BNL WORKSHOP ON MULTI-HADRON AND NONLOCAL MATRIX ELEMENTS IN LATTICE QCD
FEBRUARY 5-6, 2015

SINGLE-HADRON STATES IN A FINITE VOLUME IN THE PRESENCE OF QED INTERACTIONS

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MIT

IN COLLABORATION WITH MARTIN J. SAVAGE
PHYS. REV. D 90, 054503 (2014)

QED IS SPECIAL IN A FINITE VOLUME

GAUSS'S LAW + PERIODICITY ?

Hilf and Polley, Phys. Lett. B 131, 412 (1983)
Duncan, Eichten and Thacker, Phys. Rev. Lett. 76, 3894 (1996)
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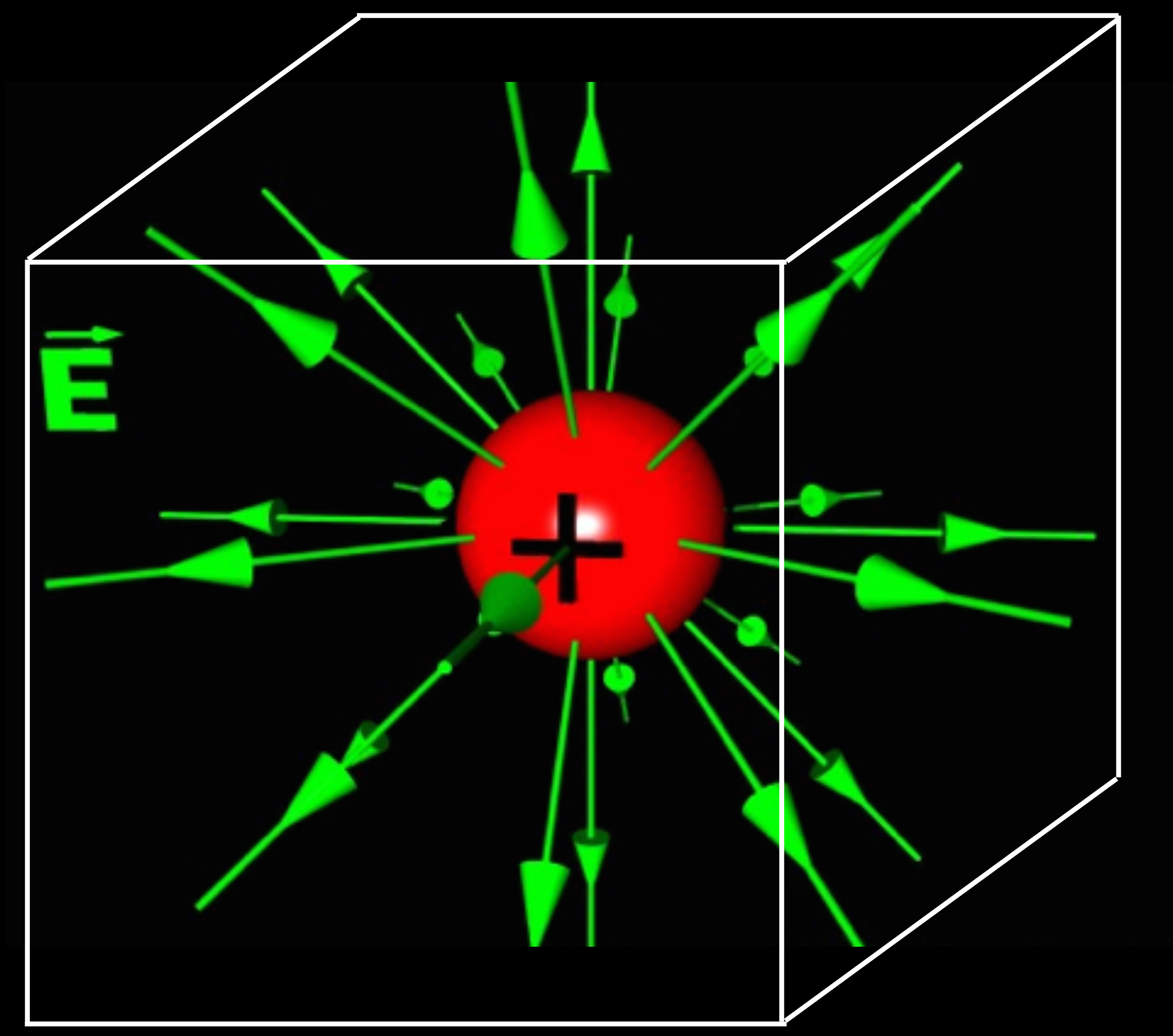

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YVES PELLETIER
(YPELLETIER@NCF.CA)

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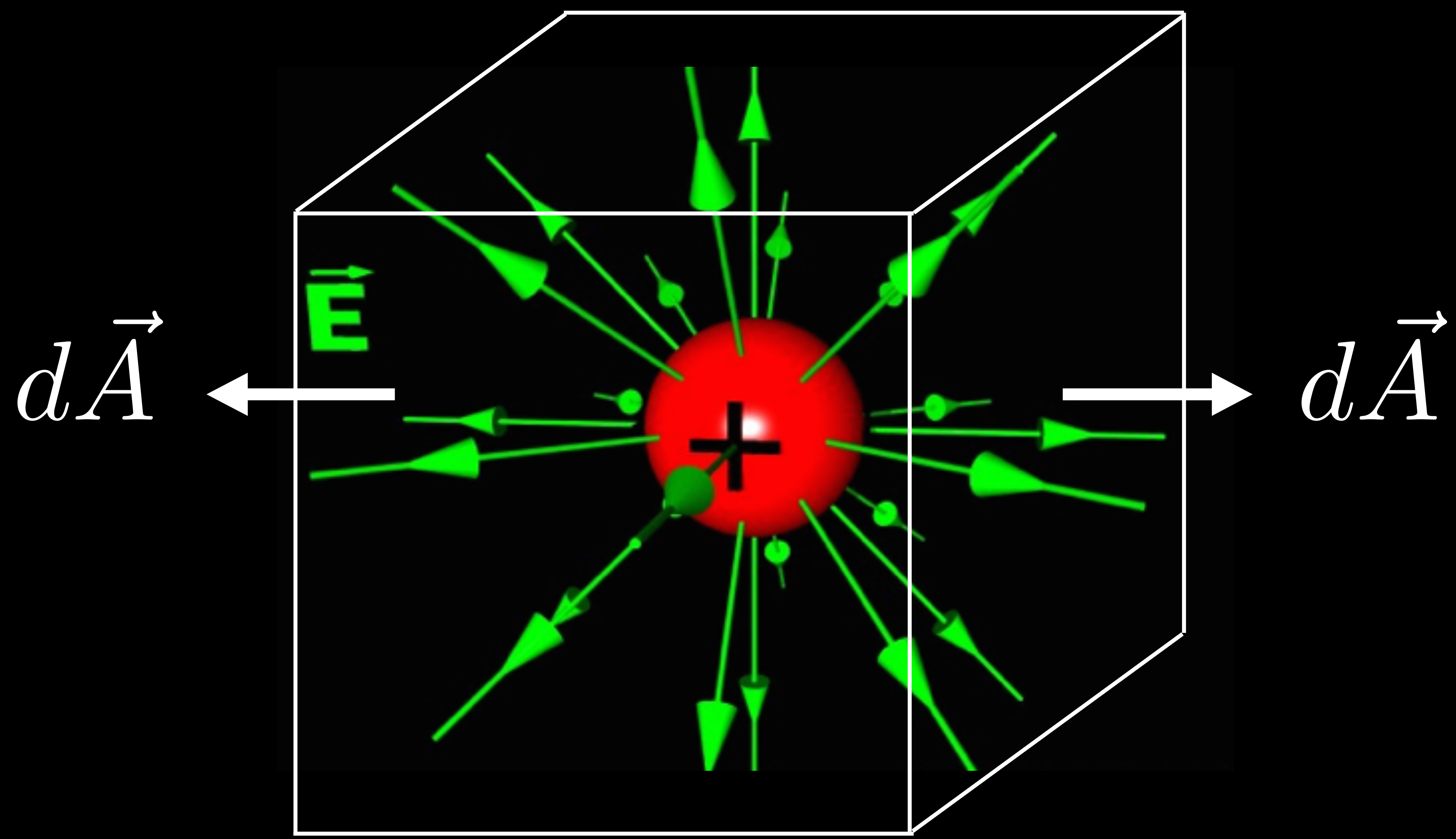

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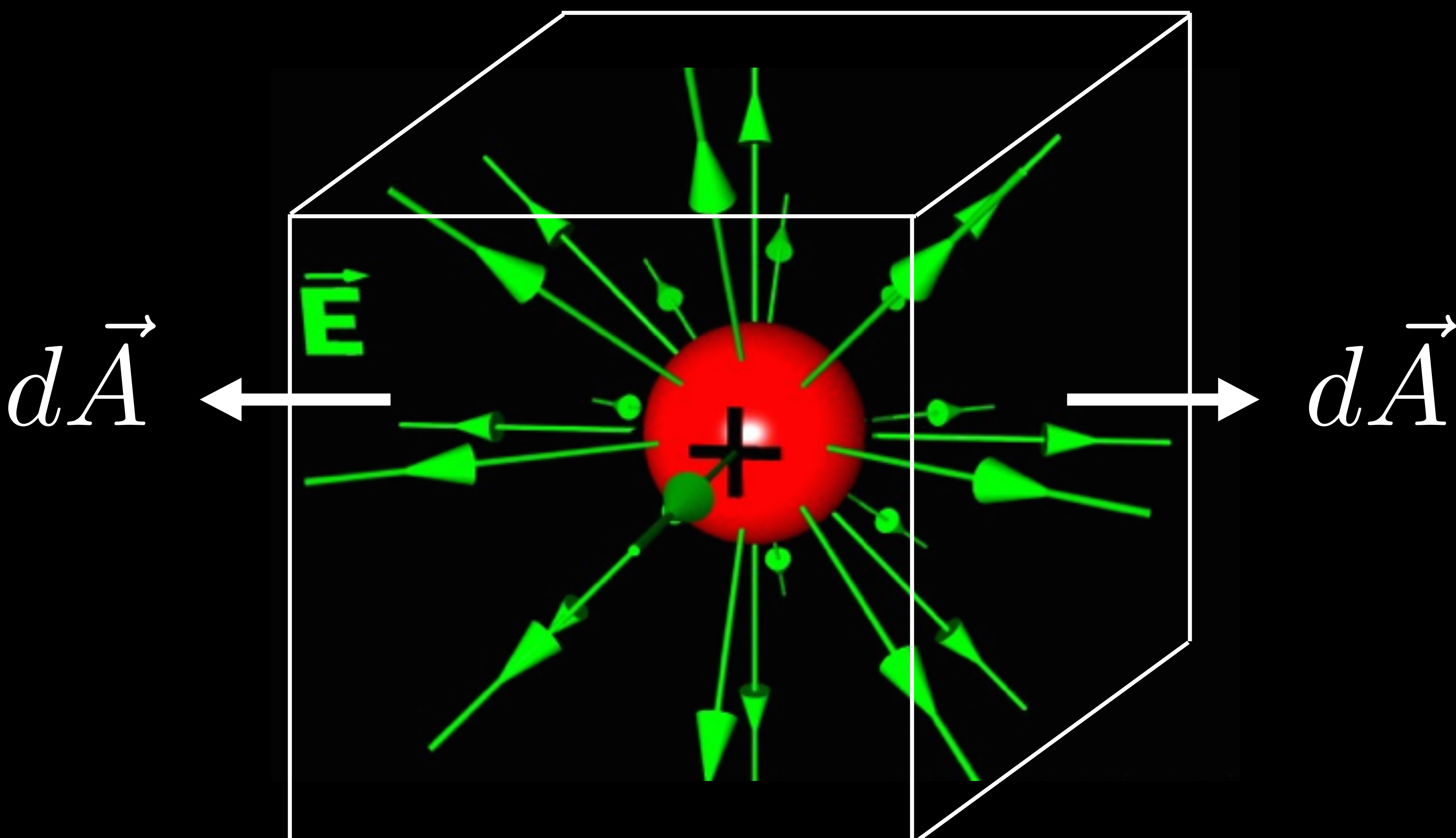
QED IS SPECIAL IN A FINITE VOLUME

$$
\begin{aligned}
\delta S &= \int d^4x \ \left[\partial_\mu F^{\mu\nu}(x) - eQ\,\overline{\psi}(x)\gamma^\nu\psi(x)\right]\delta\left(A_\nu(x)\right) \\
&= \int dt\ \frac{1}{L^3}\sum_{\mathbf{q}}\ \delta\left(\tilde{A}_\nu(t,\mathbf{q})\right)\int_{\mathrm{L}^3} d^3\mathbf{x}\ e^{i\mathbf{q}\cdot\mathbf{x}}\ \left[\partial_\mu F^{\mu\nu}(t,\mathbf{x}) - eQ\,\overline{\psi}(t,\mathbf{x})\gamma^\nu\psi(t,\mathbf{x})\right]
\end{aligned}
$$



