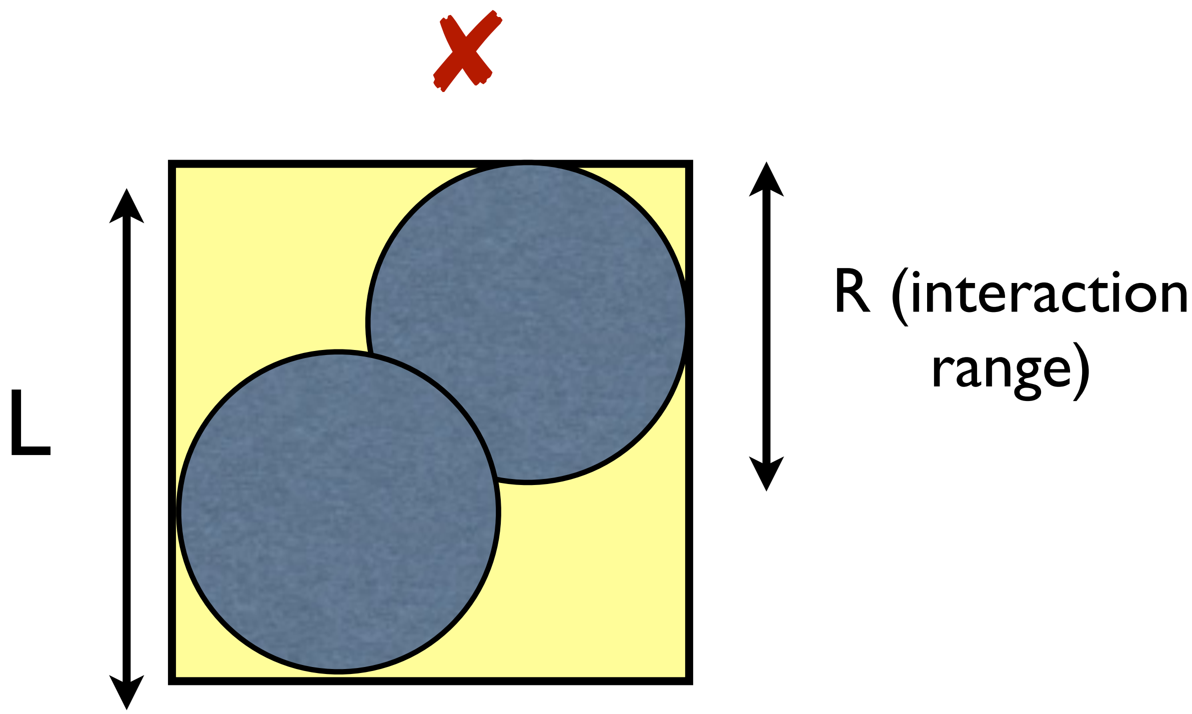
The fundamental issue

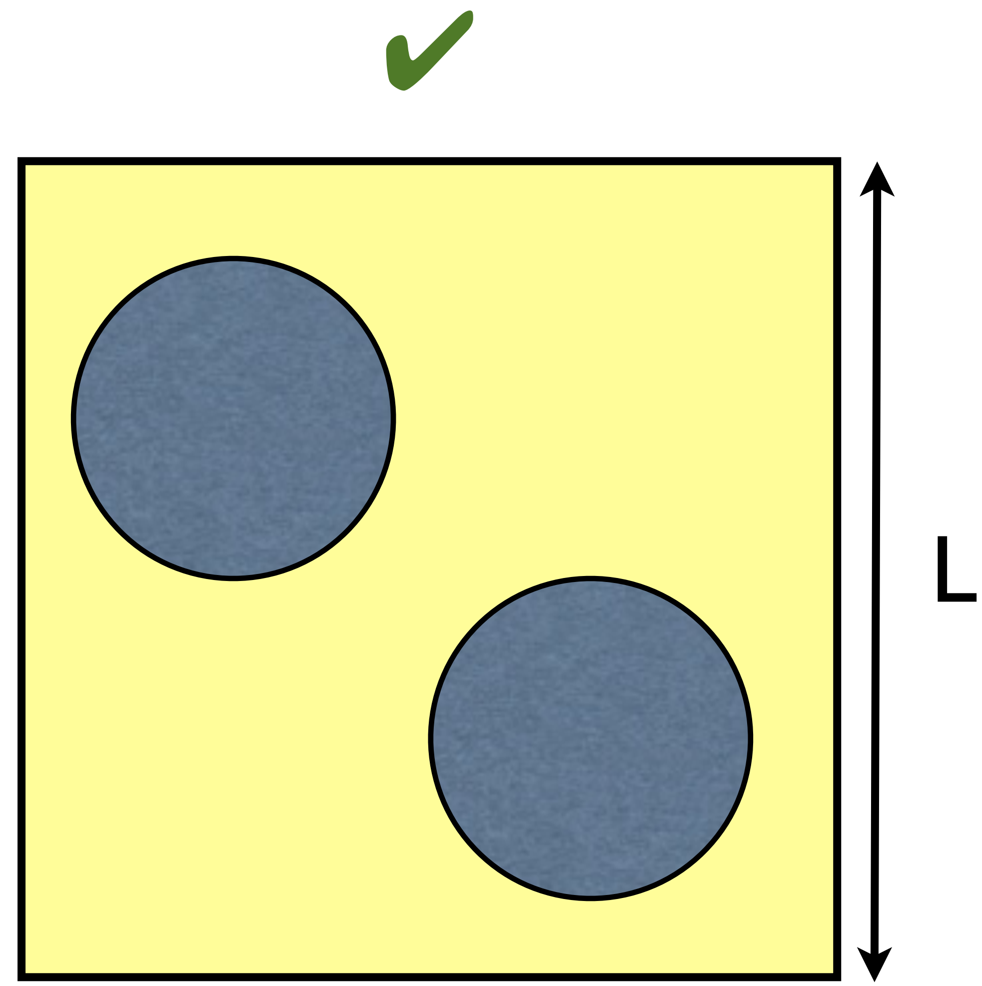
- Lattice QCD can calculate energy levels of multiple particle systems in a box
- How are these related to scattering amplitudes?



When is spectrum related to scattering amplitudes?



L<2R
No "outside" region.
Spectrum NOT related to scatt. amps.
Depends on finite-density properties

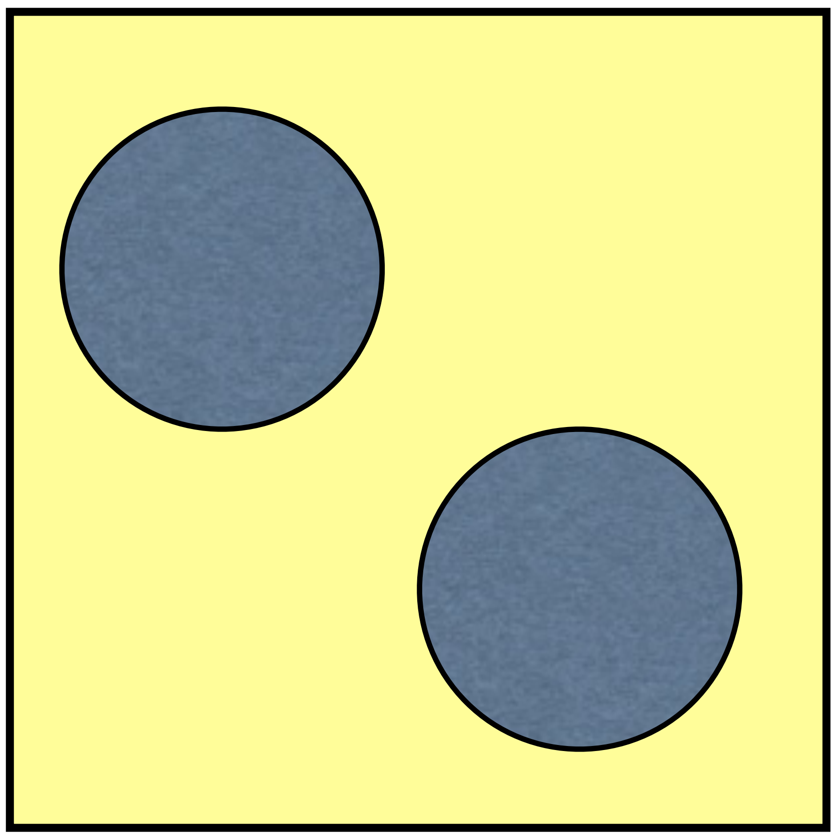


L>2R
There is an "outside" region.
Spectrum IS related to scatt. amps.
up to corrections proportional to
$e^{-M_\pi L}$

[Lüscher]