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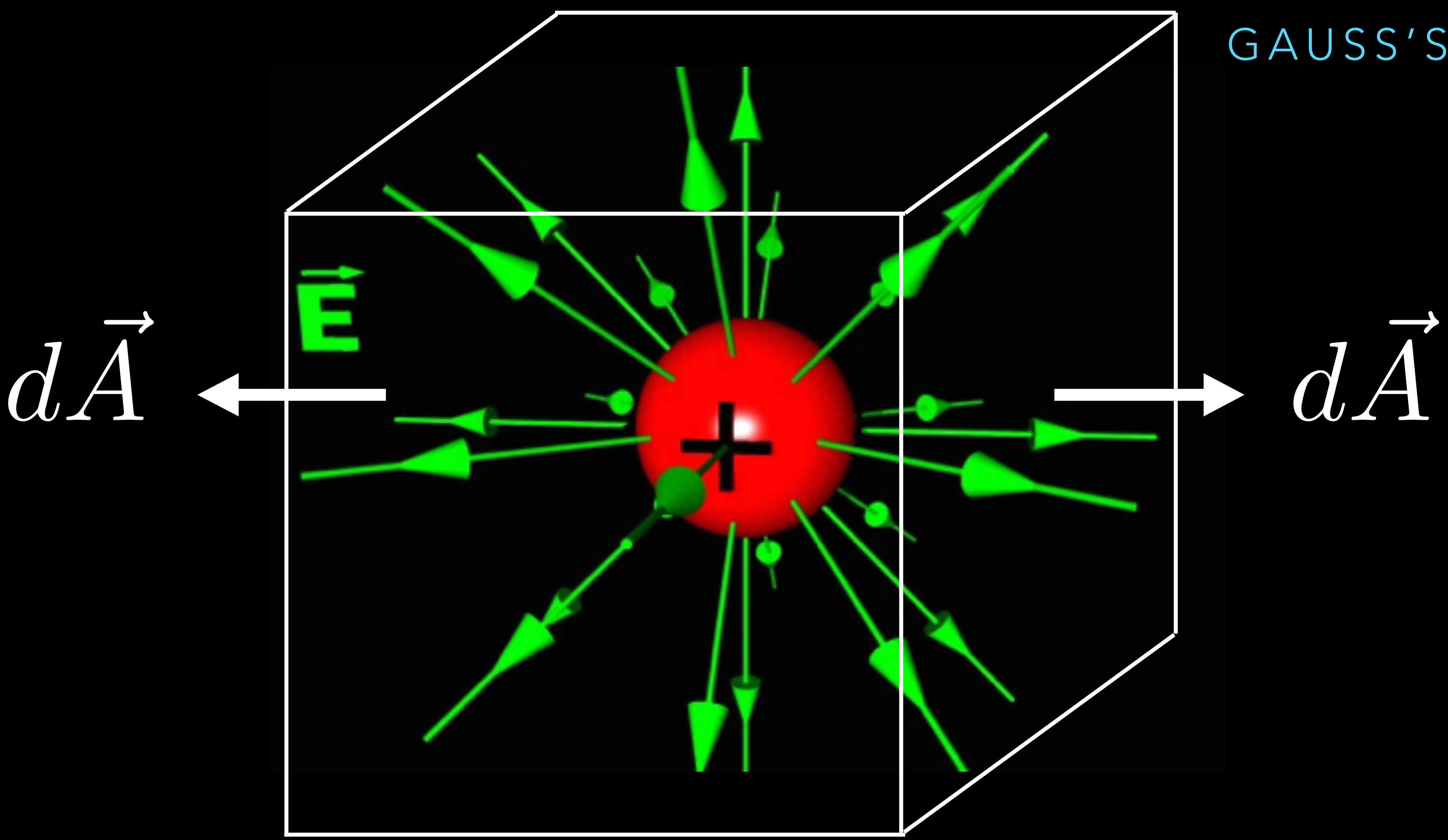
QED IS SPECIAL IN A FINITE VOLUME

$$\delta S = \int d^4x \; \left[\partial_\mu F^{\mu\nu}(x) - eQ\, \overline{\psi}(x)\gamma^\nu\psi(x)\right] \delta\left(A_\nu(x)\right)$$

$$= \int dt\, \frac{1}{L^3} \sum_{\mathbf{q}} \delta\left(\tilde{A}_\nu(t,\mathbf{q})\right) \int_{\mathrm{L}^3} d^3\mathbf{x}\; e^{i\mathbf{q}\cdot\mathbf{x}} \left[\partial_\mu F^{\mu\nu}(t,\mathbf{x}) - eQ\, \overline{\psi}(t,\mathbf{x})\gamma^\nu\psi(t,\mathbf{x})\right]$$

$$\mathbf{q} = 0 \;\rightarrow\; \partial_\mu F^{\mu\nu} = j^\nu \;\rightarrow\; \int \mathbf{E}\cdot d\mathbf{A} = \int dV\, j^0 = eQ$$

GAUSS'S LAW



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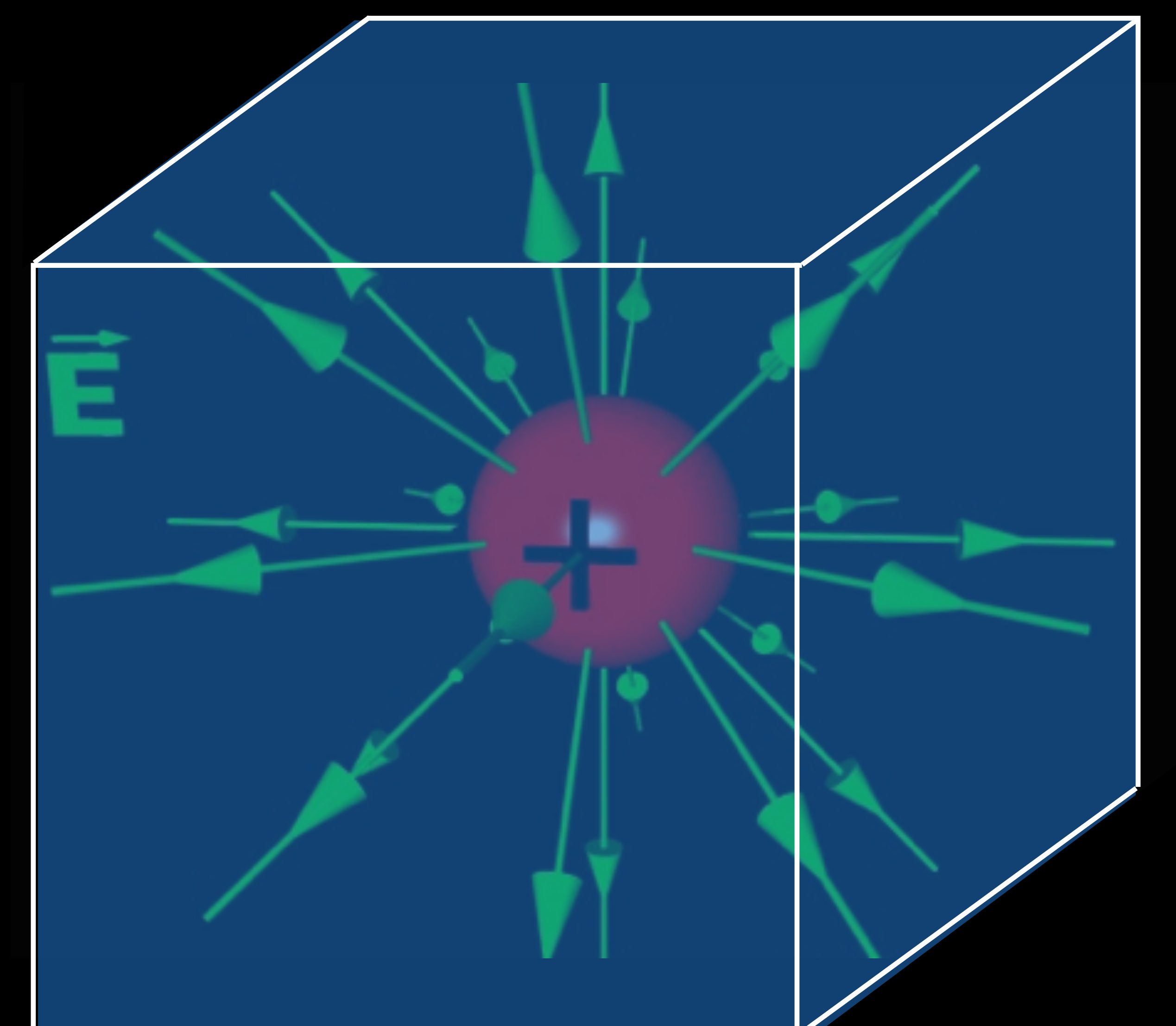
QED IS SPECIAL IN A FINITE VOLUME

SOLUTION: ADD A UNIFORM BACKGROUND CHARGE/CURRENT DENSITY

$$\mathcal{L}^{QED} \to \mathcal{L}^{QED} - b^\nu A_\nu \qquad (j^\nu \to j^\nu + b^\nu)$$

REQUIRING $\int dV\, b^0 = -eQ$ RESTORES GAUSS'S LAW.



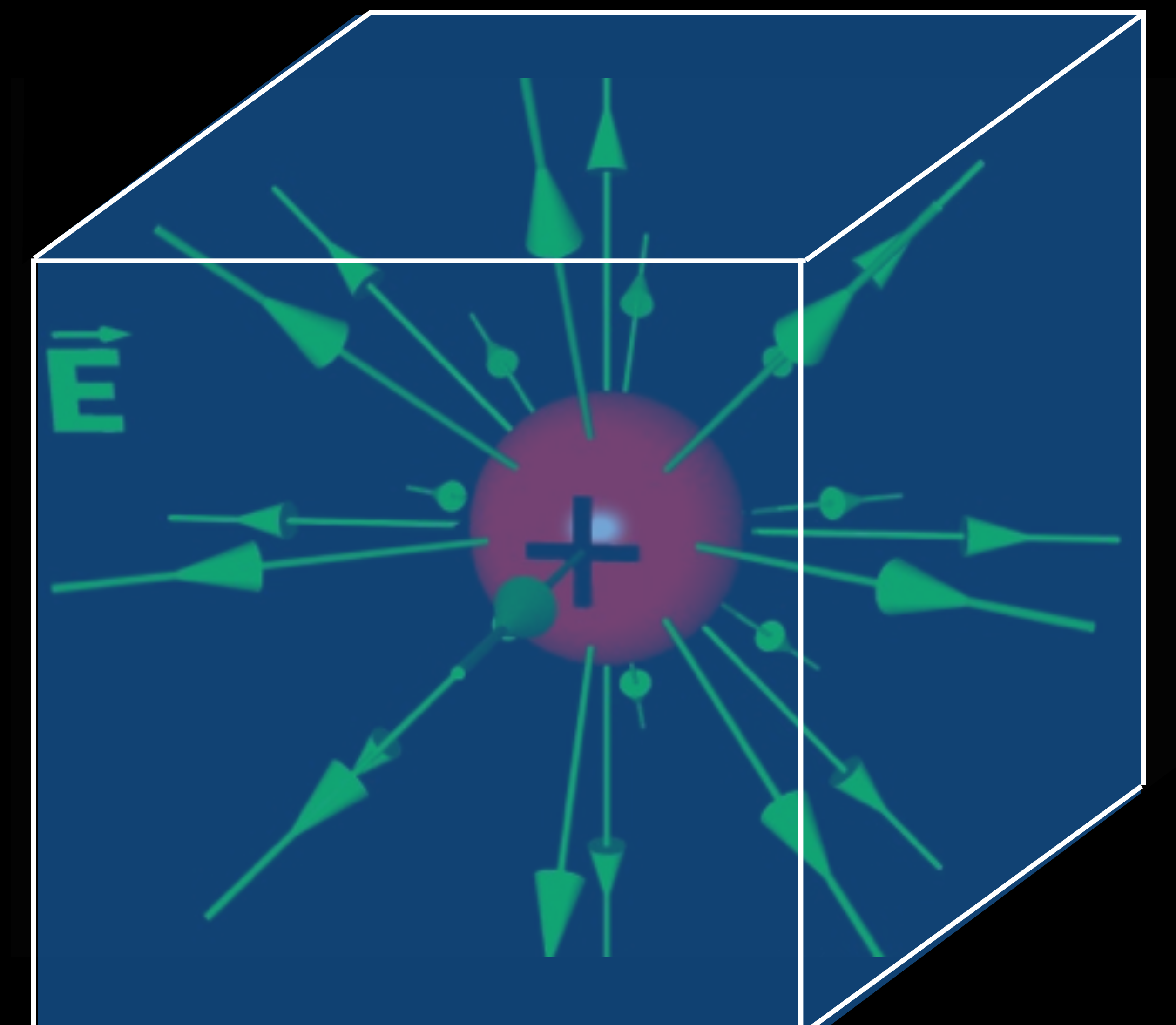
QED IS SPECIAL IN A FINITE VOLUME

HOW DO WE IMPLEMENT THIS?

$$\mathcal{L}^{QED} \to \mathcal{L}^{QED} - b^{\nu} A_{\nu}$$

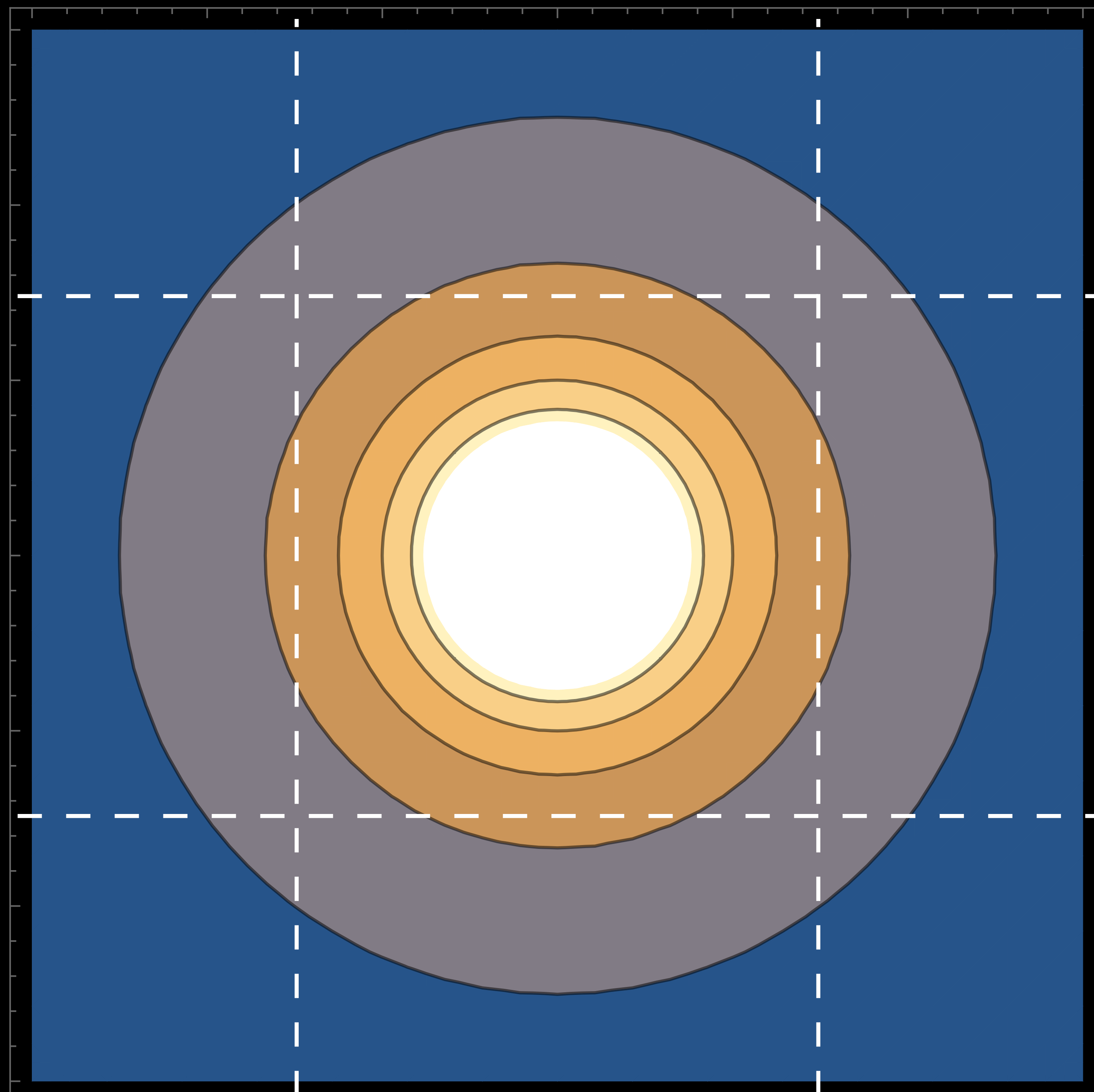
$$\frac{\delta S}{\delta b^{\nu}} = 0 \;\to\; -\int dt d^3x \; A_{\nu}(\mathbf{x}, t) = 0 \;\to\; \boxed{\widetilde{A}_{\nu}(\mathbf{q} = 0; t) = 0}$$

REMOVE THE PHOTON ZERO MODE



QED IS SPECIAL IN A FINITE VOLUME

COULOMB POTENTIAL OF A POINT CHARGE
WITHOUT THE UNIFORM CHARGE DENSITY



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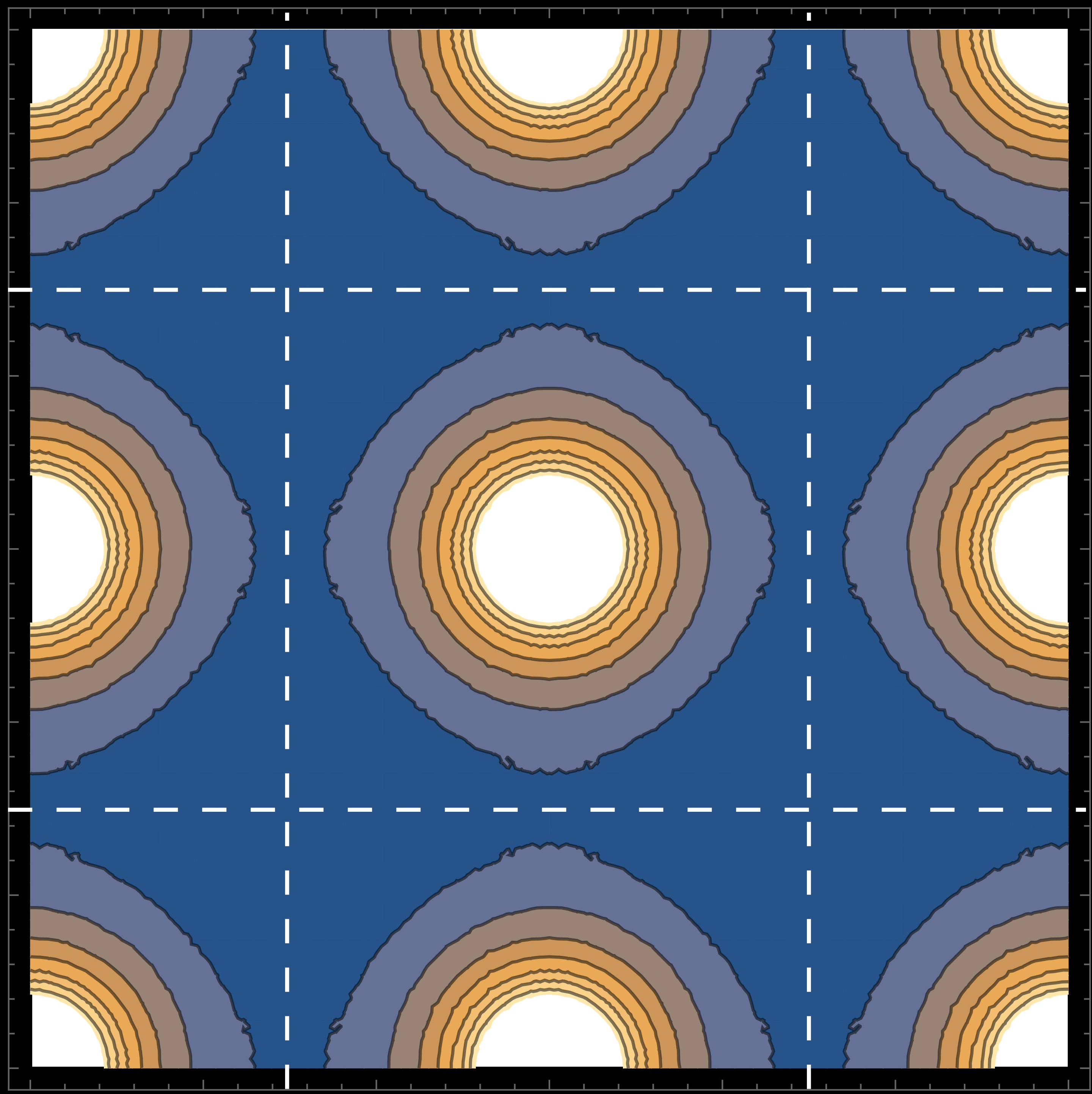


x-y plane

PREVIOUS INVESTIGATIONS OF QED FINITE-VOLUME EFFECTS ON MASSES

- VECTOR-DOMINANCE MODELS
  Bardeen, Bijnens and Gerard, Phys. Rev. Lett. 62, 1343 (1989)
  Duncan, at al., Phys. Rev. Lett. 76, 3894 (1996)
  Blum, et al., Phys. Rev. D 76, 114508 (2007)
- CHIRAL PERTURBATION THEORY
  Hayakawa and Uno, Prog. Theor. Phys. 120, 413 (2008)
  Blum, et al., Phys. Rev. D 82, 094508 (2010)
  Thomas, Lang and Young (2014), nucl-th:1406.4579
- SCALAR AND SPINOR QED
  Borsanyi, et al., arXiv:1406088.

CURRENT WORK

ZD, Martin J. Savage, Phys. Rev. D 90, 054503 (2014)

- SYSTEMATICALLY INCLUDING COMPOSITENESS CONTRIBUTIONS
- THE FIRST COMPLETE $\mathcal{O}\left(\frac{1}{L^4}\right)$ CALCULATION
- COMPLETELY GENERAL: APPLICABLE TO HADRONS AND NUCLEI

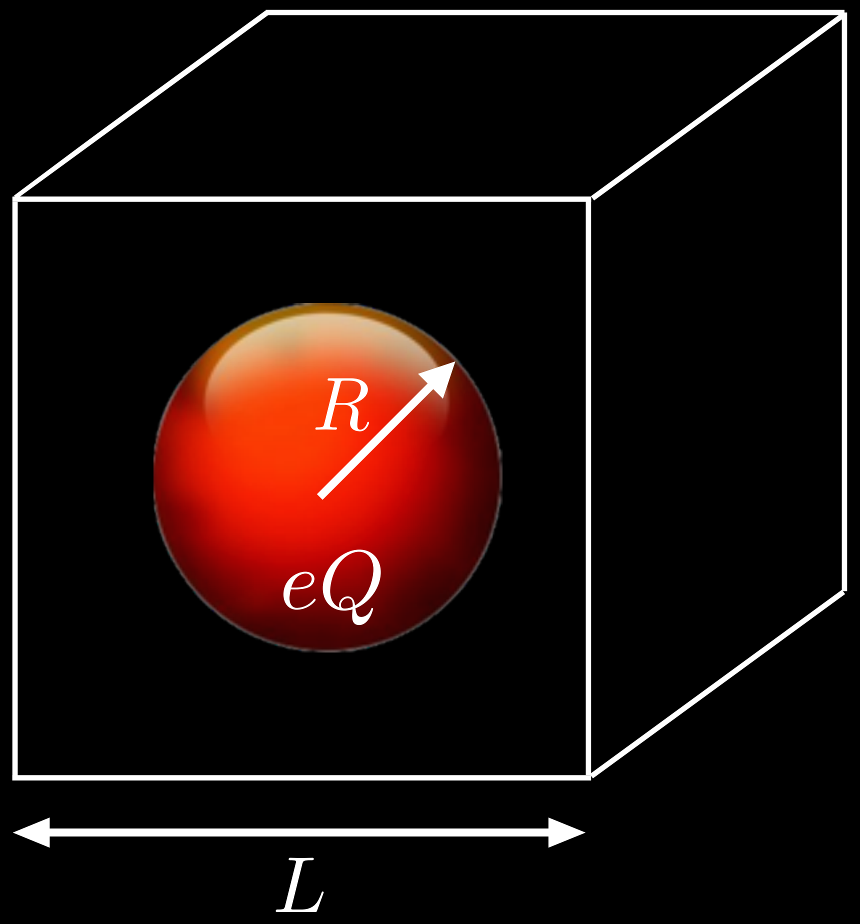
A CLASSICAL EXAMPLE: CHARGED SPHERE IN A FINITE VOLUME

SELF ENERGY

$$U(R,L) = \frac{1}{2}\int d^3r\, d^3r'\, \rho\, V(\mathbf{r}-\mathbf{r}')$$

$$\rho = \frac{eQ}{\frac{4}{3}\pi R^3}$$

$$V(\mathbf{r}-\mathbf{r}') = \frac{eQ}{4\pi|\mathbf{r}-\mathbf{r}'|}$$



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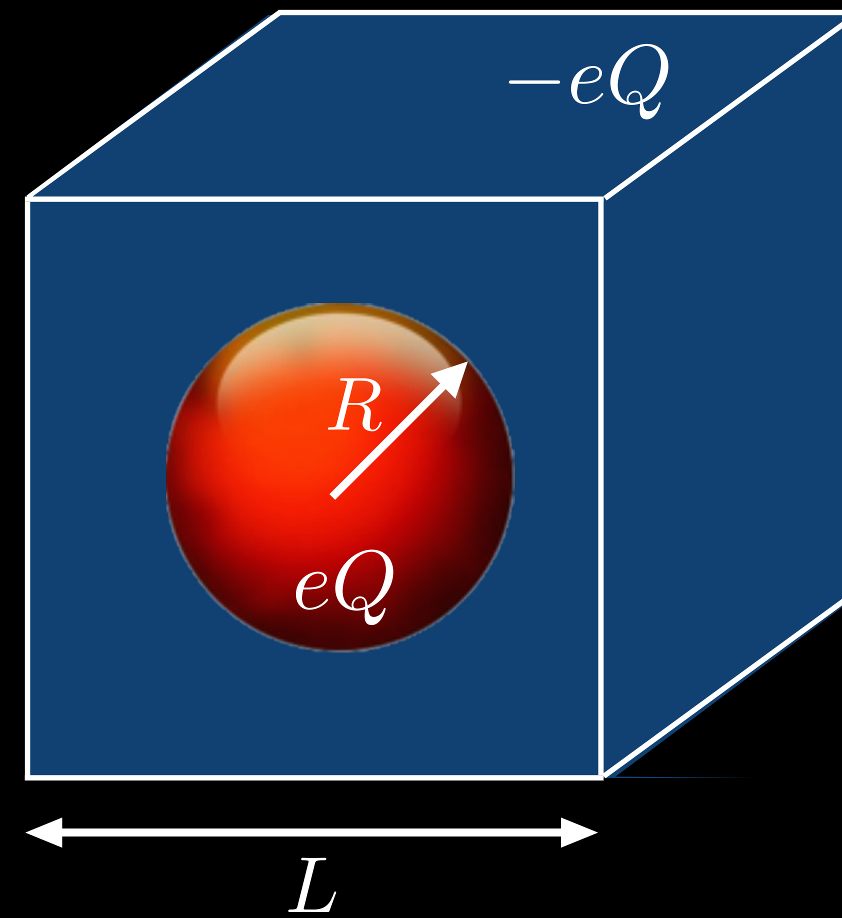

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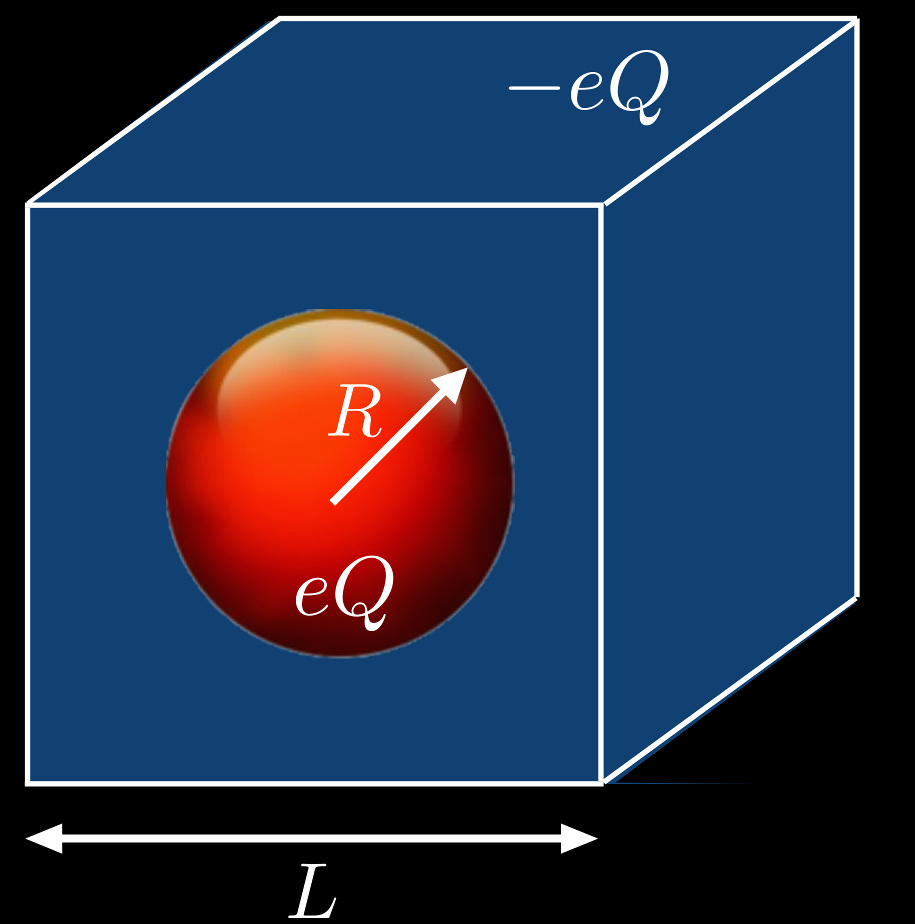

ZERO MODE REMOVED

$$V(\mathbf{r}-\mathbf{r}') = \frac{1}{L^3}\sum_{\mathbf{k}\in\frac{2\pi\mathbf{n}}{L},\mathbf{n}\neq\mathbf{0}} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{\mathbf{k}^2} = V^{(\infty)}(\mathbf{r}-\mathbf{r}') + \sum_{\mathbf{m}\neq\mathbf{0}}\int\frac{d^3k}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{\mathbf{k}^2}e^{i\mathbf{k}\cdot\mathbf{m}L}$$

POISSON RE-SUMMATION

(NOT EXACTLY: INCLUDE THE ZERO MODE BUT INTRODUCE AN IR REGULATOR. SUBTRACT OFF THE IR DIVERGENCE AT THE END.)