Systems considered today

Quantization conditions

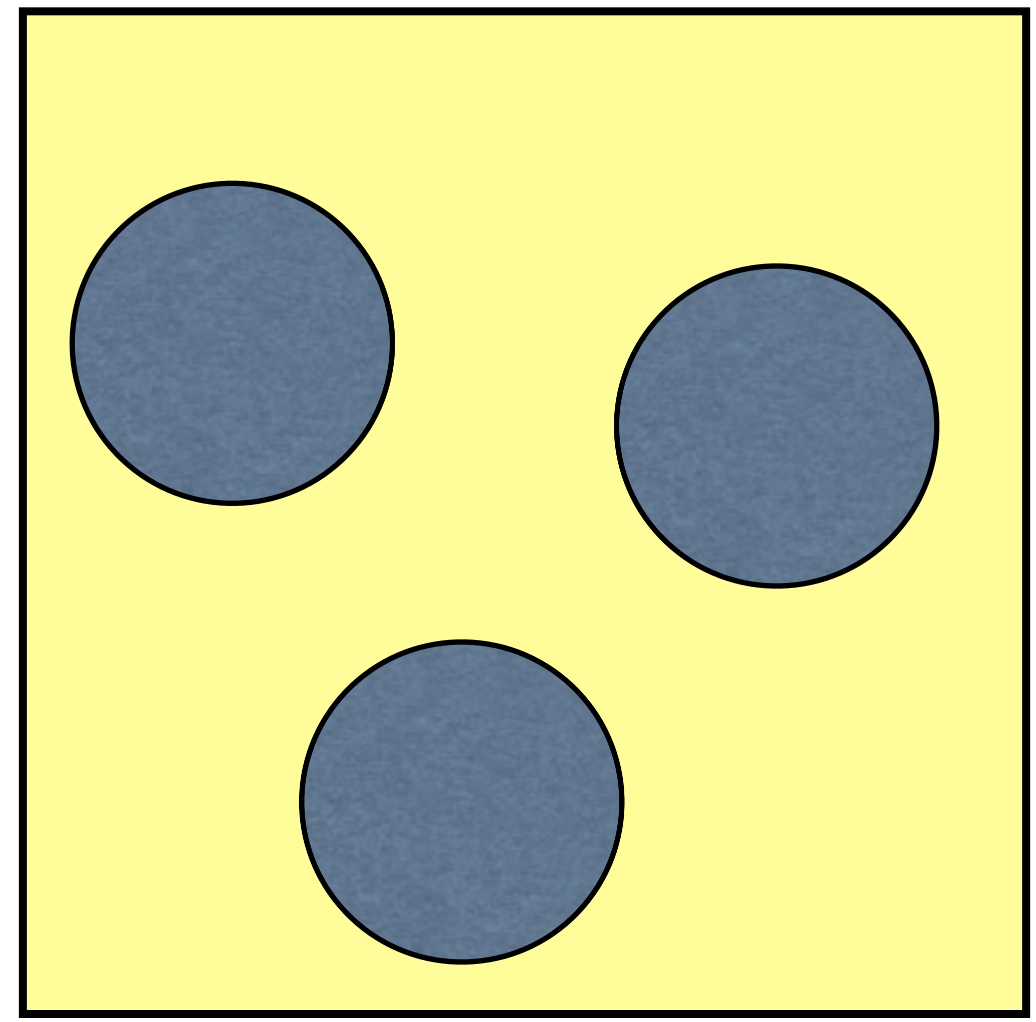


Theoretically understood;
numerical implementations mature

[Mohler, Wilson]

What about including QED?

[Beane, Davoudi]



Formalism under development

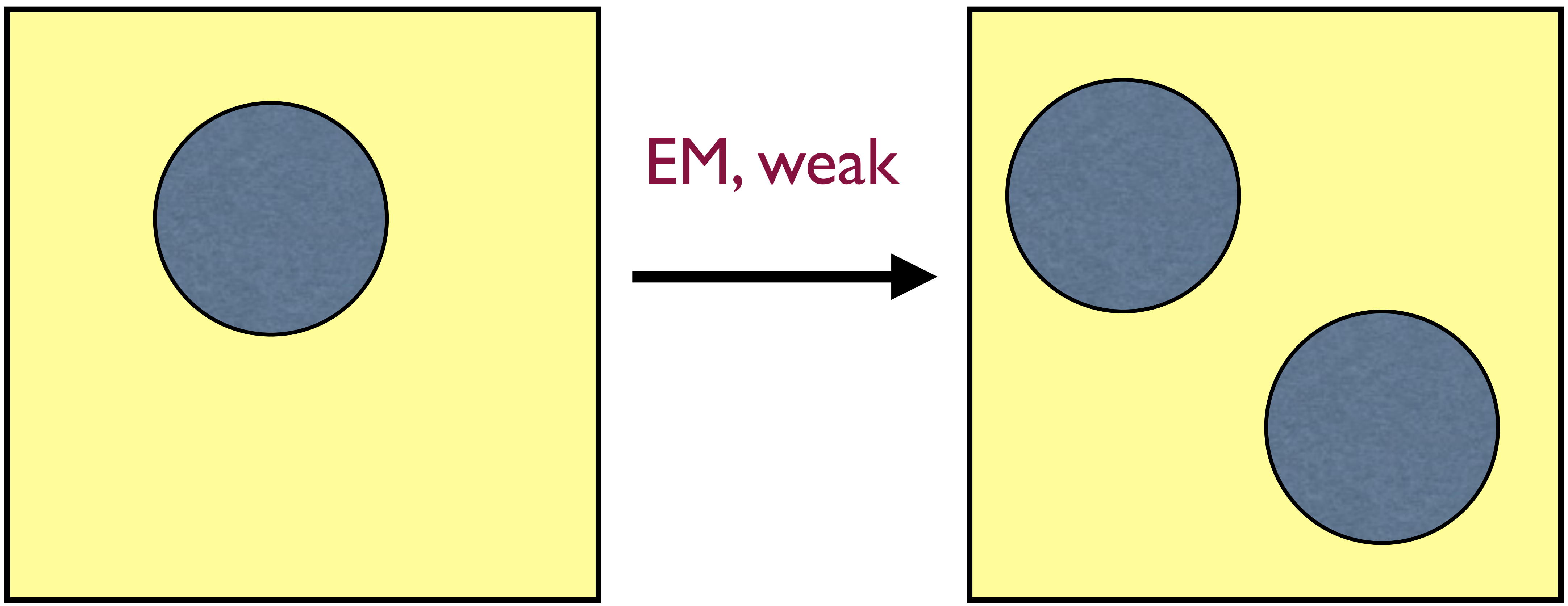
[Hansen]

How implement numerically?

[Doi]

Systems considered today

Transition amplitudes



Theoretically understood; [Agadjanov, Briceño]

numerical implementations expanding [Ishizuka, Kelly, Shultz]

Outline

- Motivation
- Theoretical status
- Key theoretical ingredients
- 2-particle quantization condition
- Future directions & challenges

Studying resonances

- Most hadrons are resonances
  - Resonances are not asymptotic states; show up in behavior of scatt. amplitudes
  - FV methods determine scattering amplitudes indirectly

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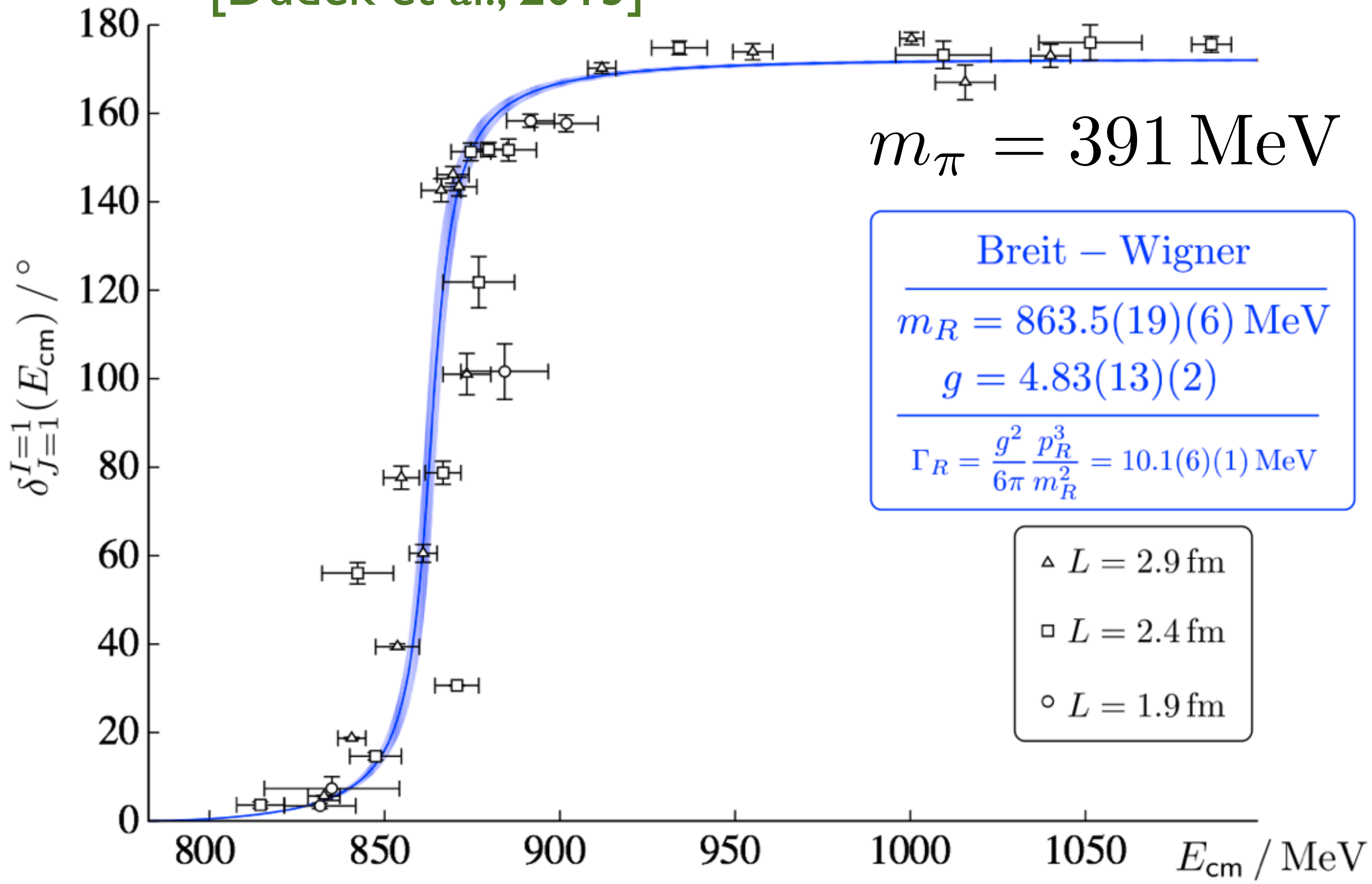
ρ resonance in ππ phase shift

[Dudek et al., 2013]



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  - FV methods aim to determine scattering amplitudes indirectly
- Many resonances have three particle decay channels

$$\omega(782) \rightarrow \pi\pi\pi \qquad K^* \longrightarrow K\pi\pi \quad N(1440) \rightarrow N\pi\pi$$

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$$\omega(782) \to \pi\pi\pi \qquad K^* \longrightarrow K\pi\pi \quad N(1440) \to N\pi\pi$$

- Most resonances have multiple decay channels

$$a_0(980) \longrightarrow \eta\pi, K\overline{K} \qquad f_0(980) \longrightarrow \pi\pi, K\overline{K}$$

Determining interactions

- For nuclear physics need NN <u>and</u> NNN interactions
  - Input for effective field theory treatments of larger nuclei & nuclear matter

- Meson interactions needed for understanding pion & kaon condensates
  - $\pi\pi$, $K\overline{K}$, $\pi\pi\pi$, $\pi K\overline{K}$, etc.