A CLASSICAL EXAMPLE: CHARGED SPHERE IN A FINITE VOLUME

SELF ENERGY

$$U(R,L) = \frac{1}{2}\int d^3r\, d^3r' \boxed{\rho}\, \boxed{V(\mathbf{r}-\mathbf{r}')}$$

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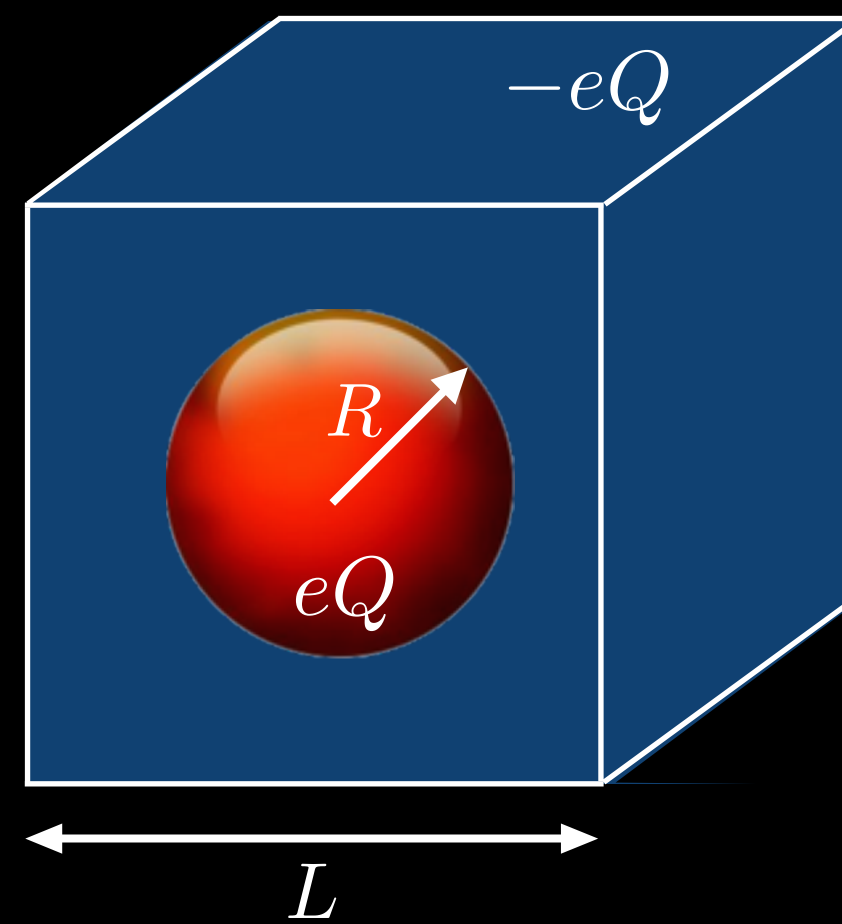
$$V(\mathbf{r}-\mathbf{r}') = \frac{eQ}{4\pi|\mathbf{r}-\mathbf{r}'|}$$



EXPANSION IN R/L

$$U(R,L) = \frac{3}{5}\frac{(Qe)^2}{4\pi R} + \frac{(Qe)^2}{8\pi L}\, c_1 + \frac{(Qe)^2}{10L}\left(\frac{R}{L}\right)^2 + \cdots$$

$$c_1 = -2.83729$$

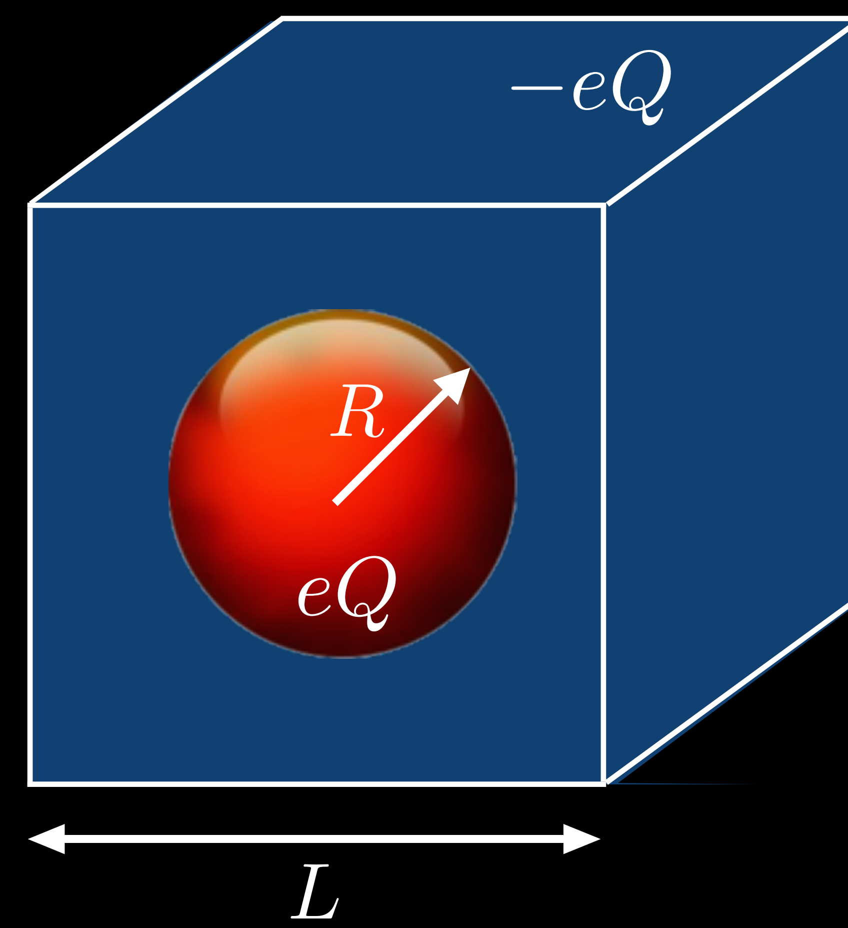
A CLASSICAL EXAMPLE: CHARGED SPHERE IN A FINITE VOLUME

SELF ENERGY

$$U(R,L) = \frac{1}{2}\int d^3r\, d^3r'\, \rho\, V(\mathbf{r}-\mathbf{r}')$$

$$\rho = \frac{eQ}{\frac{4}{3}\pi R^3}$$

$$V(\mathbf{r}-\mathbf{r}') = \frac{eQ}{4\pi|\mathbf{r}-\mathbf{r}'|}$$



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INDEPENDENT OF R

$$\frac{(Qe)^2}{6L^3}\langle r^2\rangle \text{ with } \langle r^2\rangle = \frac{3}{5}R^2$$

A NON-RELATIVISTIC EFFECTIVE FIELD THEORY APPROACH

HEAVY-FIELD FORMALISM

Isgur and Wise, Phys. Lett. B 232, 113 (1989)

Isgur and Wise, Phys. Lett. B 237, 527 (1990)

Jenkins and Manohar, Phys. Lett. B 255, 558 (1991)

Thacker and Lepage, Phys.Rev. D 43, 196 (1991)

Labelle (1992), hep-ph/9209266.

Manohar, Phys. Rev. D 56, 230 (1997)

Luke and Savage, Phys. Rev. D 57, 413 (1998)

Hill and Paz, Phys. Rev. Lett. 107, 160402 (2011)

Chen, Rupak and Savage, Nucl. Phys. A 653, 386 (1999)

COMPOSITE SPIN-0 PARTICLES

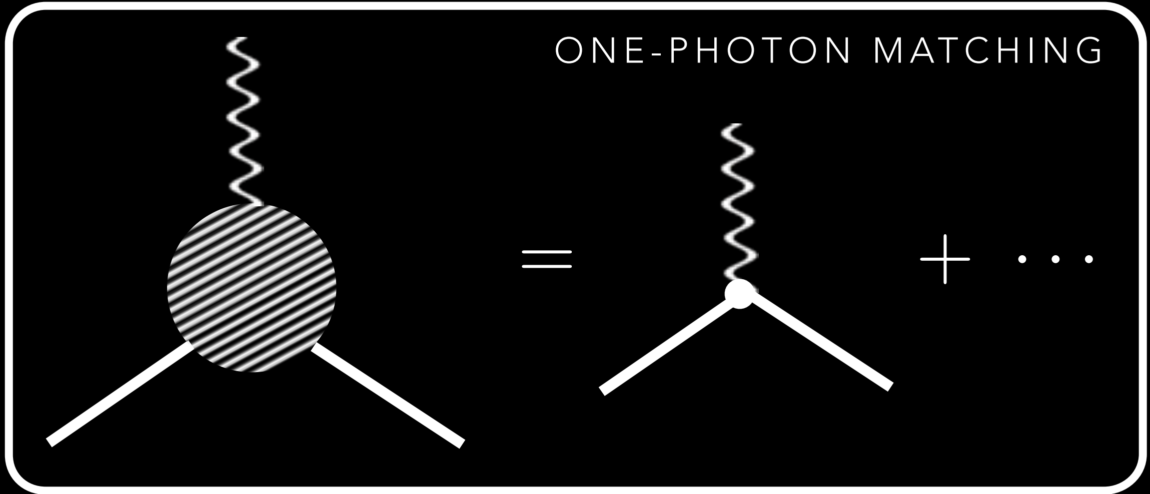
NR-QED LAGRANGIAN

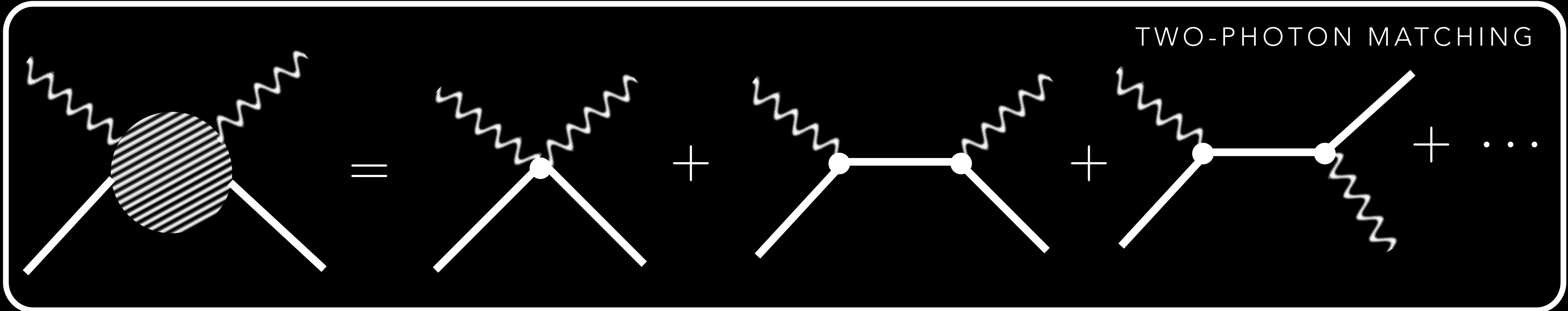
$$\mathcal{L}_\phi = \phi^\dagger \left[iD_0 + \frac{|\mathbf{D}|^2}{2m_\phi} + \frac{|\mathbf{D}|^4}{8m_\phi^3} + \frac{e\langle r^2\rangle_\phi}{6}\nabla\cdot\mathbf{E} + 2\pi\tilde{\alpha}_E^{(\phi)}|\mathbf{E}|^2 + 2\pi\tilde{\beta}_M^{(\phi)}|\mathbf{B}|^2 + iec_M\frac{\{D^i,(\nabla\times\mathbf{B})^i\}}{8m_\phi^3} + \cdots \right]\phi$$

MATCHING

$$D_0 = \partial_0 + ieQA_0$$

$$\mathbf{D} = \vec{\nabla} - ieQ\mathbf{A}$$





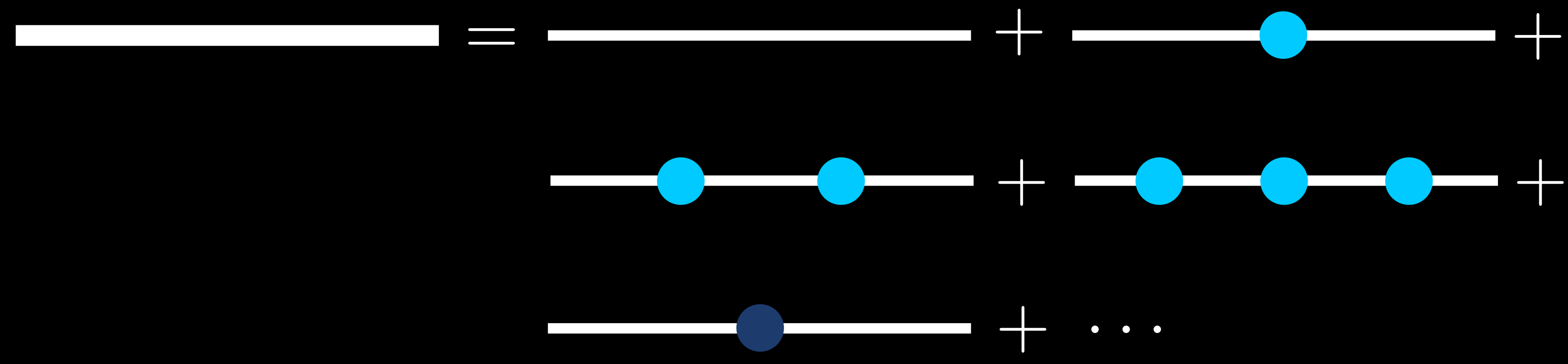
$$\tilde{\alpha}_E^{(\phi)} = \alpha_E^{(\phi)} - \frac{\alpha_e Q}{3m_\phi}\langle r^2\rangle_\phi$$

$$\tilde{\beta}_M^{(\phi)} = \beta_M^{(\phi)}$$

$$c_M = \frac{2}{3}m_\phi^2\langle r^2\rangle_\phi$$

COMPOSITE SPIN-0 PARTICLES

PROPAGATOR

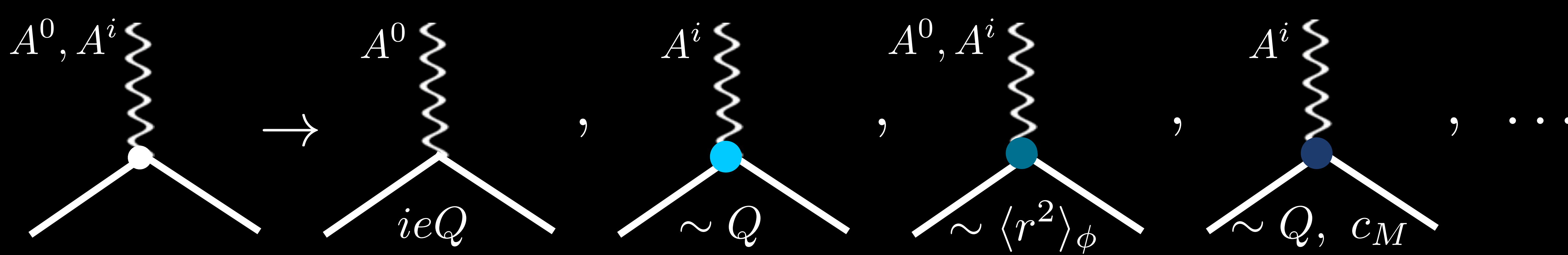


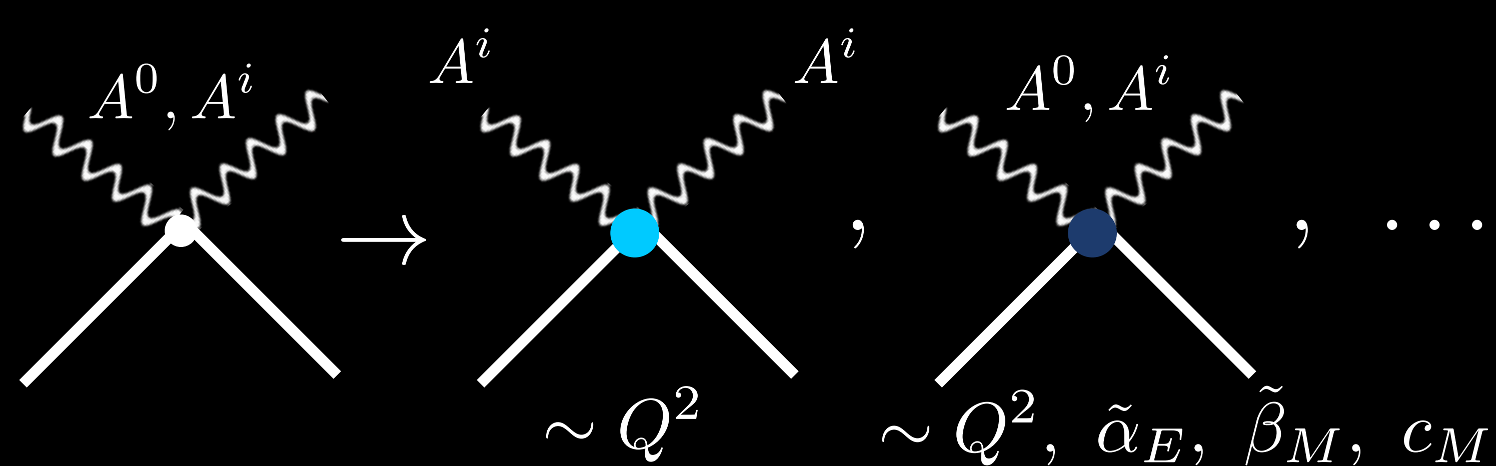
$\mathcal{O}\left(\frac{1}{m_\phi}\right)$

$\mathcal{O}\left(\frac{1}{m_\phi^2}, \langle r^2\rangle_\phi\right)$

$\mathcal{O}\left(\frac{1}{m_\phi^3}, \frac{\langle r^2\rangle_\phi}{m_\phi}, \alpha_E, \beta_M\right)$

VERTICES



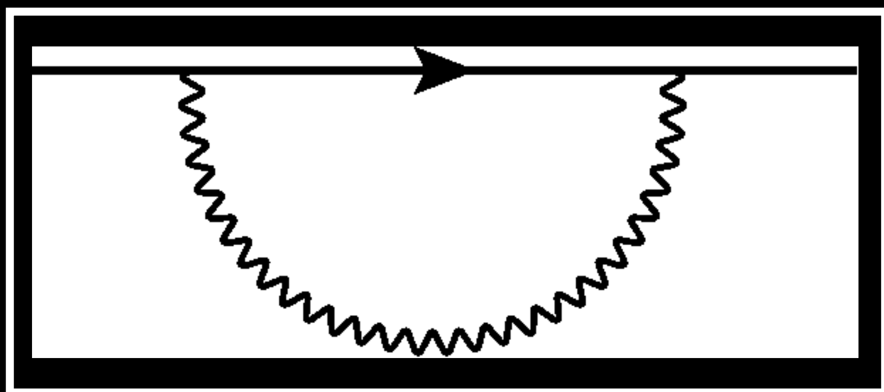


ONLY TREE LEVEL

COMPOSITE SPIN-0 PARTICLES

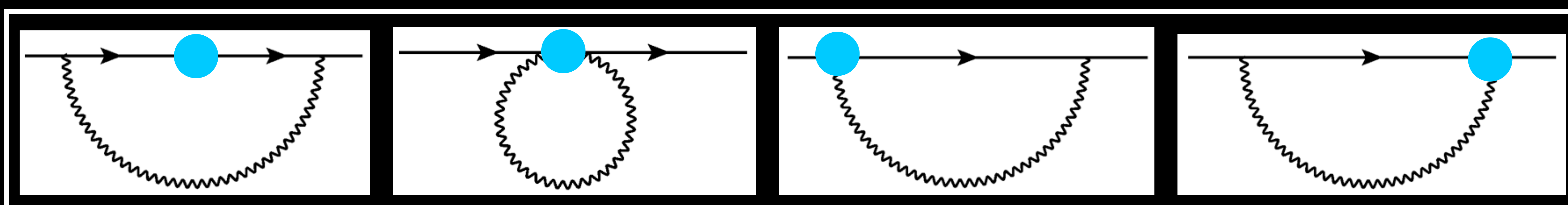
WHAT IS $\delta m_\phi = m_\phi^V - m_\phi^\infty$?

LEADING ORDER



$$\delta m_\phi^{(\mathrm{LO})} = \frac{\alpha_e Q^2}{2\mathrm{L}}\ c_1$$

NEXT-TO-LEADING ORDER



Also by:
Hayakawa and Uno, Prog. Theor. Phys. 120, 413 (2008)
Borsanyi, et al., arXiv:1406088.

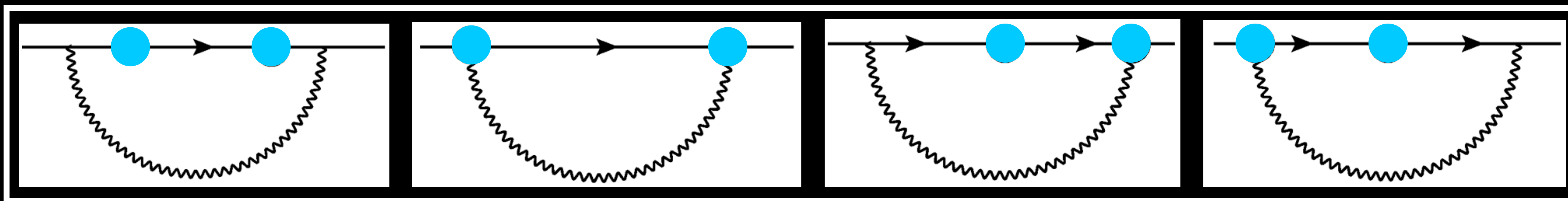
$$\delta m_\phi^{(\mathrm{NLO})} = \frac{\alpha_e Q^2}{m_\phi L^2}\ c_1$$

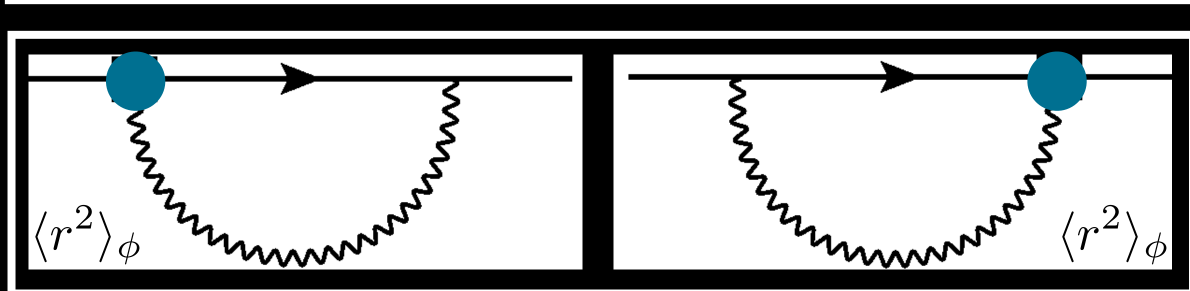
$$\hat{\sum_{\mathbf{n}\neq\mathbf{0}}} \frac{1}{|\mathbf{n}|^2} = \pi c_1, \quad \hat{\sum_{\mathbf{n}\neq\mathbf{0}}} \frac{1}{|\mathbf{n}|} = c_1 = -2.83729, \quad \hat{\sum_{\mathbf{n}\neq\mathbf{0}}} 1 = -1, \hat{\sum_{\mathbf{n}\neq\mathbf{0}}} |\mathbf{n}| = c_{-1} = -0.266596.$$

COMPOSITE SPIN-0 PARTICLES

WHAT IS $\delta m_\phi = m_\phi^V - m_\phi^\infty$?

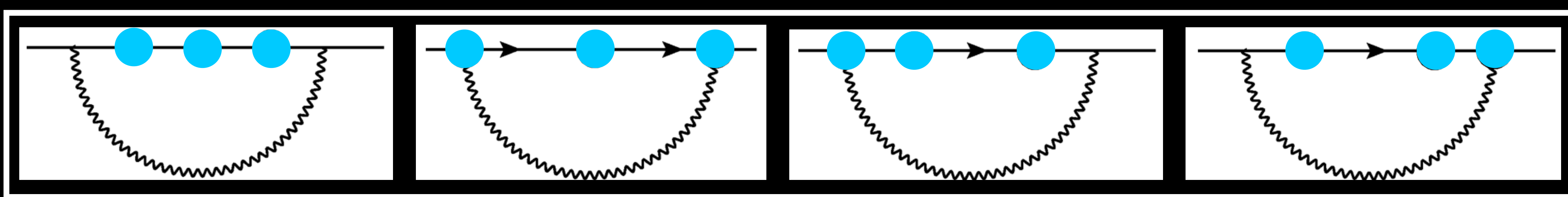
NEXT-TO-NEXT-TO-LEADING ORDER

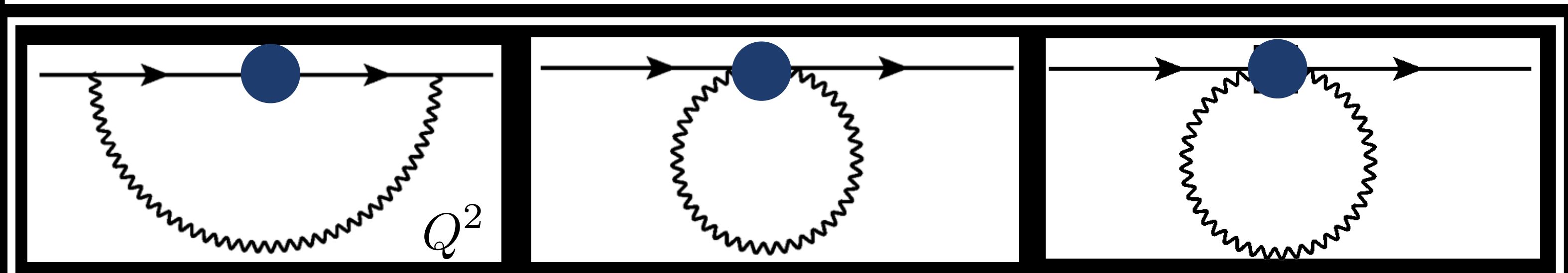




$$\delta m_\phi^{(\mathrm{N^2LO})} = \frac{2\pi \alpha_e Q}{3L^3} \langle r^2 \rangle_\phi$$

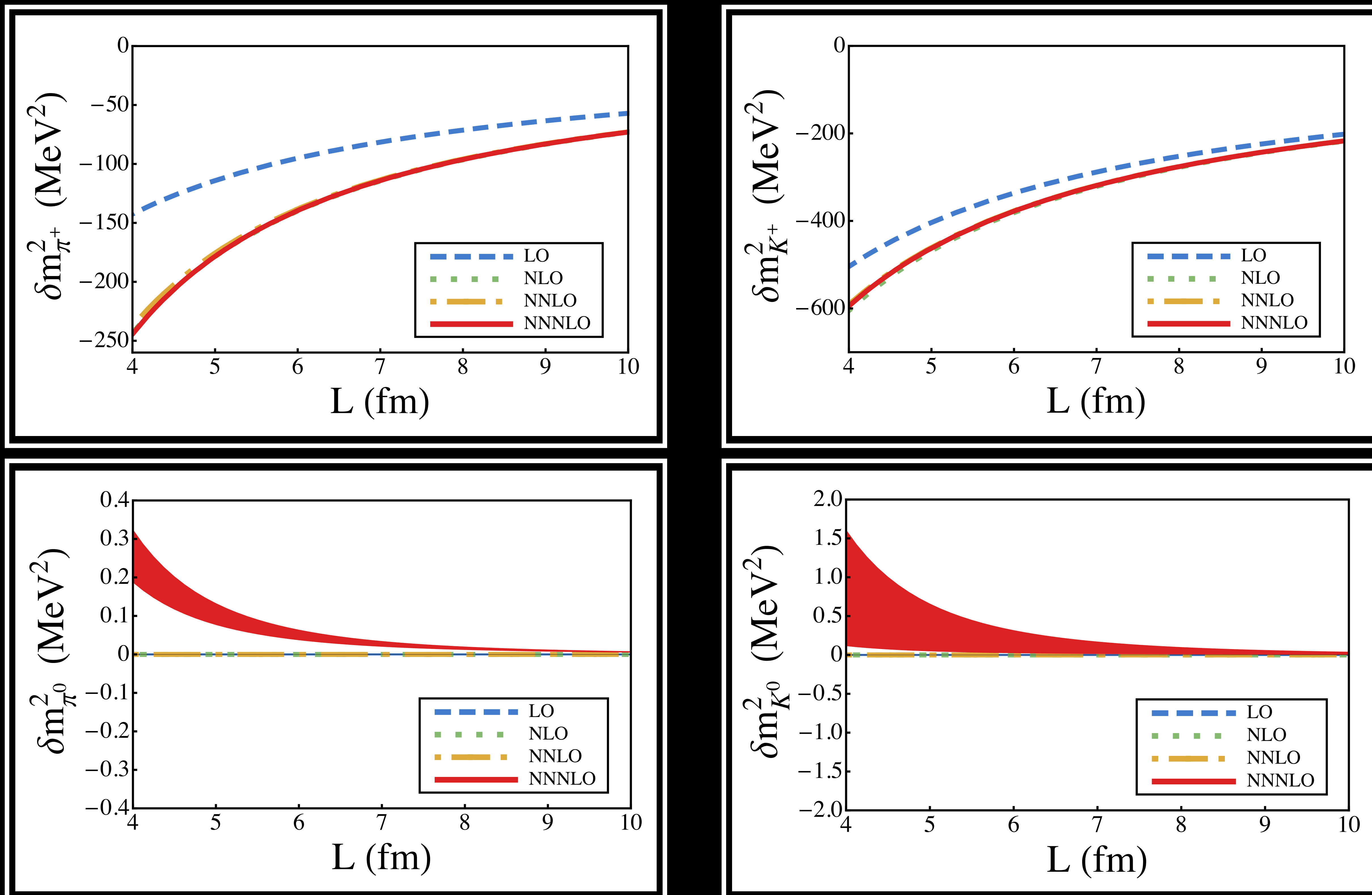
NEXT-TO-NEXT-TO-NEXT-T0-LEADING ORDER





$$\delta m_\phi^{(\mathrm{N^3LO})} = -\frac{4\pi^2}{L^4} \left(\alpha_E^{(\phi)} + \beta_M^{(\phi)}\right) c_{-1} + \frac{4\pi^2 \alpha_e Q}{3 m_\phi L^4} \langle r^2 \rangle_\phi \, c_{-1}$$