Calculating decay amplitudes

- Weak decay amplitudes allow tests of SM
  - $K \to \pi\pi$, $\pi\pi\pi$
  - $D \to \pi\pi$, $K\overline{K}$, $\eta\eta$, 4π, ….
  - $B \to K\pi$ (+ $\ell^+ \ell^-$)
  - …
- EM transition amplitudes probe hadron structure

$$\rho \longrightarrow \pi\gamma^* \qquad N\gamma^* \longrightarrow \Delta \longrightarrow N\pi$$

Theoretical status

Status for 2 particles

- Long understood in NRQM [Huang & Yang 57, ...]
- Quantization formula in QFT for energies below inelastic threshold converted into NRQM problem and solved by [Lüscher 86 & 91]
- Solution generalized to arbitrary total momentum **P**, multiple (2 body) channels, general BCs and arbitrary spins [Rummukainen & Gottlieb 85; Kim, Sachrajda & SS 05; Bernard, Lage, Meißner & Rusetsky 08; Hansen & SS 12; Briceño & Davoudi 12; …]
- Relation between finite volume $1 \to 2$ weak amplitude (e.g. $K \to \pi\pi$) and infinite volume decay amplitude determined [Lellouch & Lüscher 00]
- LL formula generalized to general **P**, to multiple (2 body) channels, and to arbitrary currents and general BCs (e.g. $\gamma^*\pi \to \rho \to \pi\pi$, $\gamma^*N \to \Delta \to \pi N$, $\gamma D \to NN$) [Kim, Sachrajda & SS 05; Christ, Kim & Yamazaki 05; Meyer 12; Hansen & SS 12; Briceño & Davoudi 12; Agadjanov, Bernard, Meißner & Rusetsky 14; Briceño, Hansen & Walker-Loud 14; …]
- Leading order QED effects on quantization condition determined [Beane & Savage 14]

Status for 3 particles

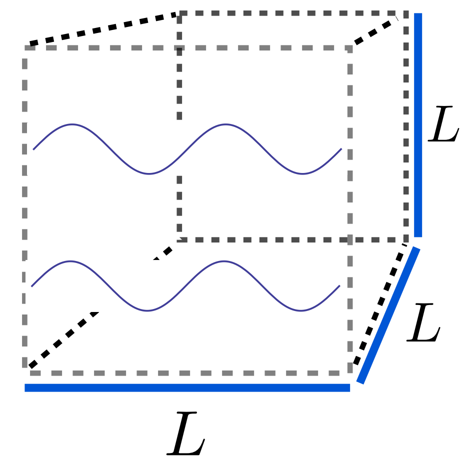
- [Beane, Detmold & Savage 07 and Tan 08] derived threshold expansion for n particles in NRQM, and argued it applied also in QFT
- [Polejaeva & Rusetsky 12] showed in NREFT that 3 body spectrum determined by infinite-volume scattering amplitudes, using integral equation
- [Briceño & Davoudi 12] used a dimer approach in NREFT, with s-wave interactions only, to determine relation between spectrum and a finite volume quantity, itself related to infinite-volume amplitudes by an integral equation
- [Hansen & SS 14, 15] derived quantization condition in (fairly) general, relativistic QFT relating spectrum and $\mathcal{M}_2$ and 3-body scattering *quantity* $K_{df,3}$; relation between $K_{df,3}$ & $\mathcal{M}_3$ via integral equations now known
- [Meißner, Rios & Rusetsky 14] determined volume dependence of 3-body bound state in unitary limit

Some key theoretical ingredients

Following method of [Kim, Sachrajda & SS 05]

Set-up

- Work in continuum (assume that LQCD can control discretization errors)



- Cubic box of size L with periodic BC, and infinite (Minkowski) time
  - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\vec{k} = \frac{2\pi}{L}\vec{n}$
  - Easily extend to other BC (e.g. twisted)

- Consider general QFT with arbitrary vertices

Methodology

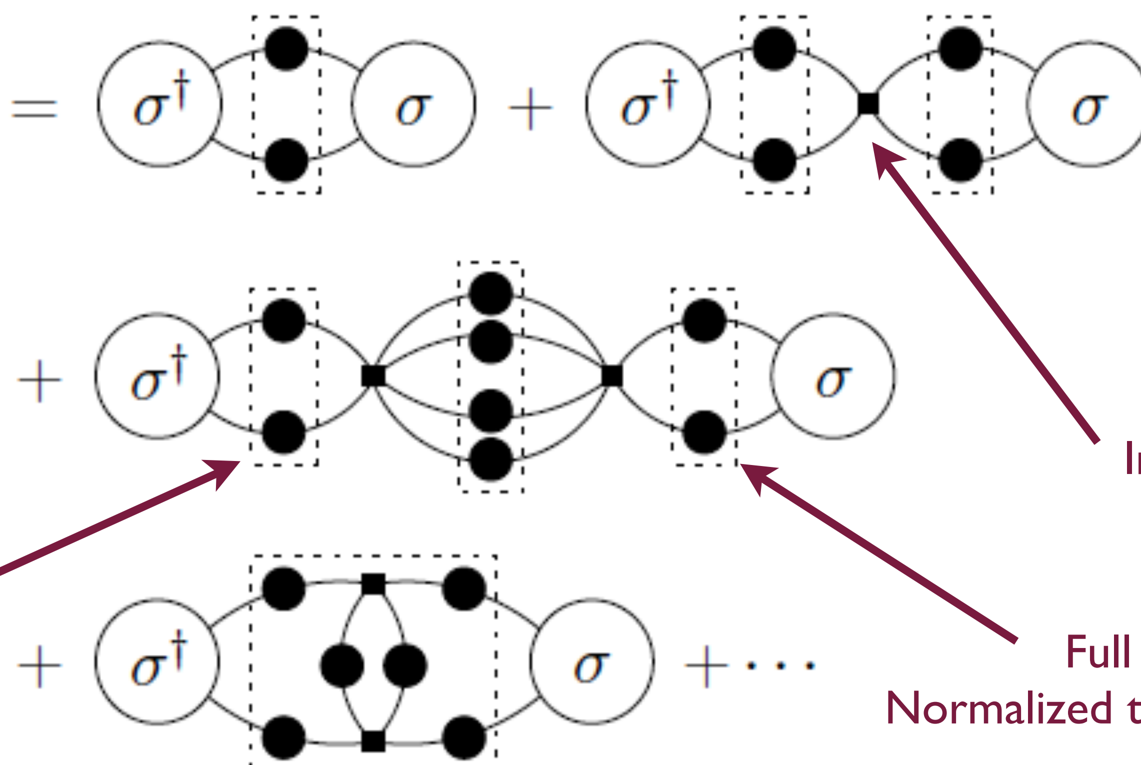
CM energy is
$E^* = \sqrt{(E^2 - P^2)}$

- Calculate (for some $\mathbf{P} = 2\pi \mathbf{n}_P / L$)

$$C_L(E, \vec{P}) \equiv \int_L d^4x \; e^{-i\vec{P}\cdot\vec{x} + iEt} \langle \Omega | T\sigma(x)\sigma^\dagger(0) | \Omega \rangle_L$$

- Poles in C_L occur at energies of finite-volume spectrum
- For 2 & 3 particle states, $\sigma \sim \pi^2$ & π^3, respectively
- Use all-orders diagrammatic expansion, e.g.

$$C_L(E, \vec{P}) =$$



Infinite-volume vertices

Boxes indicated summation over finite-volume momenta

Full propagators
Normalized to unit residue at pole

Key step 1

- Replace loop sums with integrals where possible
  - Drop exponentially suppressed terms ($\sim e^{-ML}$, $e^{-(ML)^2}$, etc.) while keeping power-law dependence

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})$$

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Exp. suppressed if g(k) is smooth
and scale of derivatives of g is ~1/M

Key step 2

- Use "sum=integral + [sum-integral]" if integrand has pole, with [KSS]

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}\ \ (q^*, q^{*'})g^*(\hat{q}^{*'}) \quad + \text{exp. suppressed}$$

q^* is relative momentum of pair on left in CM

Kinematic function

f & g evaluated for ON-SHELL momenta
Depend only on direction in CM

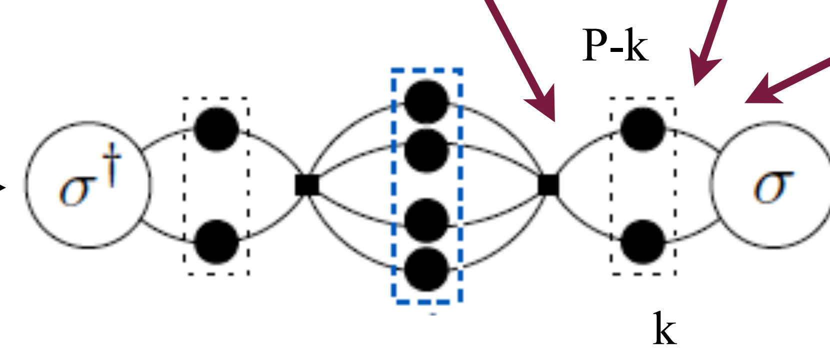
- Example

Focus on this loop

f is left-hand part of integrand

g is right-hand part of integrand

$P = (E, \vec{P}) \longrightarrow$



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$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}\ \ (q^*, q^{*\prime})g^*(\hat{q}^{*\prime})$$