VOLUME-DEPENDENCE OF KAON AND PION MASSES



COMPOSITE SPIN-1/2 PARTICLES

NR-QED LAGRANGIAN

$$\mathcal{L}_\psi = \psi^\dagger [iD_0 + \frac{|\mathbf{D}|^2}{2M_\psi} + \frac{|\mathbf{D}|^4}{8M_\psi^3} + c_F \frac{e}{2M_\psi}\sigma\cdot\mathbf{B} + c_D \frac{e}{8M_\psi^2}\nabla\cdot\mathbf{E} + ic_S \frac{e}{8M_\psi^2}\sigma\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}) +$$

$$2\pi\tilde{\alpha}_E^{(\psi)}|\mathbf{E}|^2 + 2\pi\tilde{\beta}_M^{(\psi)}|\mathbf{B}|^2 + iec_M \frac{\{D^i, (\nabla\times\mathbf{B})^i\}}{8M_\psi^3} + \cdots]\psi$$

MATCHING

$$c_F = Q + \kappa_\psi + \mathcal{O}(\alpha_e)$$

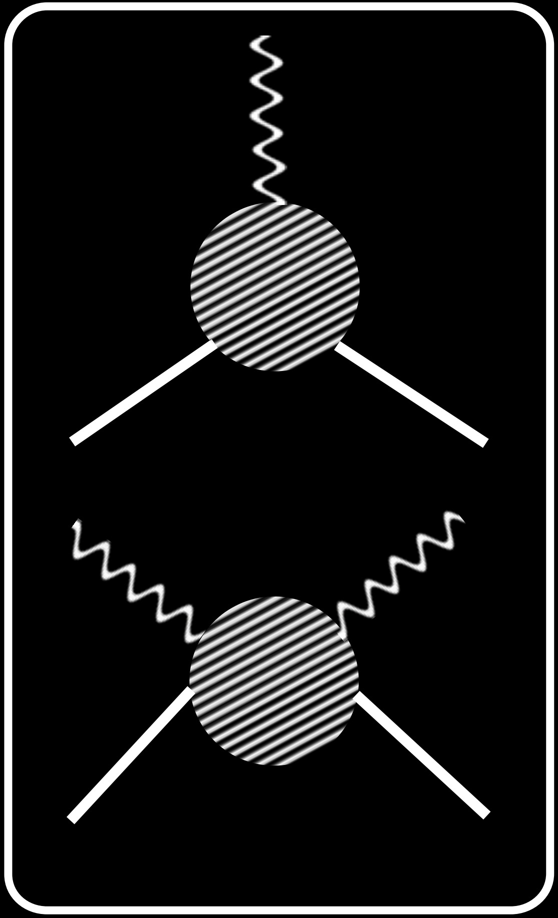
$$c_D = Q + \frac{4}{3}M_\psi^2\langle r^2\rangle_\psi + \mathcal{O}(\alpha_e)$$

$$c_S = 2c_F - Q$$

$$c_M = (c_D - c_F)/2$$

$$\tilde{\alpha}_E^{(\psi)} = \alpha_E^{(\psi)} - \frac{\alpha_e}{4M_\psi^3}\left(Q^2 + \kappa_\psi^2\right) - \frac{\alpha_e Q}{3M_\psi}\langle r^2\rangle_\psi$$

$$\tilde{\beta}_M^{(\psi)} = \beta_M^{(\psi)} + \frac{\alpha_e Q^2}{4M_\psi^3}$$

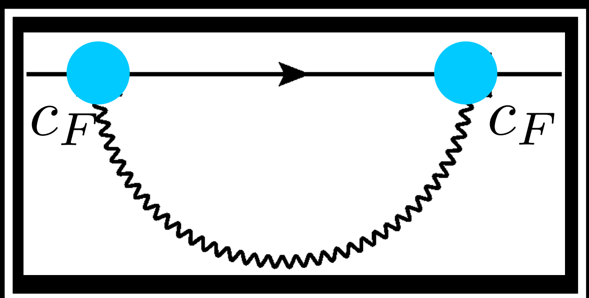


COMPOSITE SPIN-1/2 PARTICLES

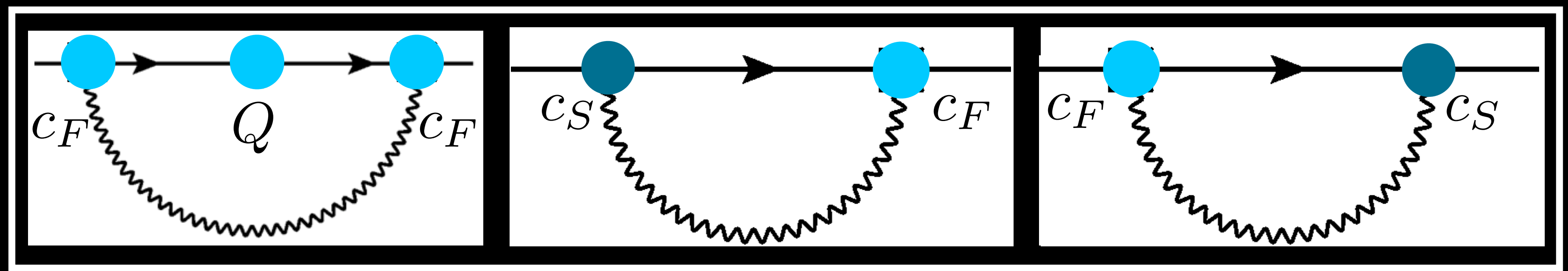
WHAT IS $\delta M_\psi = M_\psi^V - M_\psi^\infty$?

NEW CONTRIBUTIONS:

NNLO



NNNLO



RESULT

$$\delta M_\psi = \frac{\alpha_e Q^2}{2L} c_1 + \frac{\alpha_e Q^2}{M_\psi L^2} c_1 + \frac{2\pi\alpha_e Q}{3L^3}\langle r^2\rangle_\psi + \frac{\pi\alpha_e}{M_\psi^2 L^3}\left[\frac{1}{2}Q^2 + (Q+\kappa_\psi)^2\right]$$

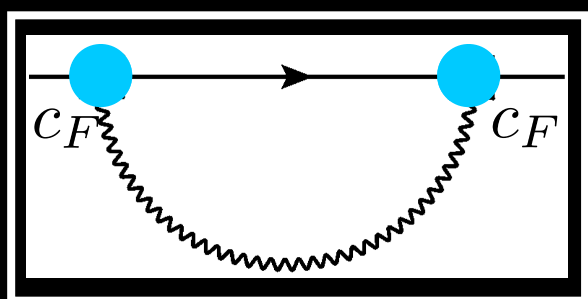
$$-\frac{4\pi^2}{L^4}\left(\tilde{\alpha}_E^{(\psi)} + \tilde{\beta}_M^{(\psi)}\right) c_{-1} + \frac{\pi^2\alpha_e Q}{M_\psi^3 L^4}\left(\frac{4}{3}M_\psi^2\langle r^2\rangle_\psi - \kappa_\psi\right) c_{-1} - \frac{\alpha_e \pi^2}{M_\psi^3 L^4}\kappa_\psi(Q+\kappa_\psi)c_{-1}$$

COMPOSITE SPIN-1/2 PARTICLES

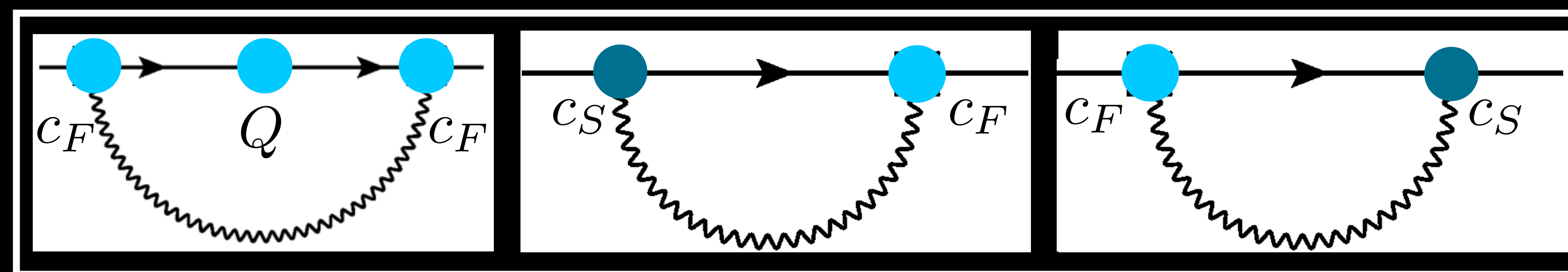
WHAT IS $\delta M_\psi = M_\psi^V - M_\psi^\infty$?

NEW CONTRIBUTIONS:

NNLO



NNNLO



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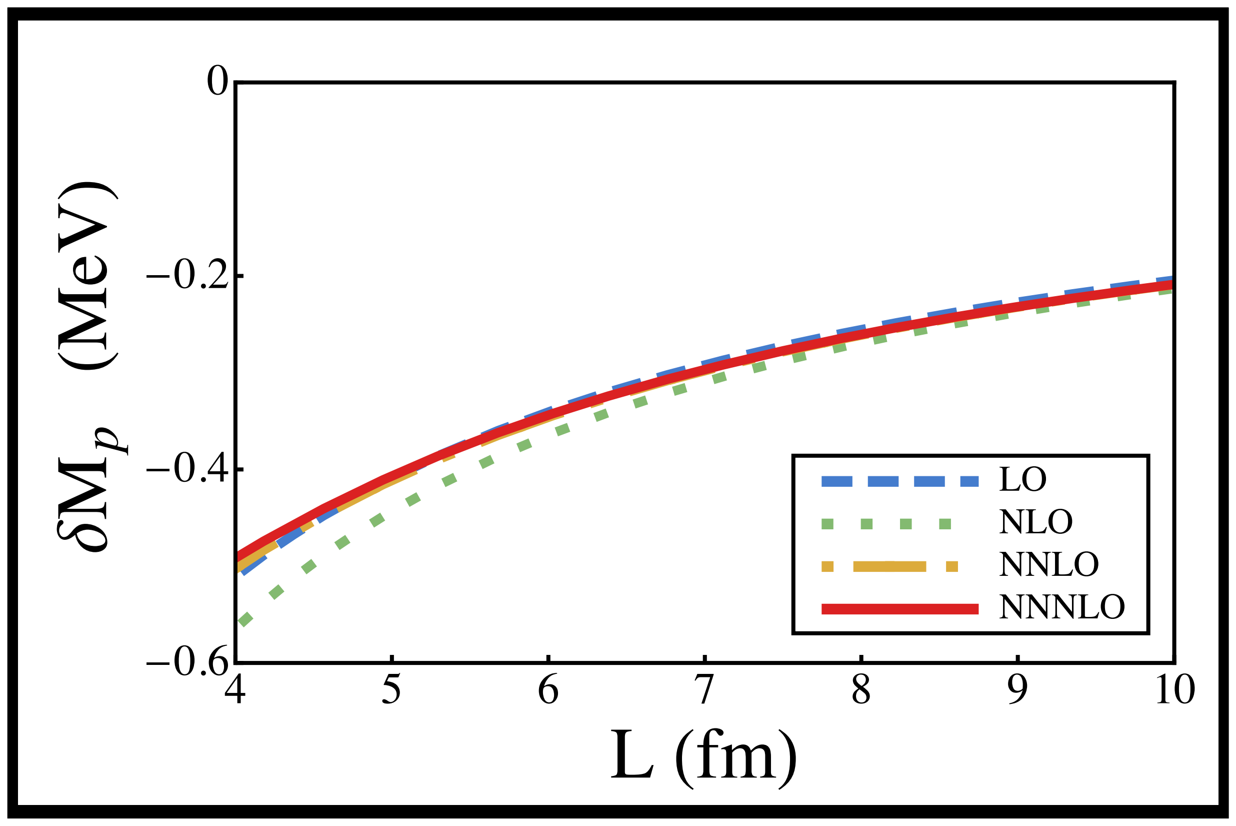
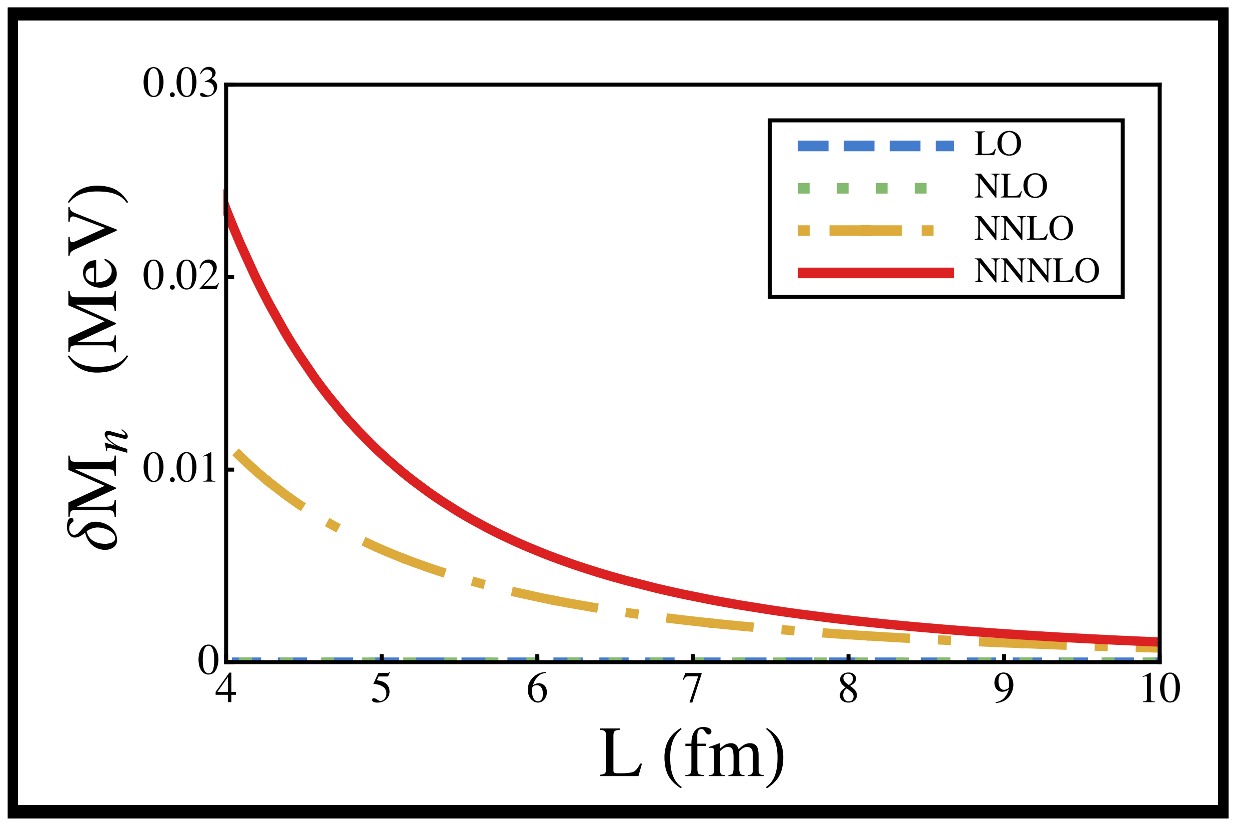
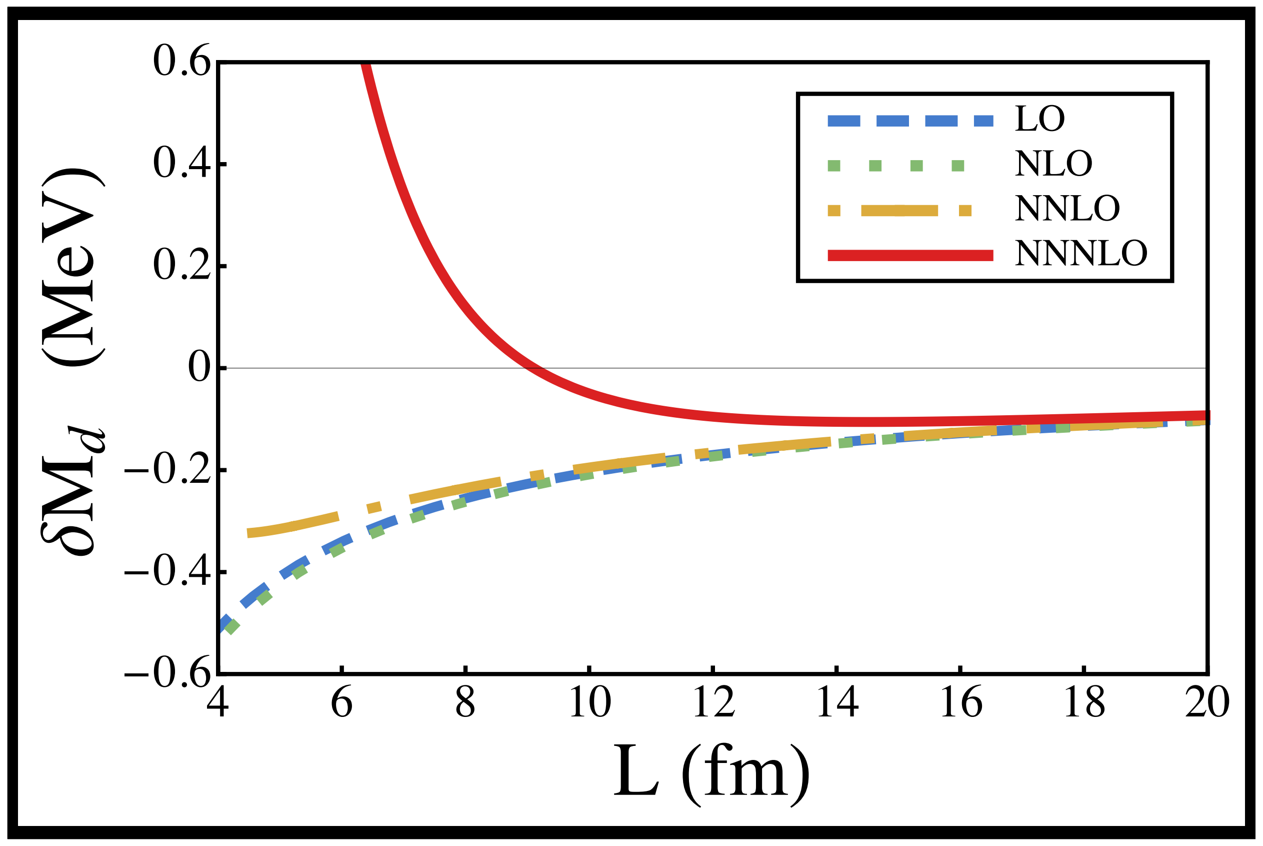
$$- \frac{4\pi^2}{L^4} \left(\tilde{\alpha}_E^{(\psi)} + \tilde{\beta}_M^{(\psi)} \right) c_{-1} + \frac{\pi^2 \alpha_e Q}{M_\psi^3 L^4} \left(\frac{4}{3} M_\psi^2 \langle r^2 \rangle_\psi - \kappa_\psi \right) c_{-1} - \frac{\alpha_e \pi^2}{M_\psi^3 L^4} \kappa_\psi (Q + \kappa_\psi) c_{-1}$$

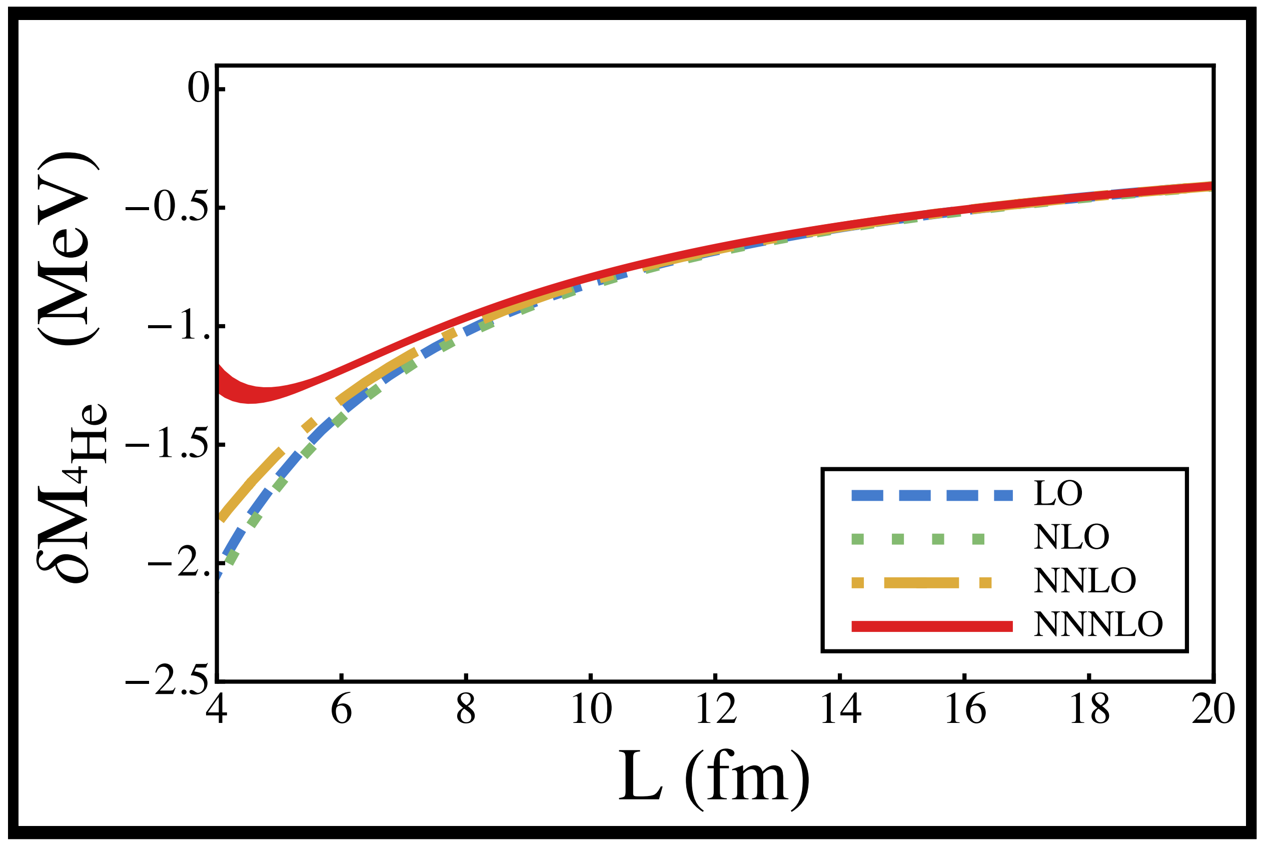
VOLUME-DEPENDENCE OF PROTON AND NEUTRON MASSES





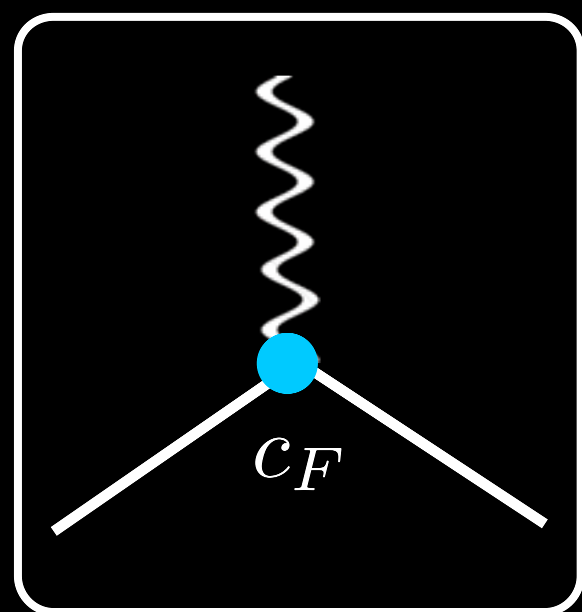
VOLUME-DEPENDENCE OF LIGHT NUCLEI: DEUTERON AND HELIUM-4

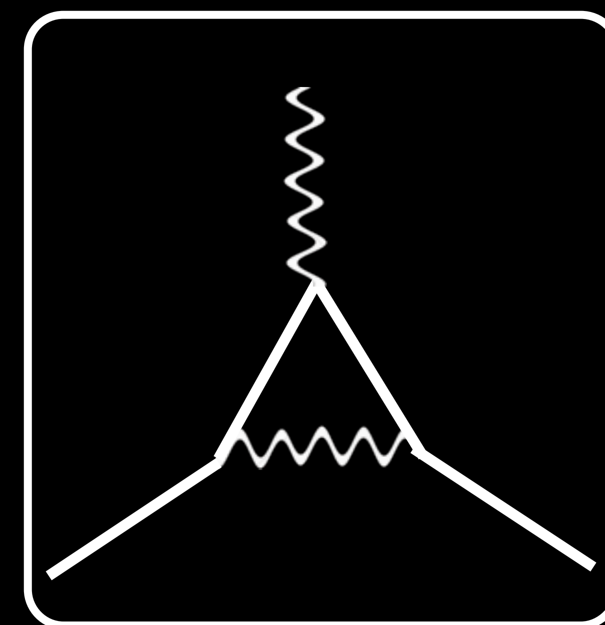




MAGNETIC MOMENT OF MUON

FINITE-VOLUME CORRECTIONS TO κ_μ?





NR-QED $\times \left(\sqrt{\frac{E}{M_\mu}}\right)^2 \longrightarrow$ QED

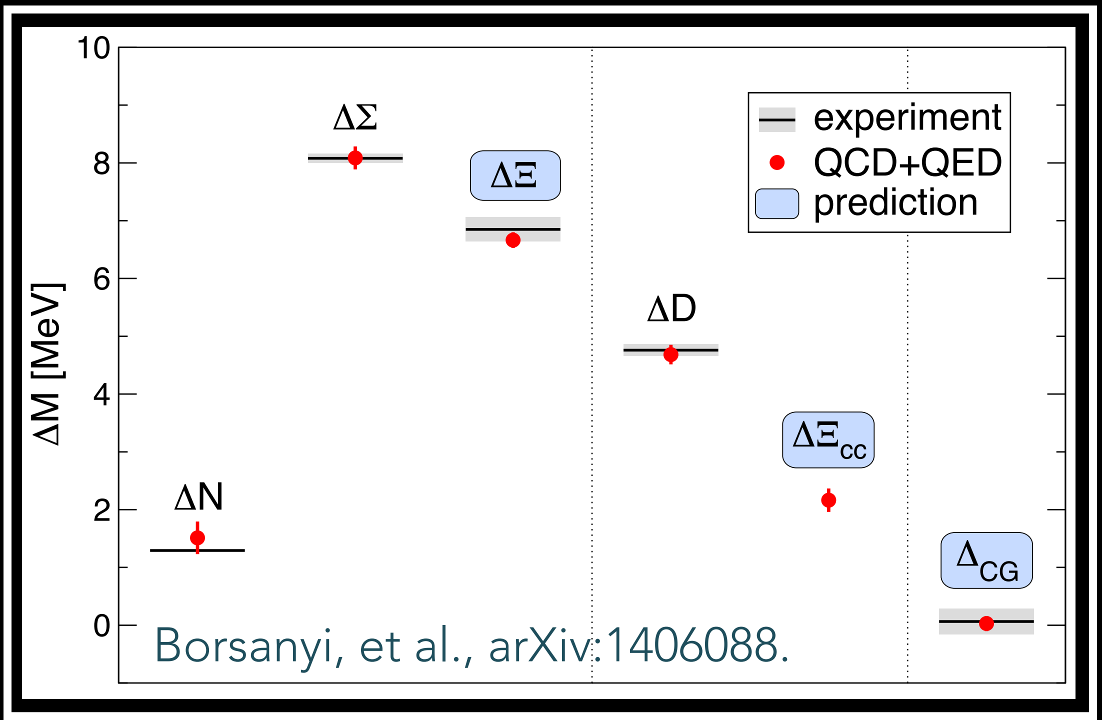
RESULT

$$\kappa_\mu \equiv \frac{g_\mu - 2}{2} = \frac{\alpha_e}{2\pi}\left[1 + \frac{\pi c_1}{M_\mu \mathrm{L}} + \mathcal{O}\left(\frac{1}{M_\mu^2 \mathrm{L}^2}\right)\right]$$