- Decomposed into spherical harmonics, $\mathcal{F}$ becomes

$$F\ \ _{\ell_1,m_1;\ell_2,m_2} \equiv \quad \eta\left[\frac{\mathrm{Re}q^*}{8\pi E^*}\delta_{\ell_1\ell_2}\delta_{m_1m_2} + \frac{i}{2\pi EL}\sum_{\ell,m} x^{-\ell}\,\mathcal{Z}^P_{\ell m}[1;x^2]\int d\Omega Y^*_{\ell_1,m_1}Y^*_{\ell,m}Y_{\ell_2,m_2}\right]$$

$x \equiv q^*L/(2\pi)$ and $\mathcal{Z}^P_{\ell m}$ is a generalization of the zeta-function

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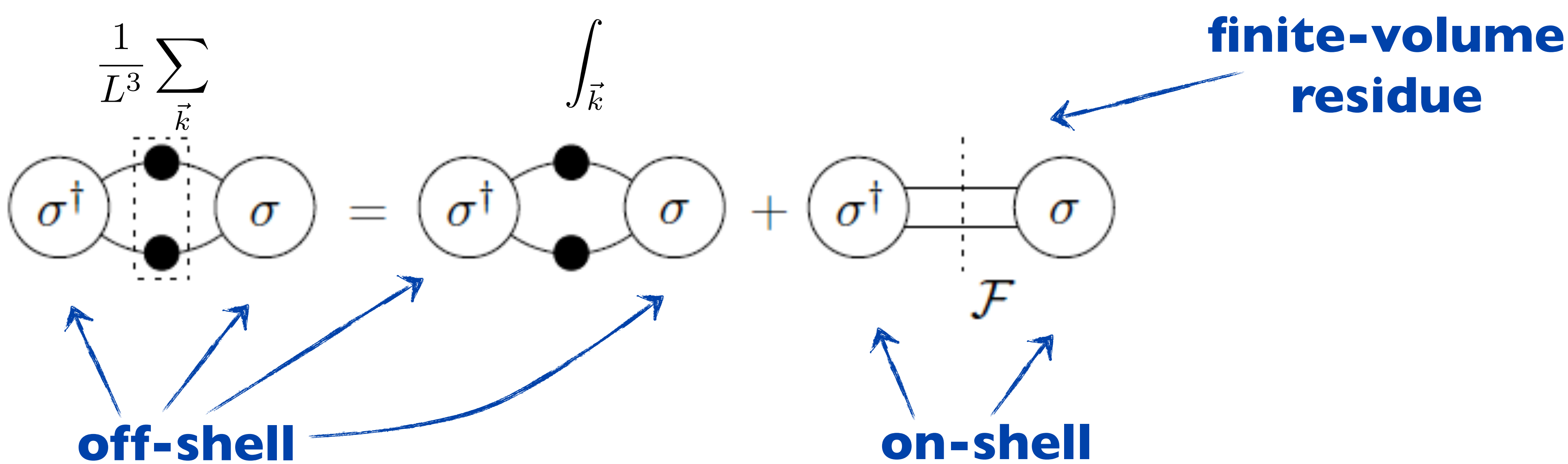
- Diagrammatically



Variant of key step 2

- For generalization to 3 particles use (modified) PV prescription instead of iε

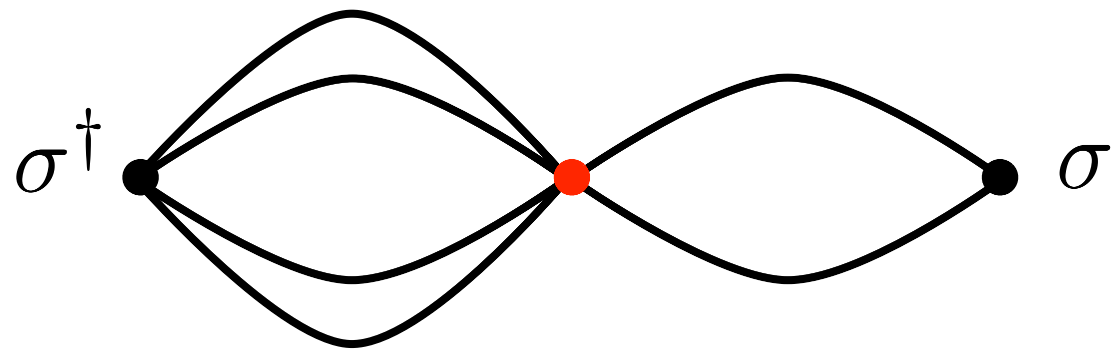
$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \widetilde{PV}\!\!\int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+\cancel{i\epsilon}}\frac{1}{(P-k)^2-m^2+\cancel{i\epsilon}}g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}_{\widetilde{PV}}(q^*, q^{*\prime})g^*(\hat{q}^{*\prime})$$

- Key properties of F_{PV} : real and no unitary cusp at threshold [see Max's talk]

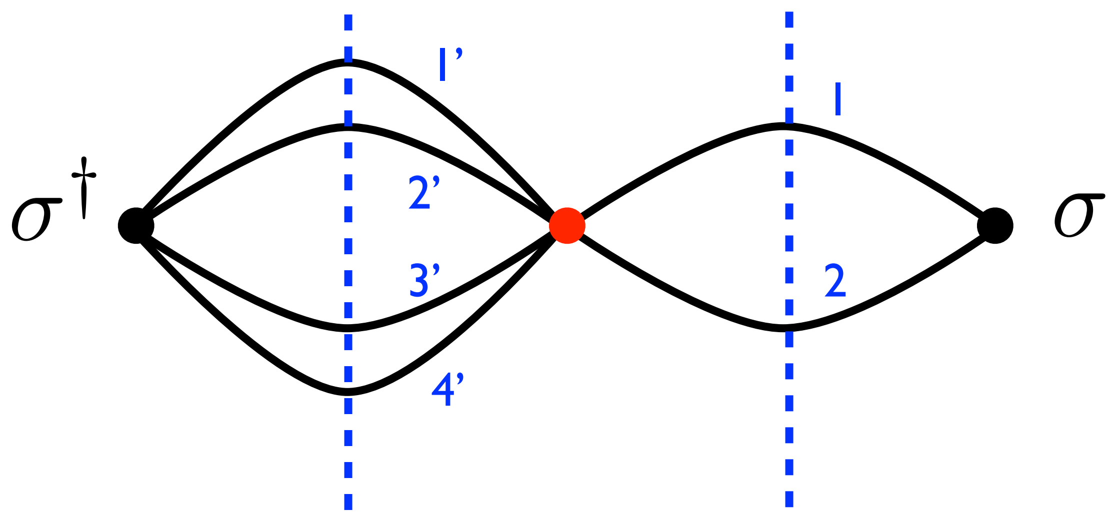
Key step 3

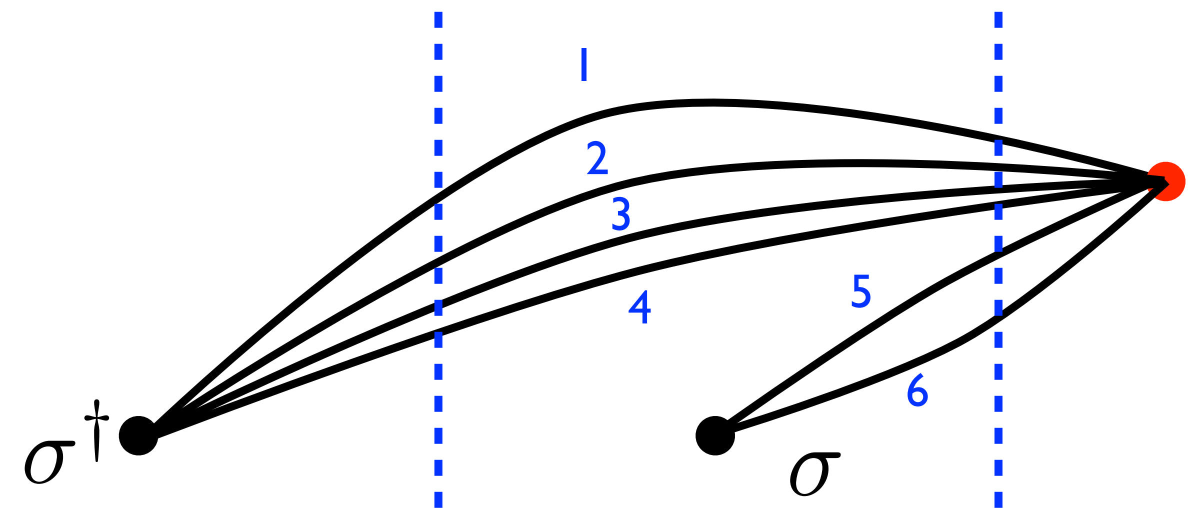
- Identify potential singularities: can use time-ordered PT (i.e. do k_0 integrals)
- Example



Key step 3

- 2 out of 6 time orderings:





$$\frac{1}{E-\omega_1'-\omega_2'-\omega_3'-\omega_4'} \qquad \frac{1}{E-\omega_1-\omega_2} \qquad\qquad \frac{1}{E-\omega_1-\omega_2-\omega_3-\omega_4} \qquad \frac{1}{\sum_{j=1,6}\omega_j}$$

On-shell energy $\omega_j = \sqrt{\vec{k}_j^2 + M^2}$

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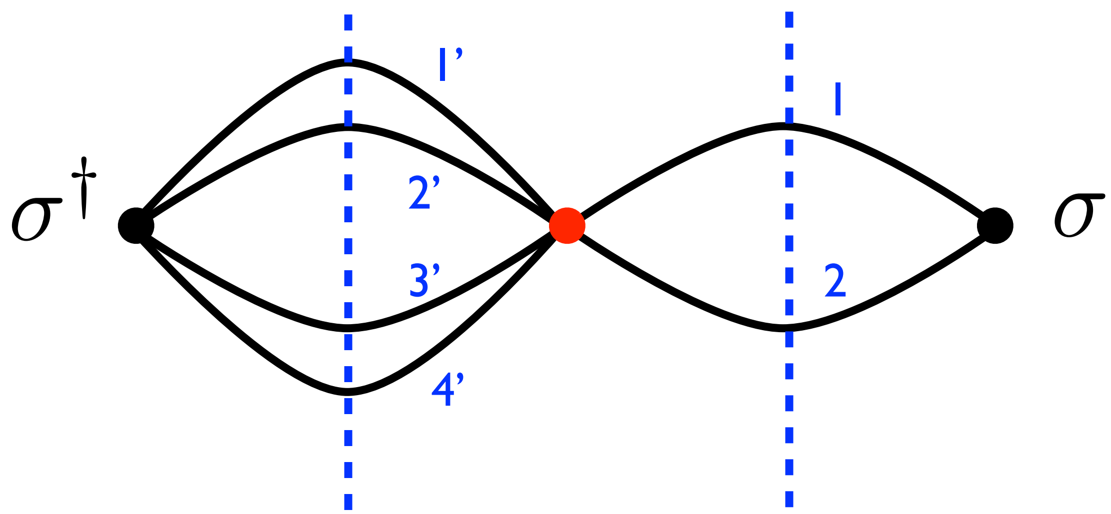
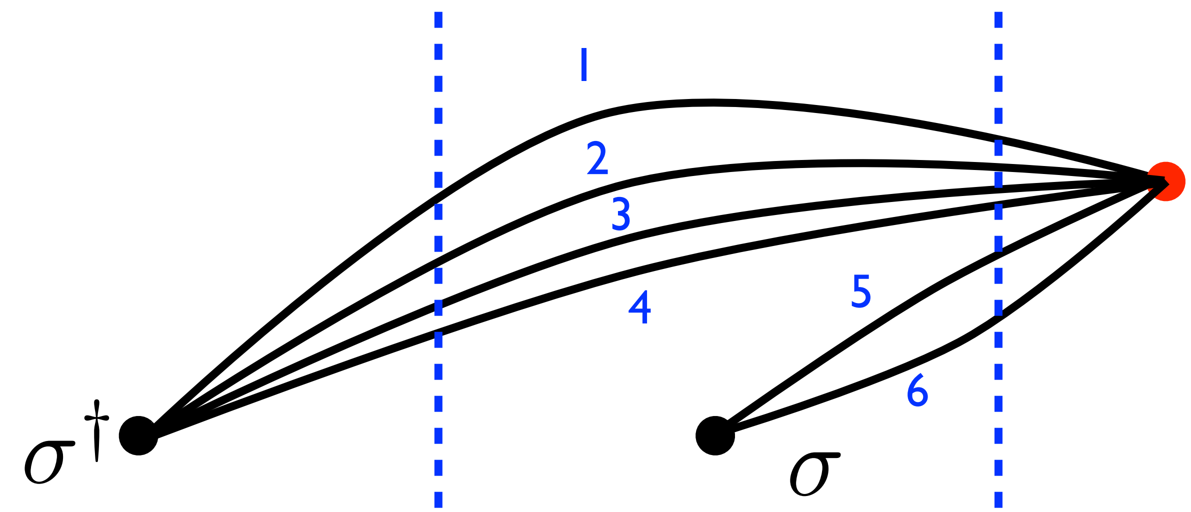


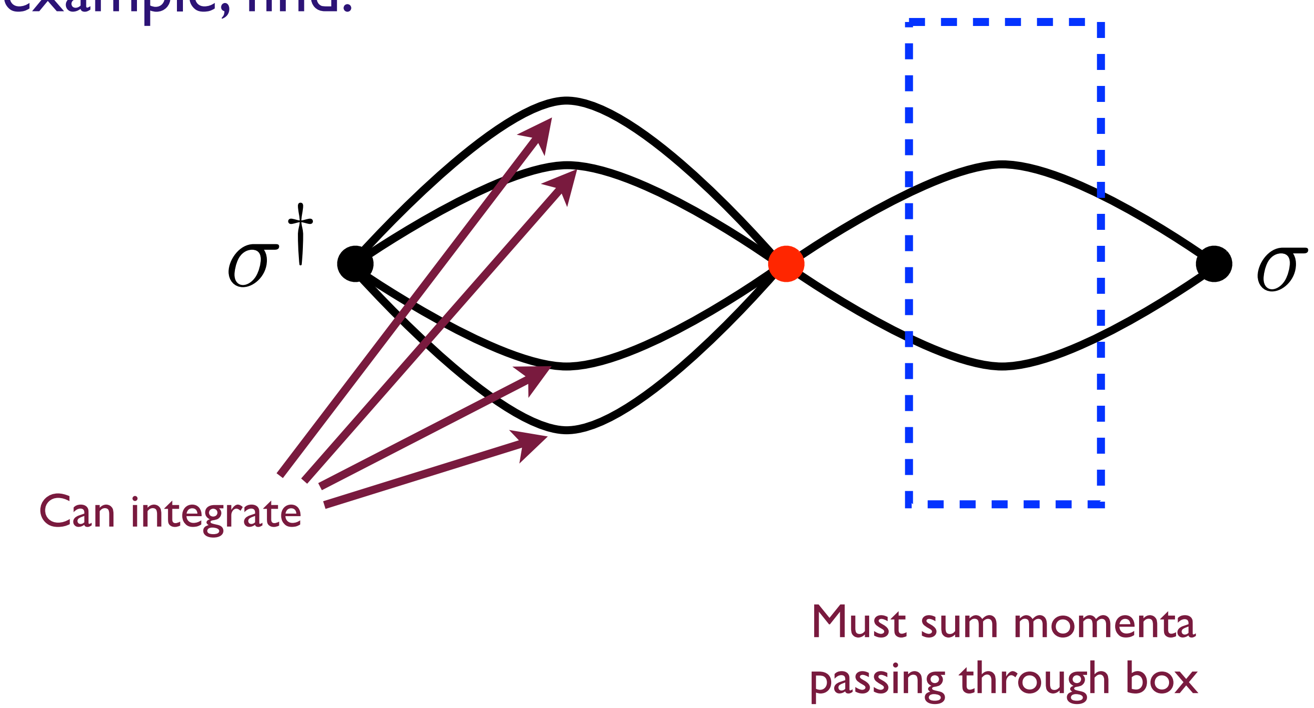


$$\frac{1}{E-\omega'_1-\omega'_2-\omega'_3-\omega'_4} \qquad \frac{1}{E-\omega_1-\omega_2} \qquad \frac{1}{E-\omega_1-\omega_2-\omega_3-\omega_4} \qquad \frac{1}{\sum_{j=1,6}\omega_j}$$

- If restrict $0 < E^* < 4M$ then only 2-particle "cuts" have singularities, and these occur only when both particles go on-shell

Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- In our example, find:



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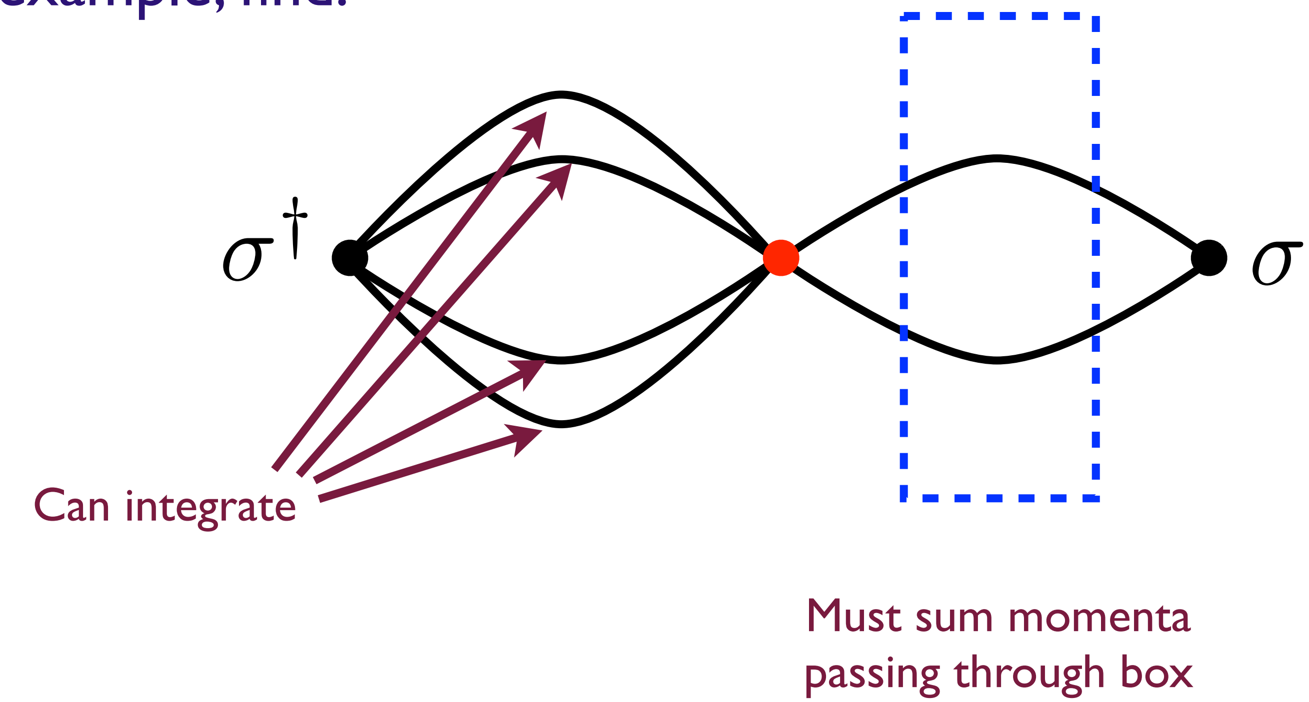