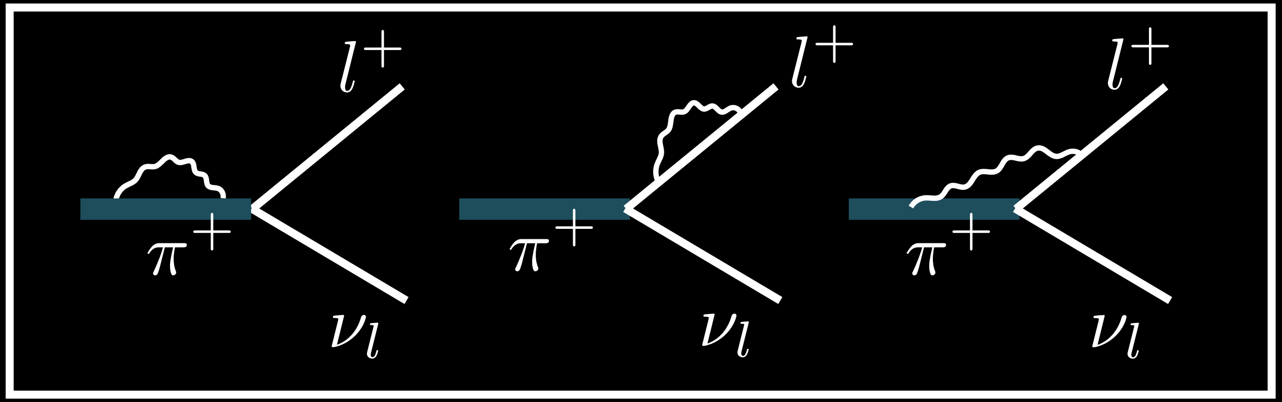
$1\mathrm{ppm}$ PRECISION REQUIRES $\sim (60\ \mathrm{nm})^3$ VOLUMES!!

WHY SHOULD WE CONTROL FINITE-VOLUME EFFECTS IN SINGLE-HADRON SECTOR?

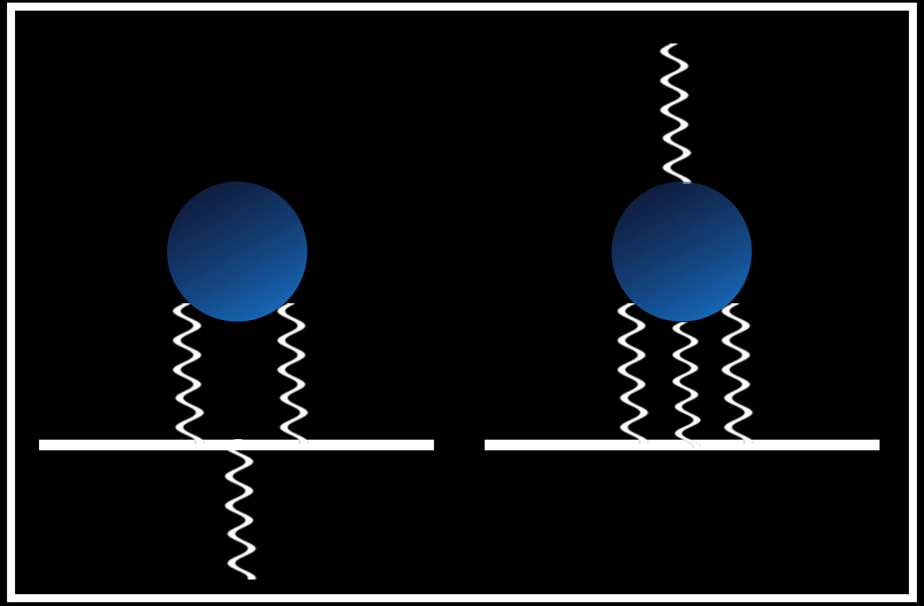
- MASS SPLITTING IN HADRONIC MULTIPLETS e.g. Blum, et al., Phys. Rev. D 82, 094508 (2010) Divitiis, et al., Phys. Rev. D 87, 114505 (2013)



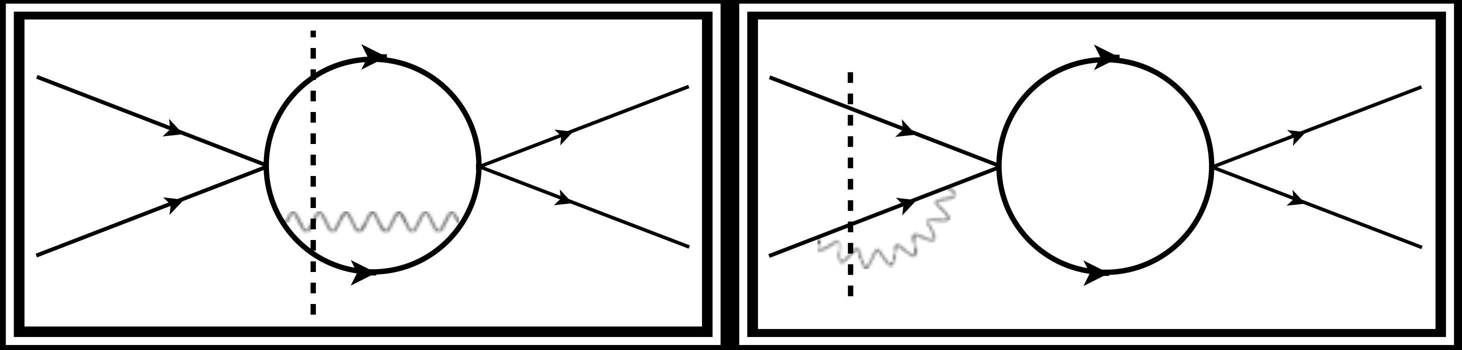
- SINGLE-HADRON MATRIX ELEMENTS e.g. Corrasco, et al., arXiv:1502.0025 (2015)



- HADRONIC CONTRIBUTIONS TO MUON $g-2$ e.g. Blum, et al., Phys. Rev. Lett. 114, 012001 (2014)

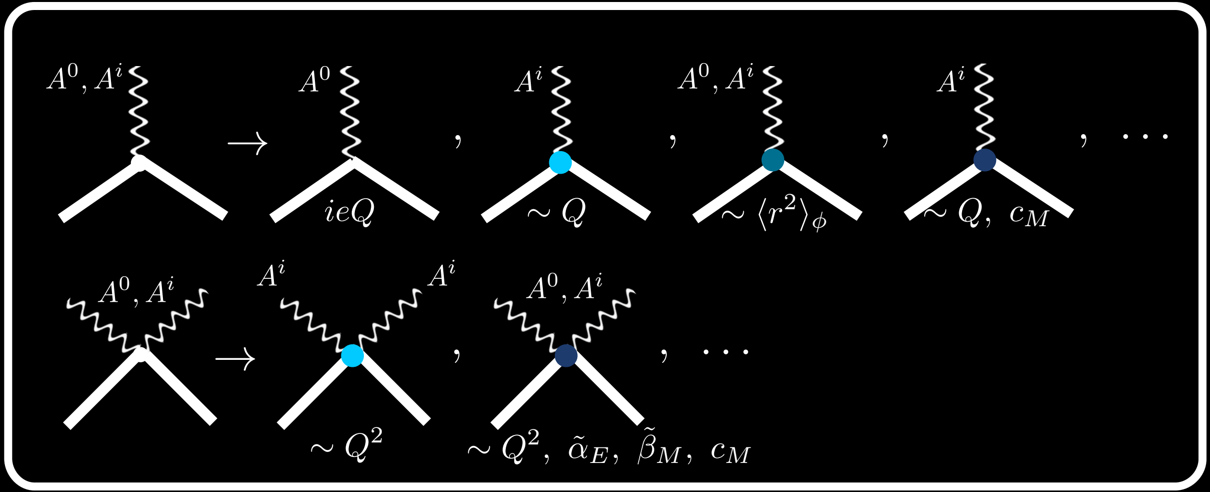


- MULTI-HADRON PROCESSES WITH QED Beane and Savage, Phys. Rev. D 90, 074511 (2014)

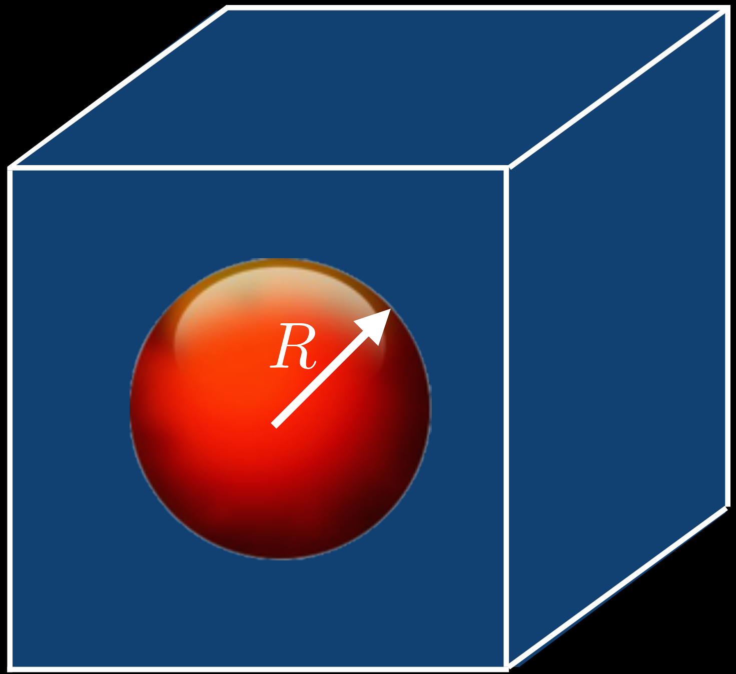


CONCLUSION

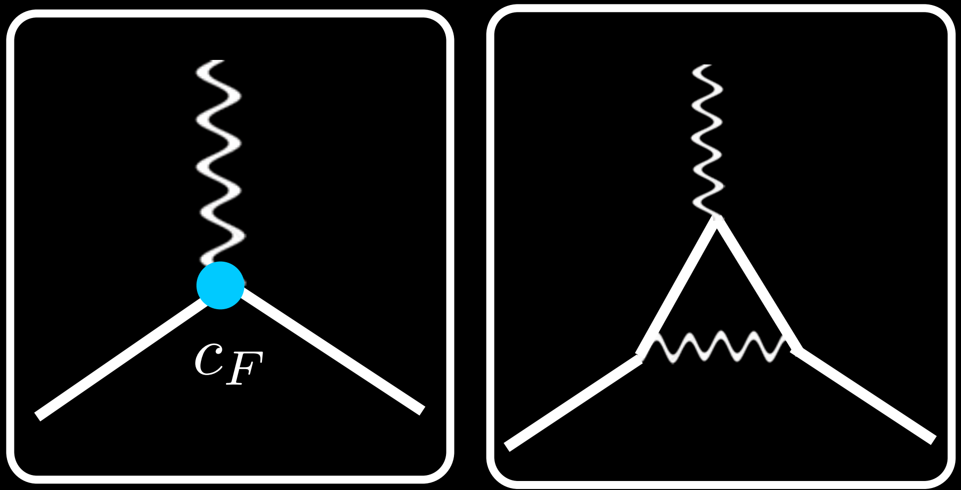
- NREFT IS A SIMPLE AND GENERAL FORMALISM TO STUDY FINITE-VOLUME QED EFFECTS.



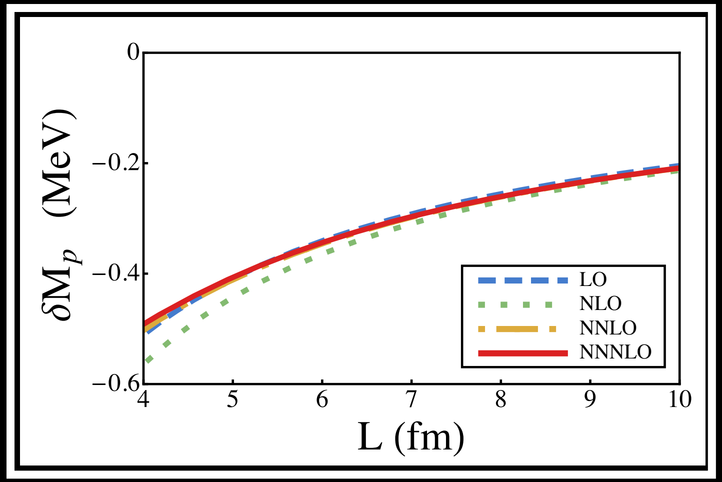
- ELIMINATION OF ZERO MODE GIVES RISE TO A SENSIBLE QED IN A FINITE VOLUME.

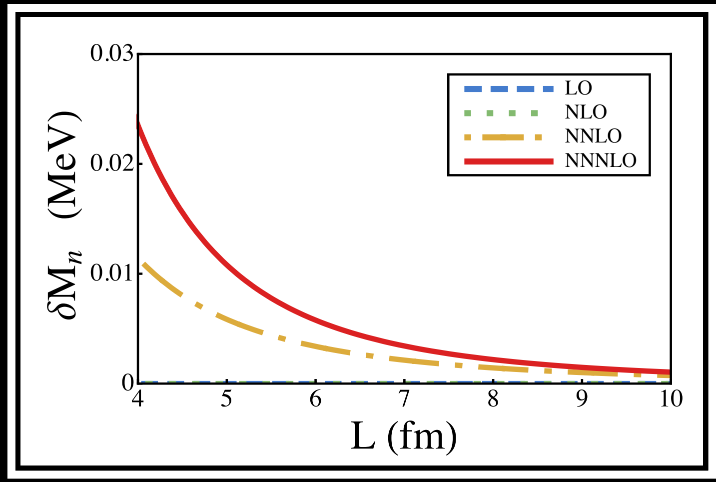


- THE DIRECT EVALUATION OF MUON MAGNETIC MOMENT TO REQUIRED PRECISION REQUIRES UNFEASIBLY LARGE VOLUMES.



- CHARGED PARTICLES ARE LARGELY AFFECTED BY THE FINITE BOUNDARY OF THE VOLUME.





- NEUTRAL PARTICLES RECEIVE CORRECTIONS TO THEIR MASSES DUE TO THEIR MAGNETIC MOMENT (IF ANY) AND THEIR POLARIZABILITIES.

THANK YOU