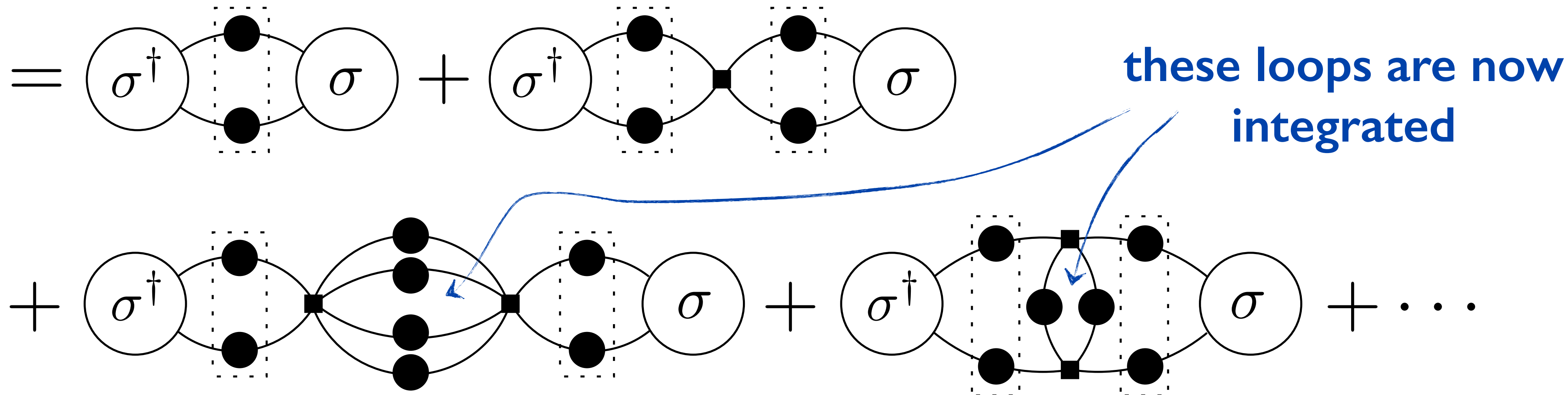
- Then repeatedly use sum=integral + "sum-integral" to simplify

2-particle quantization condition

Following method of [Kim, Sachrajda & SS 05]

- Apply previous analysis to 2-particle correlator ($0 < E^* < 4M$)

$$C_L(E,\vec{P}) =$$



- Collect terms into infinite-volume Bethe-Salpeter kernels

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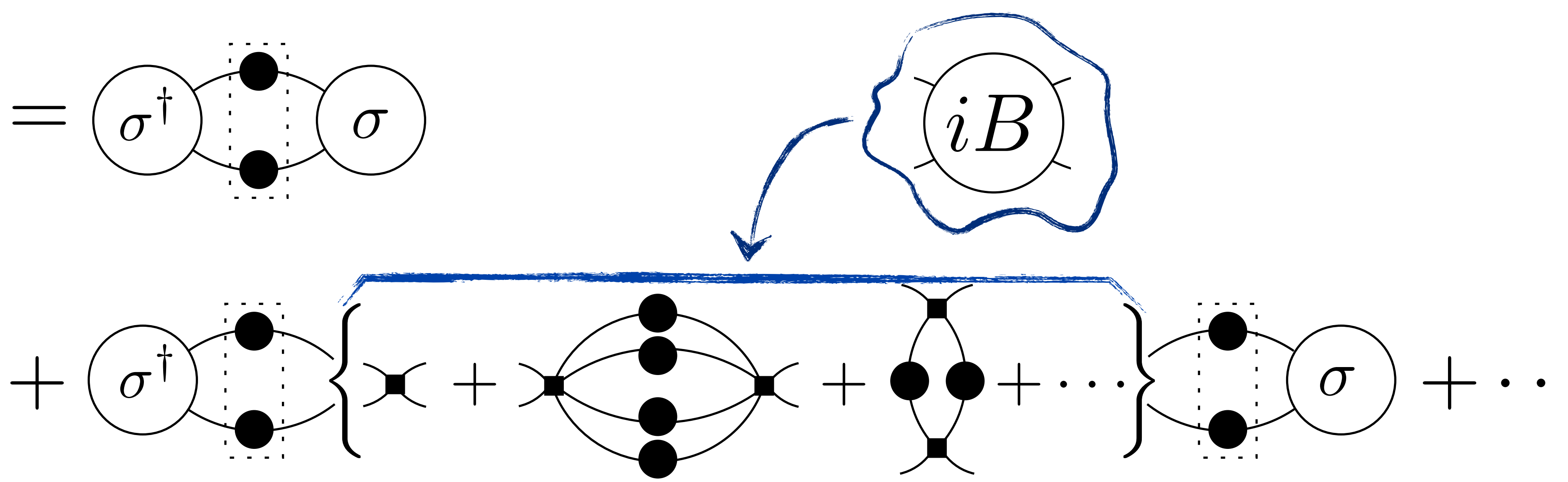

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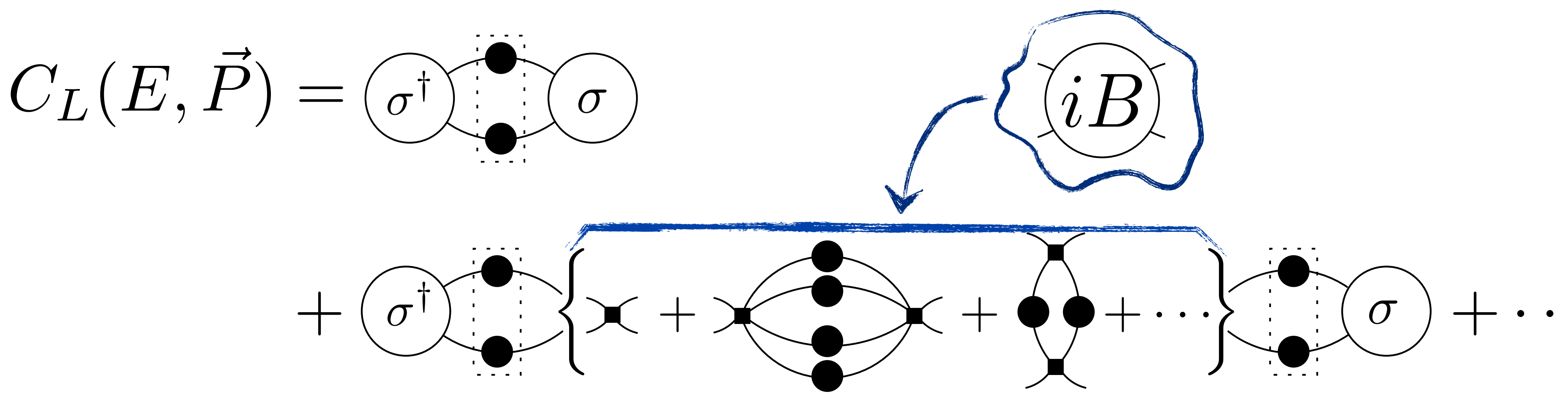
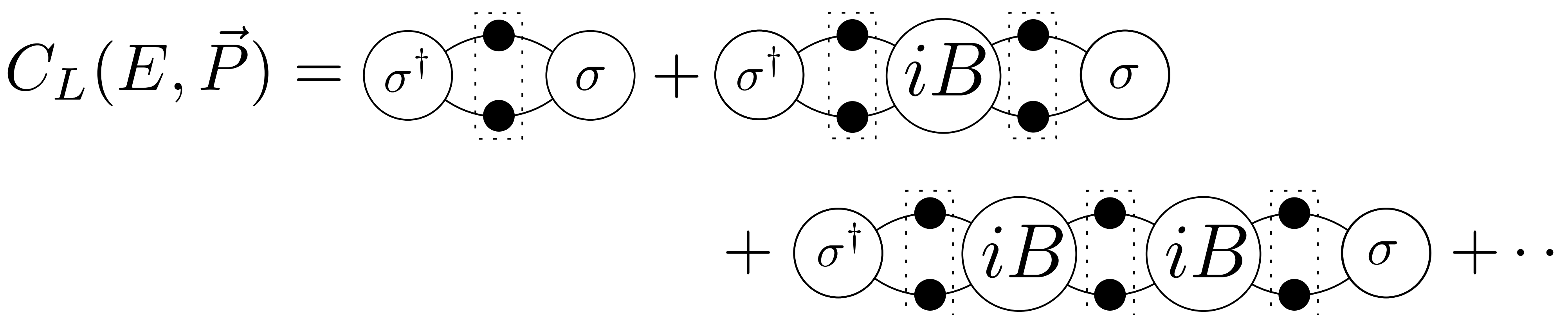
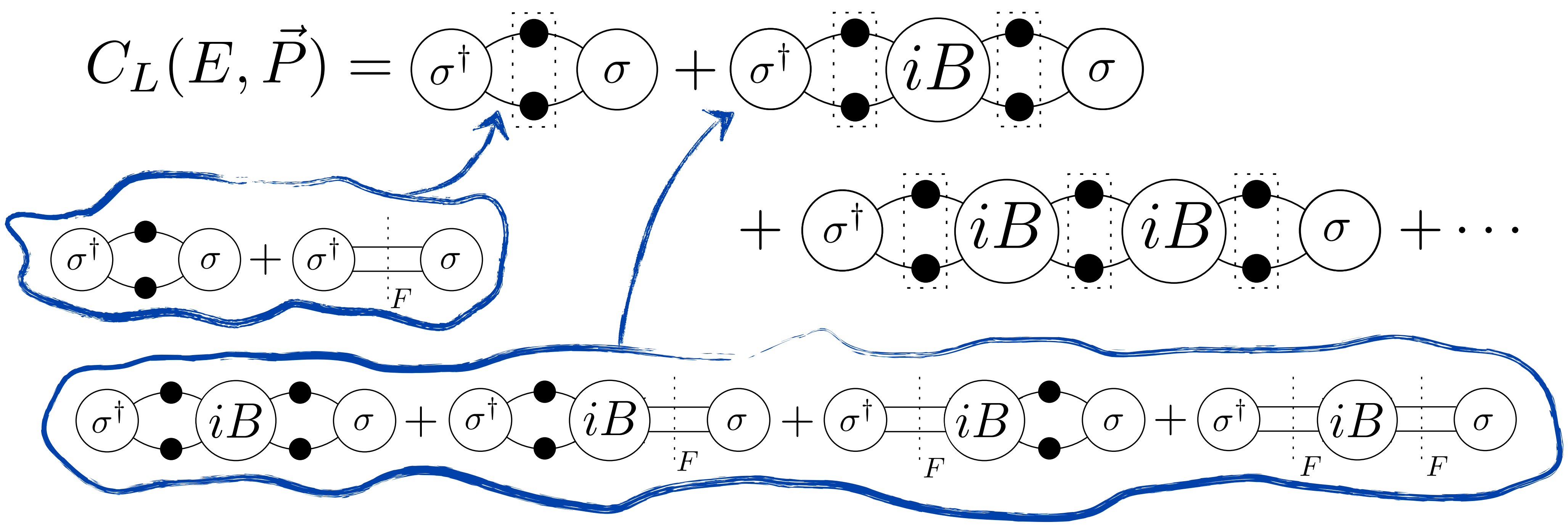


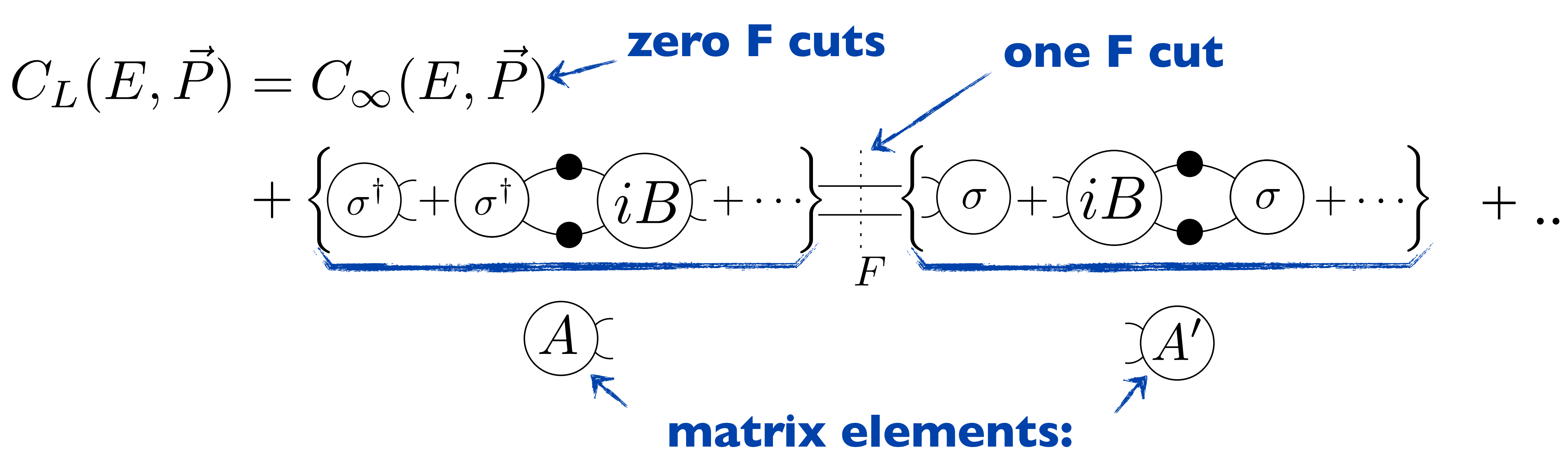
- Leading to



- Next use sum identity



- And regroup according to number of "F cuts"



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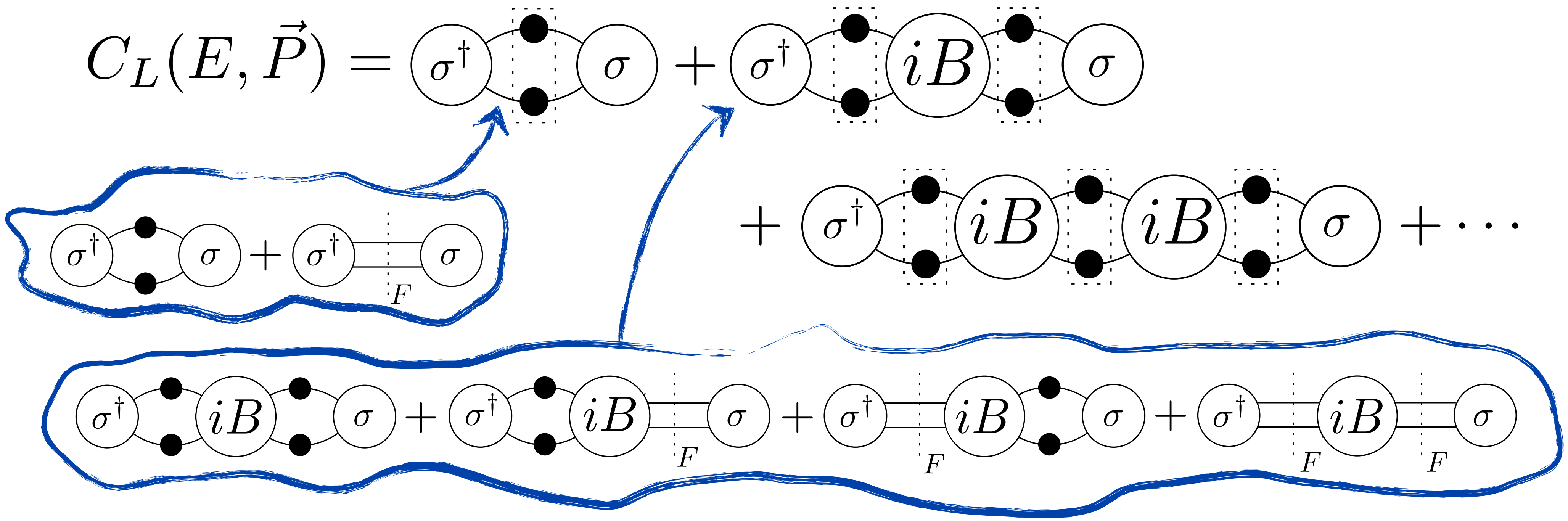
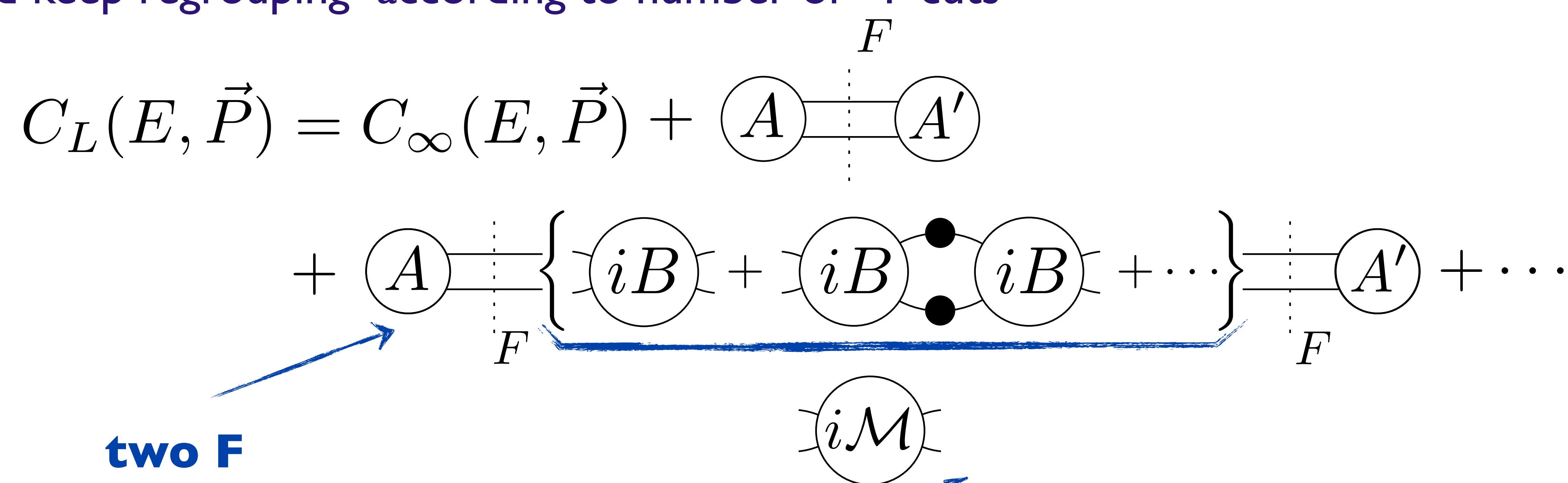


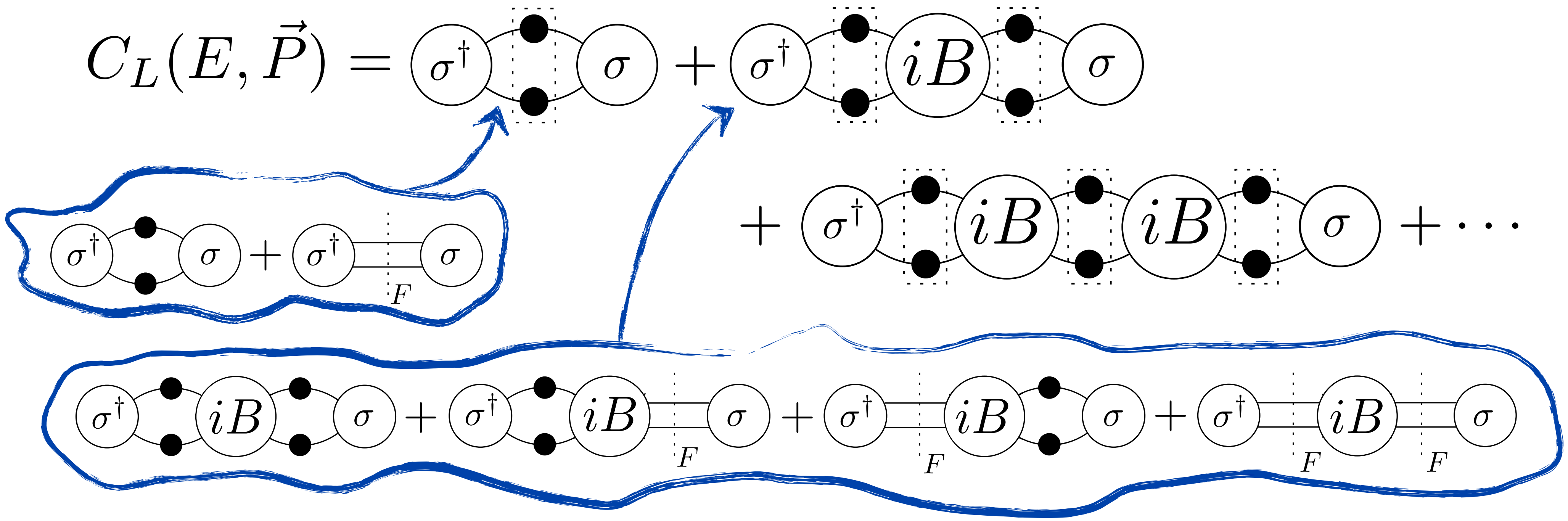
- And keep regrouping according to number of "F cuts"



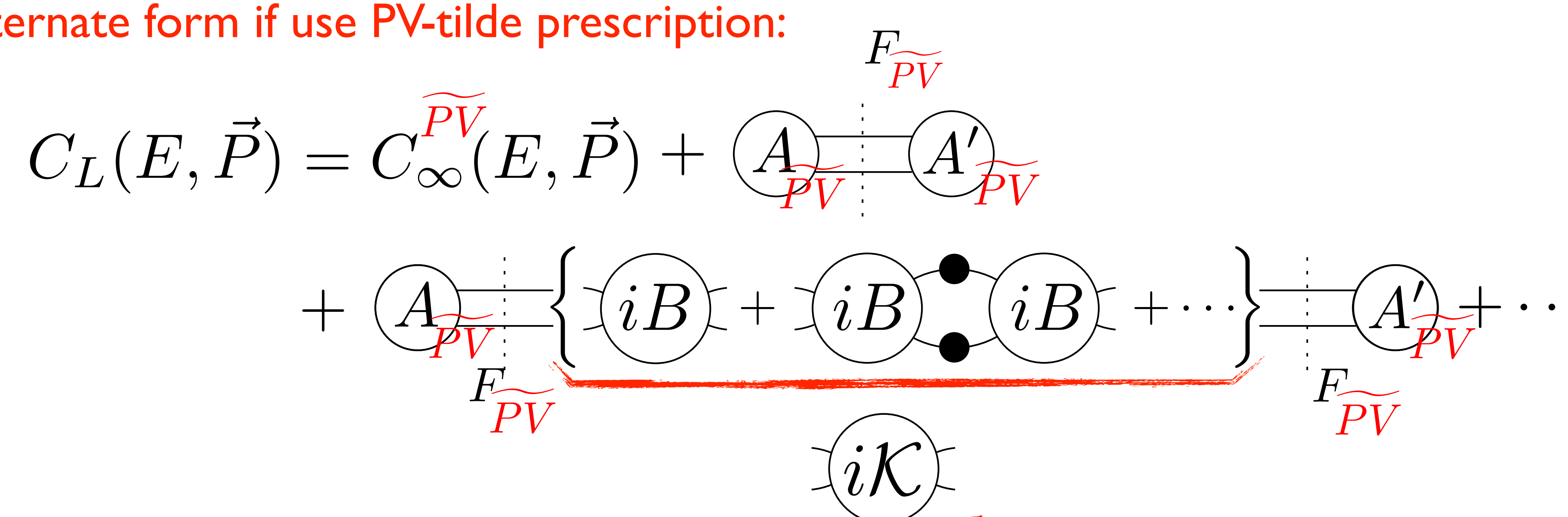
two F cuts

the infinite-volume, on-shell 2→2 scattering amplitude

- Next use sum identity



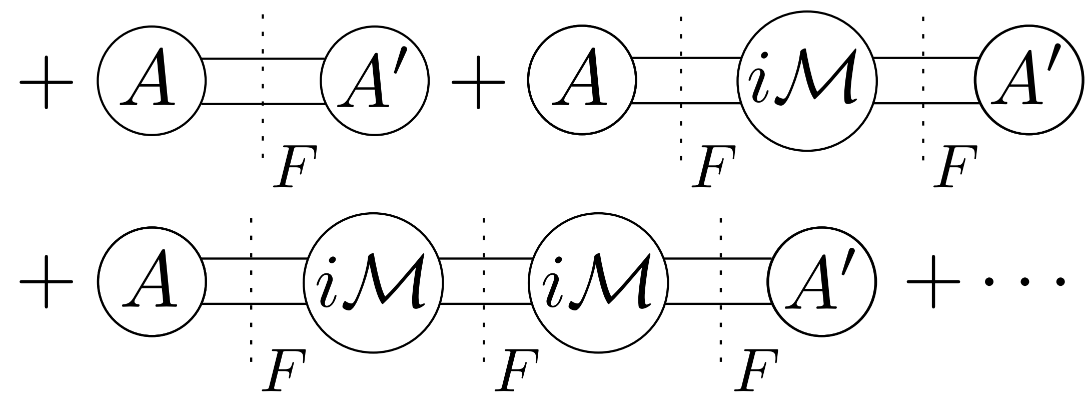
- Alternate form if use PV-tilde prescription:



the infinite-volume, on-shell
2→2 K-matrix

- Final result:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$

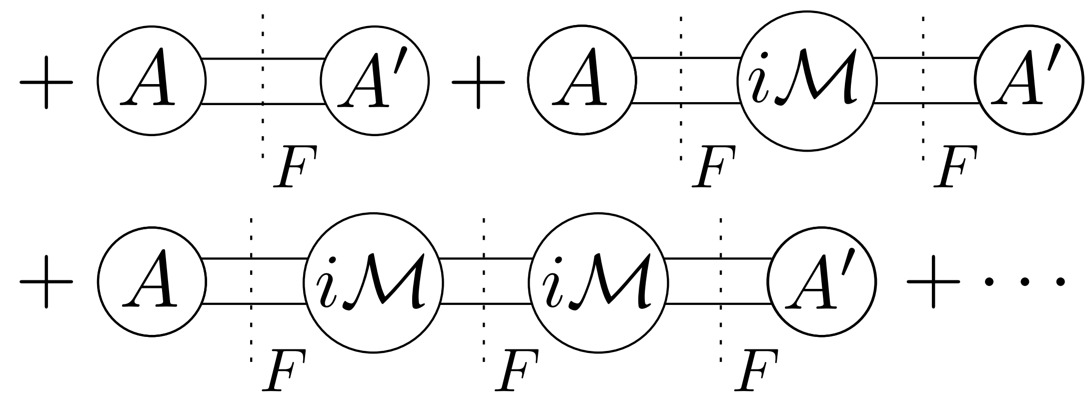


- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + \sum_{n=0}^{\infty} A' iF[i\mathcal{M}_{2\to 2} iF]^n A$$

- Correlator is expressed in terms of infinite-volume, physical quantities and kinematic functions encoding the finite-volume effects

- Final result:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$



- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + \sum_{n=0}^{\infty} A' iF[i\mathcal{M}_{2\to 2} iF]^n A$$

- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to 2} iF} A$$

no poles, only cuts

no poles, only cuts

matrices in l,m space

- $C_L(E,\vec{P})$ diverges whenever $iF \dfrac{1}{1 - i\mathcal{M}_{2\to 2} iF}$ diverges

2-particle quantization condition

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to 2} iF} A$$

- At fixed L & P, the finite-volume spectrum E_1, E_2, ... is given by solutions to

$$\boxed{\Delta_{L,\vec{P}}(E) = \det\left[(iF)^{-1} - i\mathcal{M}_{2\to 2}\right] = 0}$$

- $\mathcal{M}$ is diagonal in *l,m*: $i\mathcal{M}_{2\to 2;\ell',m';\ell,m} \propto \delta_{\ell,\ell'}\delta_{m,m'}$
- F is off-diagonal, since the box violates rotation symmetry
- To make useful, truncate by assuming that $\mathcal{M}$ vanishes above l_{max}
- For example, if l_{max}=0, obtain

$$\boxed{i\mathcal{M}_{2\to 2;00;00}(E_n^*) = [iF_{00;00}(E_n, \vec{P}, L)]^{-1}}$$

Generalization of s-wave Lüscher equation to moving frame [Rummukainen & Gottlieb]

Equivalent K-matrix form

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A' i F_{\widetilde{\mathrm{PV}}} \frac{1}{1+\mathcal{K}_2 F_{\widetilde{\mathrm{PV}}}} A$$

- At fixed L & P, the finite-volume spectrum E_1, E_2, ... is given by solutions to

$$\Delta_{L,\vec{P}}(E) = \det\left[(F_{\widetilde{PV}})^{-1} + \mathcal{K}_2\right] = 0$$

- $\mathcal{K}_2$ is diagonal in *l,m*
- F_{PV} is off-diagonal, since the box violates rotation symmetry
- To make useful, truncate by assuming that $\mathcal{K}_2$ vanishes above l_{max}
- For example, if l_{max}=0, obtain

$$i\mathcal{K}_{2;00;00}(E_n^*) = \left[iF_{\widetilde{PV};00;00}(E_n,\vec{P},L)\right]^{-1}$$

Future directions & challenges

Many challenges remain!

- Extend 1→2 work to include arbitrary spin particles (so can use for N)
  - First step in NREFT taken for $\gamma^* N \to \Delta \to \pi N$ [Agadjanov et al. 14]
- Develop general formalism for 2→2 *transitions* (e.g. resonance form factors)
- Fully develop 3 body formalism
  - Allow two particle sub channels to be resonant
  - Extend to non-identical particles, particles with spin
  - Generalize LL factors to 1→3 decay amplitudes (e.g. for $K \to \pi\pi\pi$)
  - ….
- Develop models of amplitudes so that new results can be implemented in simulations (e.g. following $K\pi$, $K\eta$ coupled channel analysis of [Dudek, Edwards, Thomas & Wilson 14])

Many challenges remain!

- Onwards to 4 particles?!?

Thank you!
Questions?

Backup Slides

3-particle correlator